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NOTE ON THE THREE SERIES THEOREM

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The Kolmogorov Three Series Theorem appears in textbooks with basically two different sets of hypotheses. The most common statement defines a sequence of truncated random variables as $Y_n^c = X_n 1_{[|X_n| \leqq c]}$ for $c > 0$. Using this method of truncation it is easy to show that all three conditions of the theorem are necessary. However, some books, including Feller vol. II, define the truncated random variables as

$$\begin{aligned} Y_n^c &= -c \qquad \text{if } X_n \leqq -c, \\ &= X_n \qquad \text{if } -c < X_n < c, \\ &= +c \qquad \text{if } X_n \geqq c. \end{aligned}$$

The theorem is stated with the same three conditions. However, in this case, only two of the conditions are necessary.

THEOREM. *Let $\{X_n\}$ be a sequence of independent random variables. The series $\sum_{n=1}^{\infty} X_n$ converges a.s. if and only if for some $c > 0$ $\sum_{n=1}^{\infty} E(Y_n^c) < \infty$ and $\sum_{n=1}^{\infty} \text{Var}\,(Y_n^c) < \infty$ where Y_n^c is defined as above.*

PROOF. In view of Feller's version of the Three Series Theorem, it is sufficient to show that $\sum_{n=1}^{\infty} E(Y_n^c) < \infty$ and $\sum_{n=1}^{\infty} \text{Var}\,(Y_n^c) < \infty$ imply $\sum_{n=1}^{\infty} P[|X_n| \geqq c] < \infty$. Let $A_n = \{w: X_n \geqq c\}$ and $B_n = \{w: X_n \leqq -c\}$. Assume $\sum_{n=1}^{\infty} P(A_n \cup B_n) = \infty$. Choose N such that $|E(Y_n^c)| < c/4$ for $n \geqq N$. Then $\text{Var}\,(Y_n^c) \geqq \int_{A_n \cup B_n} [Y_n^c - E(Y_n^c)]^2\, dP \geqq 9c^2 P(A_n \cup B_n)/16$. But this implies $\sum_{n=1}^{\infty} \text{Var}\,(Y_n^c) = \infty$, which is a contradiction. Therefore we need only verify the convergence of the two series in this case.

REFERENCE

[1] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

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