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NOTES

A BOUND FOR THE DISTRIBUTION OF THE MAXIMUM OF CONTINUOUS GAUSSIAN PROCESSES

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Let $X(t)$ be a real valued separable Gaussian process on $[0, 1]$. $E\{X(t)^2\}^{\frac{1}{2}} \leqq \Gamma$ and $E\{(X(t)-X(s))^2\} \leqq \psi(|t-s|)$, where ψ is assumed to be continuous and non-decreasing on $[0, 1]$. Consider the random function $\|X\|_\infty = \sup_{t\in[0,1]}|X(t)|$. We shall give an upper bound for the "tail" of the probability distribution of $\|X\|_\infty$.

The major result in this paper is a lemma that is very close to Fernique's lemma [1]. In fact the motivation for this work was to find a proof of this important lemma. As far as the author knows, none has been published or is otherwise available. Fernique's use of the lemma is to provide sufficient conditions for the continuity of Gaussian processes. A proof of his continuity result is given by Dudley [2]; our proof of the lemma provides an alternate proof of this result. However, the lemma has other significant applications, two of which will be mentioned below. The proof in this paper was suggested by Nisio's proof of her Theorem 1, [3]. From the corollary to the lemma a simple proof of Nisio's result will be obtained.

Our lemma is presented differently from Fernique's lemma because in many cases it yields sharper results. In the study of Hölder conditions for Gaussian processes the lemma presented in this paper enables us to improve previous results of the author [4] obtained originally by using Fernique's form of the lemma.

The following two conditions on the processes will be used:

(A) $\int_1^\infty \psi(e^{-x^2})\,dx < \infty$;

(B) $\psi^2(h)\log 1/h$ decreases monotonically as h decreases to zero from the right.

All processes studied will be assumed to satisfy condition (A) whereas results will be given for the cases when condition (B) applies and when it does not apply. In [4] it is pointed out that condition (B) is widely satisfied when (A) is satisfied. The expression in (B) occurs in the study of uniform Hölder conditions for Gaussian processes.

We now prove the lemma.

LEMMA. *Let $X(t)$ be real valued, separable Gaussian process on* $[0, 1]$. *Define ψ as above, and assume condition* (A) *is satisfied. Let $c(p)$ denote n^{2^p} for n* a fixed integer, $n > 1$; *let $a \geqq (2\beta\log n)^{\frac{1}{2}}$ where $\beta \geqq 2$. Then*

$$P\{\|X\|_\infty \geqq a\Gamma + b(2\beta)^{\frac{1}{2}}\textstyle\sum_{p=1}^\infty \psi(c(p)^{-1})(\log c(p))^{\frac{1}{2}}\} \tag{1}$$
$$\leqq n^2 \int_a^\infty e^{-x^2/2}\,dx + G(\beta, n),$$

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where

$$G(\beta, n) \leqq \sum_{p=1}^{\infty} (c(p))^2 \int_{(2\beta \log c(p))^{1/2}}^{\infty} e^{-u^2/2}\, du,$$

and $b = 1$ *if* $X(t)$ *satisfies condition* (B); *otherwise* $b = 2^{\frac{1}{2}}$.

PROOF. The proof depends upon finding the proper partition of [0, 1] on which to study the maximum of $X(t)$. Define $D(X(h))$ as $\{E(X(h)^2)\}^{\frac{1}{2}}$. Then

$$\Pr\{\max_{0 \leqq k < n^2} |X(k/n^2)| \geqq a\Gamma\} \leqq \Pr\left\{\max_{0 \leqq k < n^2} \frac{|X(k/n^2)|}{D(X(k/n^2))} \geqq a\right\}$$

$$\text{(2)} \qquad \leqq \sum_{k=0}^{n^2-1} \Pr\left\{\frac{|X(k/n^2)|}{D(X(k/n^2))} \geqq a\right\}$$

$$\leqq n^2 \int_a^{\infty} e^{-u^2/2}\, du.$$

Next, define $\zeta(k, q, p) = |X(k/c(p) + q/c(p+1)) - X(k/c(p))|$ and consider

$$\text{(3)} \qquad \Pr\left\{\max_{k,q,p} \frac{\zeta(k, q, p)}{\{2[D(\zeta(k, q, p))]^2 \log c(p+1)/q\}^{\frac{1}{2}}} \geqq \beta^{\frac{1}{2}}\right\}$$

for $k = 0, 1, \cdots, c(p)-1$; $q = 0, 1, 2, \cdots, c(p)-1$; $p = 1, 2, \cdots$.
The expression in (3) is bounded above by the following:

$$\text{(4)} \qquad \sum_{p=1}^{\infty} \sum_{k=0}^{c(p)-1} \sum_{q=0}^{c(p)-1} \int_{(2\beta \log c(p+1)/q)^{1/2}}^{\infty} e^{-u^2/2}\, du.$$

The expression in (4) is what we define as $G(\beta, n)$; it is obvious that the bound given in the statement of the theorem is satisfied. Also note that (4) converges for $\beta \geqq 2$, and furthermore it is less than 1.

Let the set of paths defined by the statement in the bracket of equation (2) be denoted by $A^c (A^c \equiv$ complement of $A)$ and let the set of paths defined by the statement in the bracket of equation (3) be denoted by B^c.

We will now consider the paths in $A \cap B$. Let t be a fixed point in [0, 1]. For each p, choose $k(p)$, so that

$$\text{(5)} \qquad 0 \leqq t - k(p)/c(p) < 1/c(p).$$

Note that the series

$$\text{(6)} \qquad X\left(\frac{k(1)}{c(1)}\right) + \sum_{p=1}^{\infty} X\left(\frac{k(p+1)}{c(p+1)}\right) - X\left(\frac{k(p)}{c(p)}\right)$$

converges absolutely on $A \cap B$ since on this set

$$\text{(7)} \qquad \left| X\left(\frac{k(p+1)}{c(p+1)}\right) - X\left(\frac{k(p)}{c(p)}\right) \right| < b(2\beta)^{\frac{1}{2}} \psi\left(\frac{1}{c(p)}\right) (\log c(p))^{\frac{1}{2}}$$

and

$$\sum_{p=1}^{\infty} \psi\left(\frac{1}{c(p)}\right)(\log c(p))^{\frac{1}{2}} \leqq (2\log n)^{\frac{1}{2}} \psi\left(\frac{1}{n^2}\right) + (\log n)^{\frac{1}{2}} \frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1} \int_{2^{\frac{1}{2}}}^{\infty} \psi(n^{-u^2})\,du$$

$$(8) \qquad \leqq (\log n)^{\frac{1}{2}} \frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1} \int_{1}^{\infty} \psi(n^{-u^2})\,du$$

$$\leqq \frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1} \int_{(\log n)^{\frac{1}{2}}}^{\infty} \psi(e^{-x^2})\,dx,$$

which converges according to condition (A). Furthermore, the series (6) converges to $X(t)$ almost everywhere on $A \cap B$. This follows because the series has a limit almost everywhere on $A \cap B$ and also, by hypothesis converges to $X(t)$ in L^2 (i.e. $\lim_{p\to\infty} E\{(X(k(p)/n(p)) - X(t))^2\} = 0$). Using this fact and equation (7) we see that

$$(9) \qquad |X(t)| < a\Gamma + b(2\beta)^{\frac{1}{2}} \textstyle\sum_{p=1}^{\infty} \psi(1/c(p))(\log c(p))^{\frac{1}{2}}$$

for $X(t) \in A \cap B$, with the possible exception of a subset of measure zero. The probability that equation (9) does not hold is less than or equal to the probability that $X(t) \in (A \cap B)^c$, which is bounded by $P(A^c) + P(B^c)$.
Therefore

$$P\{\|X\|_\infty \geqq a\Gamma + b(2\beta)^{\frac{1}{2}} \textstyle\sum_{p=1}^{\infty} \psi(1/c(p))(\log c(p))^{\frac{1}{2}}\}$$
$$\leqq n^2 \textstyle\int_a^\infty e^{-x^2/2}\,dx + G(\beta, n).$$

and the lemma is proven.

As a corollary we shall write this lemma in the form used by Fernique.

COROLLARY. *Under the same hypotheses as the lemma,*

$$(10) \qquad P\left\{\|X\|_\infty \geqq x\left(\Gamma + \frac{b\,2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1} \int_1^\infty \psi(n^{-u^2})\,du\right)\right\} \leqq Cn^2 \int_x^\infty e^{-u^2/2}\,du,$$

where $x \geqq (4\log n)^{\frac{1}{2}}$, *and* $C = 1 + \frac{1}{3}(4\log n)/(4\log n - 1)(2^{\frac{1}{2}}-1)^{-1}$.

PROOF. From (8) we see that

$$b(2\beta)^{\frac{1}{2}} \sum_{p=1}^{\infty} \psi\left(\frac{1}{c(p)}\right)(\log c(p))^{\frac{1}{2}} \leqq (\log n)^{\frac{1}{2}} b(2\beta)^{\frac{1}{2}} \frac{2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1} \int_1^\infty \psi(n^{-u^2})\,du.$$

Next we find a bound for $G(\beta, n)$ as defined in (4). Since $G(\beta, n) \leqq G(2, n)$ (recall $\beta \geqq 2$) we restrict our attention to $G(2, n)$. Using the standard upper bound for the tail of the Gaussian integral we obtain

$$G(2, n) \leqq \sum_{p=1}^{\infty} c(p) \sum_{q=0}^{c(p)-1} \left(\frac{q}{c(p+1)}\right)^2 \frac{1}{(4\log n)^{\frac{1}{2}}} 2^{-p/2}.$$

Since $\sum_{q=0}^{c(p)-1} q^2 < \int_1^{c(p)} x^2\,dx < c(p)^3/3$,

$$G(2, n) \leqq \sum_{p=1}^{\infty} \frac{1}{3(4\log n)^{\frac{1}{2}}} 2^{-p/2} = \frac{1}{3(4\log n)^{\frac{1}{2}}} \frac{1}{2^{\frac{1}{2}}-1}\,.$$

Now, using the standard lower bound for the tail of the Gaussian integral we see that

$$\frac{1}{(4\log n)^{\frac{1}{2}}} < n^2 \int_{(4\log n)^{\frac{1}{2}}}^{\infty} e^{-u^2/2}\,du\left(\frac{4\log n}{4\log n-1}\right).$$

Thus for $a = (2\beta \log n)^{\frac{1}{2}}$, we get from (1)

$$P\left\{\|X\|_\infty \geqq (2\beta\log n)^{\frac{1}{2}}\left[\Gamma + \frac{b\,2^{\frac{1}{2}}}{2^{\frac{1}{2}}-1}\int_1^\infty \psi(n^{-u^2})\,du\right]\right\}$$
$$\leqq Cn^2 \textstyle\int_{(2\beta\log n)^{\frac{1}{2}}}^{\infty} e^{-u^2/2}\,du.$$

Since this equation is true for arbitrary values of $\beta \geqq 2$ it is true for any $x \geqq (4\log n)^{\frac{1}{2}}$. This observation completes the proof of the corollary.

Note that $b\,2^{\frac{1}{2}}/(2^{\frac{1}{2}}-1)$ is approximately equal to 3.42 for $b = 1$, although it is approximately equal to 4.82 for $b = 2^{\frac{1}{2}}$. The left-hand side of Fernique's lemma is identical to (10) except that he has a 4 in place of $b\,2^{\frac{1}{2}}/(2^{\frac{1}{2}}-1)$. The value of this constant is significant, at least in the applications of this lemma by the author. That is why the lemma is useful, because in many cases the sum given in the lemma is considerably less than the integral. Consider the case when $\psi(|t-s|) = 1/(\log(|t-s|^{-1}))^\alpha$, $\alpha > \frac{1}{2}$. Then

$$\sum_{p=1}^{\infty} \psi\left(\frac{1}{n(p)}\right)(\log n(p))^{\frac{1}{2}} = (\log n)^{\frac{1}{2}} \sum_{p=1}^{\infty} \psi\left(\frac{1}{n(p)}\right)2^{p/2}$$
$$= \frac{1}{(\log n)^{\alpha-\frac{1}{2}}}\frac{1}{2^{\alpha-\frac{1}{2}}-1},$$

whereas

$$4\int_1^\infty \psi(n^{-u^2})\,du = \frac{4}{2\alpha-1}\frac{1}{(\log n)^{\alpha-\frac{1}{2}}}.$$

Thus, for example, if $\alpha = 9/2$ the constants differ by a factor of 15/2 regardless of the value of n.

More significant than this example is the fact that many of the upper bounds on the local Hölder conditions obtained in [4] using Fernique's form of this result (the corollary) can be reduced by factors of 2 to 4 by using the lemma.

It is also interesting to note that Nisio's Theorem 1 follows easily from our corollary, as we show below:

THEOREM. *Let $X(t)$ be a separable, mean continuous Gaussian process such that $E(|X(t)|^2) < \Gamma$ and such that the function ψ as defined above satisfies condition* (A). *Then*

$$P\left\{\limsup_{t\to\infty} \frac{|X(t)|}{(2\Gamma\log t)^{\frac{1}{2}}} \leqq 1\right\} = 1. \tag{11}$$

PROOF. Define $Y_k(t) = X(k+t), 0 \leqq t \leqq 1, k = 1, 2, \cdots$. Given $\varepsilon > 0$, small enough so that $(1+\varepsilon/2)^{3/2} < 1+\varepsilon$ we can choose an n so that

$$\frac{2}{2^{\frac{1}{2}}-1}\int_1^\infty \psi(n^{-u^2})\,du < (\varepsilon/2)\Gamma.$$

This is possible since $\int_1^\infty \psi(n^{-u^2})\,du = (\log n)^{-\frac{1}{2}}\int_{(\log n)^{\frac{1}{2}}}^\infty \psi(e^{-x^2})\,dx$ and condition (A) is satisfied.

From (10) we get

$$P\{\|Y_k(t)\|_\infty \geqq (2(1+\varepsilon/2)\log k)^{\frac{1}{2}}(\Gamma+\varepsilon/2\Gamma)\} \leqq 2n^2\int_{(2(1+\varepsilon/2)\log k)^{\frac{1}{2}}}^\infty e^{-u^2/2}\,du. \tag{12}$$

The integral in (12) is a term of a convergent series in k (regardless of n). Hence by the Borel-Cantelli lemma

$$P\left\{\limsup_{k\to\infty}\frac{\|Y_k(t)\|_\infty}{(2\Gamma\log k)^{\frac{1}{2}}} \leqq 1+\varepsilon\right\} = 1.$$

The theorem follows immediately.

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