The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 203–206

CHARACTERIZATION OF OPTIMAL SATURATED MAIN EFFECT PLANS OF THE 2^n FACTORIAL[1]

BY B. L. RAKTOE[2] AND W. T. FEDERER

Cornell University

1. Introduction and summary. For the 2^n factorial a treatment design to estimate the n main effects and the mean with $(n+1)$ treatment combinations is known in the literature as a *saturated main effect plan*. Let the $(n\times 1)\times n$ matrix D consisting of the 0's and 1's making up the subscripts of the observations, denote such a plan and let the $(n+1)\times(n+1)$ matrix X stand for the corresponding design matrix of -1's and 1's, then optimal (in the sense of maximum absolute value of the determinant of $X'X$) designs have been characterized in terms of the information matrix $X'X$ by many authors, such as Plackett and Burman [6] and Raghavarao [7]. Williamson [9], Mood [3], and Banerjee [2], among others, have used (0, 1)-matrices to construct optimal and weighing designs. If the elements of the first row of D are set equal to zero, then the $n\times n$ (0, 1)-matrix used in weighing designs is obtained from the last rows of D. However, D is not restricted to always include the combination having all zero levels in this paper. For a summary concerning several aspects of optimal saturated main effect plans the reader is referred to Addelman's [1] paper.

The aim of this paper is to characterize the optimal saturated main effect plans in terms of D'D rather than $X'X$. A major consequence of this is that all theory available for *semi-normalized* $(-1,1)$-matrices is applicable to *semi-normalized* $(0,1)$-matrices and vice versa. A second major consequence is that the normal equations for saturated main effect plans need not be obtained as they are readily derivable from the D matrix.

2. Relation between the $(-1, 1)$-matrix X and the $(0, 1)$-matrix D. The equation system relating the expected value of the observations for the $(n+1)$ treatment combinations and the $(n+1)$ parameters (n main effects and the mean) of a saturated main effect plan of the 2^n factorial may be written compactly as:

$$X\beta = E(Y) \tag{2.1}$$

where X is a square $(-1,1)$-matrix of order $n+1$, β is the $(n+1)$-column vector of parameters with the mean as its first element and the remaining components being the main effects, and Y is the $(n+1)$-column vector of observations taken at $(n+1)$ treatment combinations. Note that putting the mean as the first component in β implies that the first column of X consists of $+1$'s. Such a matrix is termed a *semi-normalized* $(-1,1)$-matrix.

Now let X be a square semi-normalized $(-1,1)$- matrix of order $n+1$ and perform

Received February 10, 1969; revised May 9, 1969.

[1] Paper No. BU-189 of the Biometrics Unit and No. 577 of the Department of Plant Breeding and Biometry, Cornell University.

[2] On leave from the Department of Mathematics and Statistics, University of Guelph.

column operations on X by postmultiplying X with the matrix G and calling the resulting product X^*, i.e.

$$XG = X^* \tag{2.2}$$

where G is the $(n+1)\times(n+1)$ upper triangular matrix:

$$G = \tfrac{1}{2}\begin{bmatrix} 2 & \vdots & \mathbf{1}' \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{I} \end{bmatrix}$$

Here $\mathbf{1}$ is an n-column vector of 1's, $\mathbf{0}$ is an n-column vector of 0's and I is an $n\times n$ identity matrix.

The following theorem may be easily verified.

THEOREM 2.1.

(a) $X^* = [\mathbf{1} \vdots D]$, *where* $\mathbf{1}$ *is an* $(n+1)$-column vector *of* $+1$'*s and* D *is the* $(n+1)\times n$ $(0,1)$-*matrix, the rows of which are the treatment combinations at which the observations* Y *were taken. (In other words, the effect of* G *on* X *is identical to setting all* -1'*s equal to* 0 *and leaving all* $+1$'*s unaltered in* X.)

(b) $|G| = 2^{-n}$

(c) $G^{-1} = \begin{bmatrix} \mathbf{1} & \vdots & -\mathbf{1}'_{1\times n} \\ \cdots & \vdots & \cdots \\ \mathbf{0}_{n\times 1} & \vdots & 2\mathbf{I}_{n\times n} \end{bmatrix}$

(d) $X = X^*G^{-1}$

(e) $|X^*| = 2^{-n}|X|$ or $|X| = 2^n\,|X^*|$

(f) *Equation* (2.1) *may be rewritten as*: $X^*G^{-1}\beta = E(Y)$

(g) $X'X = \begin{bmatrix} \alpha & \vdots & z' \\ \cdots & \vdots & \cdots \\ z & \vdots & \mathbf{Z} \end{bmatrix}$,

where $\alpha = n+1$.

$$\mathbf{z}_{n\times 1} = -\alpha\,\mathbf{1}_{n\times 1} + 2\,\mathbf{D}'_{n\times(n+1)}\mathbf{1}_{(n+1)\times 1}$$

$$\mathbf{Z} = \alpha\,\mathbf{J}_{n\times n} - 2\,\mathbf{D}'_{n\times(n+1)}\mathbf{J}_{(n+1)\times n} - 2\,\mathbf{J}_{n\times(n+1)}\mathbf{D}_{(n+1)\times n} + 4\,\mathbf{D}'_{n\times(n+1)}\mathbf{D}_{(n+1)\times n}$$

Here J *is a matrix of* $+1$'*s of appropriate dimensions.*

THEOREM 2.2. *If* $\|X^*\|$ *denotes the absolute value of* $|X^*|$ *then* $\|X^*\| \leqq 2^{-n}(n+1)^{\frac{1}{2}(n+1)}$ *with equality holding when* X *is a Hadamard matrix.*

PROOF. The proof of this theorem follows immediately from the (e) part of Theorem 2.1 and from Hadamard's theorem mentioned in Muir and Metzler ([4] page 761).

Theorem 2.2 leads to an important conclusion concerning (0, 1)-matrices, a class of matrices celebrated in combinational mathematics (e.g. see Ryser [8]). In the class of all semi-normalized square (0, 1)-matrices the maximum absolute value of the determinant of a matrix of this class is at most $2^{-n}(n+1)^{\frac{1}{2}(n+1)}$ with equality holding when the (0, 1)-matrix is obtained from a semi-normalized Hadamard matrix by setting all -1's equal to 0. Hence given any semi-normalized $(-1,1)$-matrix X of order $n+1$ of a saturated main effect plan we see immediately that $||X||$ is maximum when $||X^*||$ is maximum (or equivalently $||X'X||$ is maximum when $||X^{*\prime}X^*||$ is maximum) or vice versa. Formally we summarize the consequences of Theorem 2.2 in the following theorem:

THEOREM 2.3. *The study of optimal saturated main effect plans in the sense of maximum* $||X||$ *(or of* $||X'X||$*) is equivalent to the study of* X^* *in the sense of maximum* $||X^*||$ *(or of* $||X^{*\prime}X^*||$*). (Note that* X *is a semi-normalized* $(-1,1)$-matrix *and* X^* *is a semi-normalized* $(0,1)$-matrix. *Also note that* X^* *is the treatment combination matrix* D *bordered with a column of* $+1$*'s on the left.)*

3. Characterization of optimal plans. Having exhibited the important relationship between X and X^* above, our purpose in this section is to characterize the $(n+1)\times(n)$ array D (i.e. the treatment combination matrix) itself. The optimal saturated main effect plans fall into two categories:

(i) *Orthogonal plans* (or Plackett-Burman patterns) with $X'X=(n+1)I$, I being an $(n+1)\times(n+1)$ identity matrix. To this category belong all plans for which $(n+1)=4t$, since for Hadamard matrices to exist $(n+1)$ must be of the form $4t$, except for $n+1=2$.

(ii) *Nonorthogonal plans* (or Raghavarao weighing designs). Here there are two distinct cases as given by Raghavarao [7], namely, when:

(a) $(n+1)=4t+2$ with $X'X=(n-1)I+2J$
(b) $(n+1)=2t+1$ with $X'X=nI+J$ where I is an $(n+1)\times(n+1)$ identity matrix and J is an $(n+1)\times(n+1)$ matrix of $+1$'s.

The three cases outlined above are characterized in terms of D in three separate theorems which can be proved using matrix algebra.

THEOREM 3.1. *If* $X'X=(n+1)I$, *then* $D'D=\frac{1}{4}(n+1)I+\frac{1}{4}(n+1)J$ *and since* $(n+1)$ *must be of the form* $4t$ *we have* $D'D=tI+tJ$.
(The results of this theorem can be deduced from the theorem given by Paley [5] which proves that $n+1=4t$ is a necessary condition for a Hadamard matrix to exist, apart from $n+1=2$.)

THEOREM 3.2. *If* $X'X=(n-1)I+2J$ *with* $(n+1)=4t+2$ *then* $D'D=\frac{1}{4}(n-1)I+\frac{1}{4}(n+7)J=tI+(t+2)J$.

THEOREM 3.3. *If* $X'X=nI+J$ *with* $n+1=2t+1$ *then* $D'D=\frac{1}{4}nI+\frac{1}{4}(n+4)J=\frac{1}{2}tI+\frac{1}{2}(t+2)J$.

Incidentally, from Theorem 3.1, there follows an important conclusion with respect to the construction of Hadamard matrices. Since X can be obtained from X^* by setting all 0's equal to -1's and X^* is nothing else but D bordered on the left by a column of $+1$'s it follows immediately that when there exists a D satisfying Theorem 3.1 then there also exists a Hadamard matrix X. Hence the following theorem is an important equivalence.

THEOREM 3.4. *The construction of a Hadamard matrix is equivalent to the construction of an orthogonal main effect plan* (*i.e. a D such that Theorem* 3.1 *is satisfied*). (*Again note that a problem concerning* $(-1, 1)$-matrices *has been reduced to a problem concerning* $(0, 1)$-matrices.)

4. Discussion. Instead of considering only saturated main effect plans, one may also consider other saturated and unsaturated ones in the context of Theorem 2.1 and derive conditions on the resulting D to yield classes of optimal plans. This extension is under investigation.

5. Acknowledgment. We appreciate the suggestions of a referee, which made the paper more concise.

REFERENCES

[1] ADDELMAN, S. (1963). Techniques for constructing fractional replicate plans. *J. Amer. Statist. Assoc.* **58** 45–71.

[2] BANERJEE, K. S. (1948). Weighing designs and balanced incomplete blocks. *Ann. Math. Statist.* **19** 394–399.

[3] MOOD, A. M. (1948). On Hotelling's weighing problem. *Ann. Math. Statist.* **20** 432–446.

[4] MUIR, T. and METZLER, W. H. (1933). *Treatise on the Theory of Determinants.* Longmans, Green and Co., New York.

[5] PALEY, R. E. A. C. (1933). On orthogonal matrices. *J. Math. and Phys.* **12** 311–320.

[6] PLACKETT, R. L. and BURMAN, J. P. (1946). The design of optimum multifactorial experiments. *Biometrika* **33** 305–325.

[7] RAGHAVARAO, D. (1959). Some optimum weighing designs. *Ann. Math. Statist.* **30** 295–303.

[8] RYSER, H. J. (1963). *Combinatorial mathematics.* Carus Mathematical Monograph No. 14, The Mathematical Association of America and Wiley, New York.

[9] WILLIAMSON, J. (1946). Determinants whose elements are 0 and 1. *Amer. Math. Monthly* **53** 427–434