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MONOTONICITY PROPERTIES OF THE MULTINOMIAL DISTRIBUTION

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0. Summary. Let $X = (X_1, \cdots, X_k)$ have the multinomial distribution, given by

$$\Pr\{X = x\} = n! \prod_{i=1}^{k} (p_i^{x_i}/(x_i!)) \tag{0.1}$$

where $x = (x_1, \cdots, x_k)$, $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} p_i = 1$, and let

$$C(p_1, \cdots, p_m) = \Pr\{X_i \geqq s_i;\ i = 1, \cdots, m\} \tag{0.2}$$

where $\sum_{i=1}^{m} s_i \leqq n$ and $m \leqq \min(k-1, n)$. We show that $C(p_1, \cdots, p_m)$ is nondecreasing in p_i for $i = 1, \cdots, m$ and that for $s_i = s_j$,

$$C(p_1, \cdots, p_m) \leqq C_{ij}(p_1, \cdots, p_m) \quad \text{and} \tag{0.3}$$

$$C(p_1, \cdots, p_m) \geqq C_{ijt}(p_1, \cdots, p_m) \tag{0.4}$$

where $C_{ij}(p_1, \cdots, p_m)$ is obtained from $C(p_1, \cdots, p_m)$ by substituting $p = \frac{1}{2}(p_i + p_j)$ for p_i and p_j and $C_{ijt}(p_1, \cdots, p_m)$ is obtained from $C(p_1, \cdots, p_m)$ by substituting t for p_i and $p_i + p_j - t$ for p_j where $0 \leqq t \leqq \min(p_i, p_j)$. These and similar results are shown. An application of these results to a multiple decision problem is indicated.

1. Monotonicity properties. Let $C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) = \Pr\{X_i \geqq s_i, i = 1, \cdots, m; X_j = s_j, j = m+1, \cdots, k\}$. From Lemma 2.2 of [1] we have

$$\begin{aligned} C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) &= P\{X_j = s_j (j = m+1, \cdots, k), \\ &\quad \textstyle\sum_1^m X_i = n'\} \times P\{X_i \geqq s_i (i = 1, \cdots, m) \mid \sum_1^m X_i = n'\} \\ &= p_0^{n'} u \int_0^{p'_1} \cdots \int_0^{p'_m} (\textstyle\prod_1^m t_i^{s_i - 1})(1 - t_0)^{n' - s_0} \prod_1^m dt_i \\ &= u \int_0^{p_1} \cdots \int_0^{p_m} (\textstyle\prod_1^m (t_i^{s_i - 1})(p_0 - t_0)^{n' - s_0} \prod_1^m dt_i \end{aligned} \tag{1.1}$$

where $s_1, \cdots, s_m$ are positive integers, $n' = n - \sum_{m+1}^{k} s_j$, $s_0 = \sum_1^m s_i \leqq n'$,

$$p_0 = \textstyle\sum_1^m p_i,\ t_0 = \sum_1^m t_i,\ p_i' = p_i/p_0 \quad \text{for} \quad i = 1, \cdots, m,$$

$$u = (n!/n'!)(\textstyle\prod_{m+1}^{k} p_j^{s_j}/s_j!)/B(s_1, \cdots, s_m, n' - s_0 + 1)$$

and $B(\cdot)$ denotes the beta function. Differentiating (1.1) with respect to p_1, putting $p_2 = a - p_1$ where a is fixed, we have

$$\begin{aligned} &\partial C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k)/\partial p_1 \\ &= u \int_0^{p_3} \cdots \int_0^{p_m} (\textstyle\prod_3^m t_i^{s_i - 1}) \{p_1^{s_1 - 1} \int_0^{p_2} t_2^{s_2 - 1} \\ &\quad \times (p_{01} - \textstyle\sum_2^m t_i)^{n' - s_0} dt_2 - p_2^{s_2 - 1} \int_0^{p_1} t_1^{s_1 - 1} \\ &\quad \times (p_{02} - t_1 - \textstyle\sum_3^m t_i)^{n' - s_0} dt_1\} \prod_3^m dt_i \end{aligned} \tag{1.2}$$

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where we write $p_{0i} = p_0 - p_i$. The quantity inside the braces on the right hand side of (1.2) can be written as

$$V = p_1^{s_1-1} \int_{p_1}^{a} (x-p_1)^{s_2-1} (p_0 - x - \sum_3^m t_i)^{n'-s_0} dx - p_2^{s_2-1} \int_{p_2}^{a} (x-p_2)^{s_1-1} (p_0 - x - \sum_3^m t_i)^{n'-s_0} dx.$$

Let $s_1 = s_2$. As $p_1(x-p_1) \geqq (\leqq) p_2(x-p_2)$ according as $p_1 \leqq (\geqq) p_2$ for $x \leqq p_1 + p_2$ we have that $V \geqq (\leqq) 0$ according as $p_1 \leqq (\geqq) p_2$. As the above results for p_1 and p_2 hold for any pair (i, j) we have from (1.2) for $s_i = s_j (i, j \leqq m)$,

$$C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) \leqq C_{ij}(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) \quad \text{and} \tag{1.3}$$

$$C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) \geqq C_{ijt}(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) \tag{1.4}$$

where $C_{ij}(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k)$ is obtained from $C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k)$ by substituting $p = \frac{1}{2}(p_i + p_j)$ for p_i and p_j and $C_{ijt}(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k)$ is obtained from $C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k)$ by substituting t for p_i and $p_i + p_j - t$ for p_j where $0 \leqq t \leqq \min(p_i, p_j)$. For $s_i = s_j = 0$ equality holds in (1.3) and in (1.4).

From (1.3) and (1.4) we have for $s_1 = s_2 = \cdots = s_m$,

$$C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) \leqq C(q, \cdots, q; s_{m+1}, \cdots, s_k) \quad \text{and} \tag{1.5}$$

$$C(p_1, \cdots, p_m; s_{m+1}, \cdots, s_k) \geqq C(t, \cdots, t, p_0 - (m-1)t; s_{m+1}, \cdots, s_k) \tag{1.6}$$

where $q = p_0/m$ and $0 \leqq t \leqq \min(p_1, \cdots, p_m)$.

From (1.5) and (1.6) we have, putting $m = k-1$ and $s_1 = s_2 = \cdots = s_k$,

$$\begin{aligned} &\Pr\{X_i \geqq X_k, i = 1, \cdots, k-1; p_1, \cdots, p_{k-1}, p_k\} \\ &\qquad \leqq \Pr\{X_i \geqq X_k, i = 1, \cdots, k-1; q, \cdots, q, p_k\} \quad \text{and} \end{aligned} \tag{1.7}$$

$$\begin{aligned} &\Pr\{X_i \geqq X_k, i = 1, \cdots, k-1; p_1, \cdots, p_{k-1}, p_k\} \\ &\qquad \geqq \Pr\{X_i \geqq X_k, i = 1, \cdots, k-1; t, \cdots, t, p_0 - (k-2)t, p_k\} \end{aligned} \tag{1.8}$$

where $p_0 = \sum_1^{k-1} p_i$, $q = p_0/(k-1)$ and $0 \leqq t \leqq \min(p_1, \cdots, p_{k-1})$.

It is seen that (0.3) and (0.4) follow from (1.3) and (1.4), respectively. As a corollary to (0.3) and (0.4) we have from (1.5) and (1.6) for $s_1 = s_2 = \cdots = s_m$,

$$C(p_1, \cdots, p_m) \leqq C(q, \cdots, q) \quad \text{and} \tag{1.9}$$

$$C(p_1, \cdots, p_m) \geqq C(t, \cdots, t, p_0 - (m-1)t) \tag{1.10}$$

where q, p_0 and t are defined following (1.6).

From (1.1) we get

$$\begin{aligned} C(p_1, \cdots, p_m) = &\{\int_0^{p_1} \cdots \int_0^{p_m} (\prod_1^m t_i^{s_i-1})(1-t_0)^{n-s_0} \prod_1^m dt_i\} / \\ &\{B(s_1, \cdots, s_m, n-s_0+1)\} \end{aligned} \tag{1.11}$$

where $s_1, \cdots, s_m$ are positive integers. It is clear from (1.11) that $C(p_1, \cdots, p_m)$ is nondecreasing in p_i for $i = 1, \cdots, m$.

For the opposite tail probabilities of the multinomial distribution we have from Lemma 2.3 of [1]

$$(1.12)\qquad D(p_1, \cdots, p_m) = \Pr\{X_i \leqq s_i - 1 \,;\, i = 1, \cdots, m\}$$
$$= \frac{1}{B(s_1, \cdots, s_m, n-s_0+1)} \int_{p_1}^{1-\Sigma_2^m p_i} \int_{p_2}^{1-\Sigma_3^m p_i - t_1} \cdots$$
$$\int_{p_m}^{1-\Sigma_1^{m-1} t_i} \left(\prod_1^m t_i^{s_i-1}\right)(1-t_0)^{n-s_0} \prod_1^m dt_i$$

where $s_1, \cdots, s_m$ are positive integers, $s_0 = \sum_1^m s_i \leqq n$ and $m \leqq \min(k-1, n)$. It is clear from (1.12) that $D(p_1, \cdots, p_m)$ is decreasing in p_i for $i = 1, \cdots, m$. As we proved (0.3) and (0.4), we can show that for $s_i = s_j$,

$$(1.13)\qquad D(p_1, \cdots, p_m) \leqq D_{ij}(p_1, \cdots, p_m) \quad \text{and}$$

$$(1.14)\qquad D(p_1, \cdots, p_m) \geqq D_{ijt}(p_1, \cdots, p_m)$$

where $D_{ij}(p_1, \cdots, p_m)$ is obtained from $D(p_1, \cdots, p_m)$ by substituting $p = \frac{1}{2}(p_i + p_j)$ for p_i and p_j and $D_{ijt}(p_1, \cdots, p_m)$ is obtained from $D(p_1, \cdots, p_m)$ by substituting t for p_i and $p_i + p_j - t$ for p_j where $0 \leqq t \leqq \min(p_i, p_j)$. From (1.13) and (1.14) we have for $s_1 = s_2 = \cdots = s_m$,

$$(1.15)\qquad D(p_1, \cdots, p_m) \leqq D(q, \cdots, q) \quad \text{and}$$

$$(1.16)\qquad D(p_1, \cdots, p_m) \geqq D(t, \cdots, t, p_0 - (m-1)t)$$

where q, p_0 and t are defined following (1.6).

2. Application. The inequalities in Section 0 have application in a problem of selecting the "least probable event", that is the cell with the smallest probability, from a multinomial population with K cells. The application will appear in a forthcoming paper.

REFERENCES

[1] Olkin, I. and Sobel, M. (1965). Integral expressions for tail probabilities of the multinomial and the negative multinomial distributions. *Biometrika* **52** 167–179.