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AN IMPROVED FORMULA FOR THE ASYMPTOTIC VARIANCE OF SPECTRUM ESTIMATES[1]

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0. Summary. When proving results on the asymptotic behavior of estimates of the spectrum of a stationary time-series, it is invariably assumed that as the sample size T tends to infinity, so does the truncation point M_T, but at a slower rate, so that $M_T T^{-1}$ tends to zero. This is a convenient assumption mathematically in that, in particular, it ensures consistency of the estimates, but it is unrealistic when such results are used as approximations to the finite case where the value of $M_T T^{-1}$ cannot be zero. We derive a formula for the asymptotic variance on the assumption that $M_T T^{-1}$ tends to a constant γ; a more accurate approximation to the variance in the finite case is then obtained by using this formula with γ equal to the actual value of $M_T T^{-1}$. Numerical comparisons are made in the white noise case.

1. Notation and quoted results. The notation used and the conditions assumed here are those of Parzen (1957), and the formula for asymptotic variance in Section 2 is obtained by extending the proof in Section 5 of that paper to allow for a weaker condition on the bandwidth of the estimate.

We suppose that we have a sample of length T from a real-valued time-series $x(t)$. The time-series may be either continuous or discrete; the former is generally assumed in Section 1 and Section 2, and the discrete case theory is formed mainly by simply replacing integrals by summations. The numerical comparisons in Section 5 are based on discrete samples.

The model for the time-series

$$x(t) = m(t) + y(t) \tag{1.1}$$

is assumed, where $m(t)$ is a linear combination of known functions, the coefficients of which are either known or estimated by regression analysis, and where $y(t)$ is a stochastic process with zero mean possessing a finite variance $E[|y(t)|^2]$, and which is wide sense stationary, i.e. the product moment $E[y(t)y(t+v)]$ depends only on v, and defines the *covariance function* $R(v)$. $R(v)$ is assumed to be summable (absolutely integrable) and continuous; and it follows that $y(t)$—or $x(t)$—has associated with it the continuous and even *spectral density function* (or *power spectrum*) $f(\omega)$ such that

$$R(v) = \int_{-\infty}^{\infty} e^{i\omega v} f(\omega)\, d\omega \tag{1.2}$$

$$f(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\omega v} R(v)\, dv.$$

The standard form of estimate of the spectral density is

$$f_T^*(\omega) = (2\pi)^{-1} \int_{-T}^{T} e^{-i\omega v}\, k(B_T v) R_T(v)\, dv \tag{1.3}$$

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where $R_T(v)$ is the *sample covariance function*:

$$R_T(v) = T^{-1}\int_0^{T-|v|} Y_T(t)Y_T(t+|v|)\,dt, \qquad |v| < T \tag{1.4}$$
$$= 0, \qquad |v| \geqq T$$

$$Y_T(t) = x(t) - m_T, \tag{1.5}$$

m_T being the least squares estimate of $m(t)$, B_T (the *bandwidth* of the estimate) is a positive function of T such that

$$B_T \to 0, \qquad TB_T \to \gamma^{-1}, \qquad 0 \leqq \gamma < 1 \tag{1.6}$$

as $T \to \infty$, and the function $k(z)$ is even, bounded (by κ say), square integrable, and is equal to 1 at $z = 0$. The classical situation normally considered is where $\gamma = 0$, but this restriction is later relaxed.

The important theorem (Theorem 5A, page 339) proved in Parzen's paper is that (with $\gamma = 0$),

$$\lim_{T\to\infty} TB_T \operatorname{Cov}[f_T{}^*(\omega_1), f_T{}^*(\omega_2)] = f^2(\omega_1)\int_{-\infty}^{\infty} k^2(z)\,dz\,\{1+\delta(0,\omega_1)\}\,\delta(\omega_1,\omega_2) \tag{1.7}$$

where $\delta(\omega_1, \omega_2) = 1$ if $\omega_1 = \omega_2$ and 0 otherwise. As a detail, the factor $\{1+\delta(0,\omega_1)\}$ $\delta(\omega_1,\omega_2)$ should more correctly read $\{\delta(\omega_1,\omega_2)+\delta(\omega_1,-\omega_2)\}$ in the continuous case and $\{1+\delta(\pi,|\omega_1|)\}\{\delta(\omega_1,\omega_2)+\delta(-\omega_1,\omega_2)\}$ in the discrete case.

It is usual for the function $k(z)$ to be chosen equal to zero for $|z| > 1$. In this case we write $M_T = B_T{}^{-1}$; M_T is the *truncation point* of the estimate, so called because it is sufficient to calculate $R_T(v)$ only for $|v| \leqq M_T$.

2. Amended formula for the asymptotic variance. We first observe that, due to the summability of $R(v)$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |R(v)R(v+u)|\,dv\,du \leqq A_1 \tag{2.1}$$

for some finite A_1. For we have $\int_{-\infty}^{\infty} |R(v)|\,dv = A_1{}^{\frac{1}{2}}$ say, and on squaring this,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |R(v)R(v_1)|\,dv\,dv_1 \leqq A_1.$$

On writing $v_1 = v+u$, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |R(v)R(v+u)|\,dv\,du \leqq A_1$.

It therefore follows that, given $\varepsilon > 0$, there exists a number $N(\varepsilon)$ such that

$$\int_{|v|>n}\int_{|u|>n} |R(v)R(v+u)|\,dv\,du < \varepsilon \qquad \text{for all } n \geqq N(\varepsilon). \tag{2.2}$$

Now in proving our result, we follow through Parzen's proof to the stage of his equation (5.18), which gives

$$\lim_{T\to\infty} TB_T \operatorname{Cov}(f_T{}^*(\omega_1), f_T{}^*(\omega_2)) = \lim_{T\to\infty} V(\omega_1,\omega_2,B_T,T) \qquad \text{where} \tag{2.3}$$

$$V(\omega_1,\omega_2,B_T,T) = \pi^{-2}B_T\int_0^T du_2\int_{-u_2}^{T-u_2} du_1 \cos\omega_1(u_1+u_2)\cos\omega_2 u_2 \cdot k(B_T u_2)k(B_T u_2 + B_T u_1)\int_{-T}^{T} du\, U_T(u, u_1+u_2, u_2)R(u)R(u+u_1) \tag{2.4}$$

and where the function $U_T(u, v_1, v_2)$ is defined as

$$
(2.5)\quad
\begin{aligned}
U_T(u, v_1, v_2) &= 0, & u &\leqq -T+v_2,\\
&= 1-T^{-1}(v_2+u), & -T+v_2 &\leqq u \leqq \min(0, v_2-v_1),\\
&= 1-T^{-1}\max(v_1, v_2), & \min(0, v_2-v_1) &\leqq u \leqq \max(0, v_2-v_1),\\
&= 1-T^{-1}(v_1+u), & \max(0, v_2-v_1) &\leqq u \leqq T-v_1,\\
&= 0 & T-v_1 &\leqq u.
\end{aligned}
$$

No part of the calculations leading to this stage makes use of the condition that $TB_T \to \infty$.

We observe that the expression on the right-hand-side of (2.4) is absolutely integrable. For its modulus is less than

$$\pi^{-2}B_T \int_0^\infty du_2 \int_{-\infty}^{\infty} du_1 \cdot 1 \cdot |k(B_T u_2)k(B_T u_2 + B_T u_1)| \int_{-\infty}^{\infty} du \cdot 1 \cdot |R(u)R(u+u_1)|$$

since it is apparent from (2.5) that $0 \leqq U_T(u, u_1+u_2, u_2) \leqq 1$. Writing $z = B_T u_2$, this is

$$
(2.6)\quad
\begin{aligned}
&\pi^{-2}\int_0^\infty dz \int_{-\infty}^{\infty} du_1 \, |k(z)k(z+B_T u_1)| \int_{-\infty}^{\infty} du \, |R(u)R(u+u_1)|\\
&\leqq \pi^{-2}\int_0^\infty dz \, k^2(z) \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du \, |R(u)R(u+u_1)| \quad \text{(by Schwarz' inequality)}\\
&\leqq \pi^{-2}A_1 \int_0^\infty dz \, k^2(z) = A < \infty
\end{aligned}
$$

say, from (2.1) and using the fact that $k(z)$ is square integrable.

We next show that the intervals of integration of both u and u_1 in (2.4) can effectively be replaced by fixed finite intervals as $T > \infty$, i.e. given $\varepsilon > 0$, fixed intervals can be found such that the limit of the multiple integral is affected by less than ε. Firstly, let the interval for u be $(-n_0, n_0)$ where $n_0 = N(\varepsilon A_1/4A)$, as defined by (2.2); the difference between (2.4) and the resulting expression is less than $\pi^{-2}\int_0^\infty k^2(z)(4A)^{-1}\varepsilon A_1 \, dz = \frac{1}{4}\varepsilon$, using (2.6). To show that the interval for u_1 can also be made $(-n_0, n_0)$, we temporarily adjust the limits of integration for u_2 to $-n_0$ and $T-n_0$. This causes no difficulty, because integrating with respect to u_2 from $-n_0$ to 0 and from $T-n_0$ to T gives at most $\pi^{-2}B_T \cdot 2n_0 \kappa^2 A_1 < \frac{1}{4}\varepsilon$ for sufficiently large T. For $T > 2n_0$, the proposed range of integration for u_1, i.e. $(-n_0, n_0)$ is now always a subset of the given range. The maximum error caused by using these fixed limits is again less than $\pi^{-2}\int_0^\infty dz \, k^2(z)(4A)^{-1}\varepsilon A_1 = \frac{1}{4}\varepsilon$. On replacing the original limits of integration for u_2 (again with maximum error $\varepsilon/4$) we now have that for sufficiently large T, (2.4) differs by less than $4(\varepsilon/4) = \varepsilon$ from

$$
(2.7)\quad
\begin{aligned}
&\pi^{-2}B_T \int_0^T du_2 \int_{-n_0}^{n_0} du_1 \cos\omega_1(u_1+u_2)\cos\omega_2 u_2 \, k(B_T u_2)k(B_T u_2 + B_T u_1)\\
&\qquad\qquad \cdot \int_{-n_0}^{n_0} du \, U_T(u, u_1+u_2, u_2)R(u)R(u+u_1).
\end{aligned}
$$

By the change of variable $z = B_T u_2$ and the formula $2\cos A \cos B = \cos(A+B) + \cos(A-B)$, this is equal to

$$\tfrac{1}{2}\pi^{-2}\int_0^{B_T T} dz \int_{-n_0}^{n_0} du_1$$

$$\cdot\{\cos[z(B_T^{-1}(\omega_1-\omega_2))+u_1\omega_1]+\cos[z(B_T^{-1}(\omega_1+\omega_2))+u_1\omega_1]\} \tag{2.8}$$

$$\cdot\, k(z)k(z+B_T u_1)\int_{-n_0}^{n_0} du\, U_T(u, u_1+B_T^{-1}z, B_T^{-1}z)R(u)R(u+u_1).$$

The essential difference between Parzen's proof and this is in the treatment of the U_T term. In the case covered by Parzen, this tended to 1 and could thus be deleted from the working. Here, we see that with u and u_1 now being restricted to a finite fixed interval and with $B_T T \to \gamma^{-1}$, for any z

$$U_T(u, u_1+B_T^{-1}z, B_T^{-1}z) \sim 1-(B_T T)^{-1}|z| \to 1-\gamma|z|. \tag{2.9}$$

The remaining few steps in Parzen's proof may now be reproduced to obtain the conclusion that $\lim_{T\to\infty} TB_T \operatorname{Cov}[f_T^*(\omega_1), f_T^*(\omega_2)]$ is zero if $|\omega_1| \neq |\omega_2|$, and otherwise is

$$f^2(\omega)K(\gamma)(1+\delta(|\omega_1|, |\omega_2|) \tag{2.10}$$

where

$$K(\gamma) = \int_{-\gamma^{-1}}^{\gamma^{-1}} k^2(z)(1-\gamma|z|)\,dz. \tag{2.11}$$

A more detailed proof of this result is to be found in Chapter 3 of Neave (1966), and in Neave (1968b).

3. Some computed values. To see how the factor $(1-\gamma|z|)$ affects the formula, Table 1 shows the values of $K(\gamma)$ for six values of γ (including the classical case of $\gamma = 0$) and probably the two best known weighting functions $k(z)$—those due to Parzen and Tukey:

Parzen:
$$k_P(z) = 1-6z^2+6|z|^3, \qquad |z| \leqq \tfrac{1}{2}$$
$$= 2(1-|z|)^3, \qquad \tfrac{1}{2} \leqq |z| \leqq 1$$
$$= 0, \qquad |z| \geqq 1$$

Tukey:
$$k_T(z) = \tfrac{1}{2}(1+\cos\pi z), \qquad |z| \leqq 1$$
$$= 0, \qquad |z| \geqq 1.$$

TABLE 1
Values of $K(\gamma)$

γ	Parzen	Tukey
0.0	0.53929	0.75000
0.1	0.53009	0.73276
0.2	0.52089	0.71553
0.3	0.51170	0.69829
0.4	0.50250	0.68106
0.5	0.49330	0.66382

TABLE 2

True values and approximations to Var $(f_T{}^*(\omega))$ *in the white noise case*

A. Parzen weighting function; $\gamma = .2$

ω	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$
$K(0)(1+\delta(\omega,0))$	.21572					.10786				
$K(\gamma)(1+\delta(\omega,0))$	.20836					.10418				
					True values					
$T=50$	.20850	.18408	.13961	.11315	.10592	.10502	.10481	.10460	.10454	.10454
$T=200$	.20837	.10578	.10434	.10424	.10422	.10421	.10420	.10420	.10420	.10420
$T=1000$	.20836	.10420	.10418	.10418	.10418	.10418	.10418	.10418	.10418	.10418

B. Parzen weighting function; $\gamma = .4$

ω	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$
$K(0)(1+\delta(\omega,0))$	.43143					.21572				
$K(\gamma)(1+\delta(\omega,0))$	.40200					.20100				
					True values					
$T=50$	.40213	.27265	.20565	.20248	.20168	.20147	.20136	.20131	.20129	.20128
$T=200$	.40201	.20154	.20112	.20112	.20103	.20103	.20102	.20102	.20102	.20102
$T=1000$	.40200	.20102	.20100	.20100	.20100	.20100	.20100	.20100	.20100	.20100

TABLE 2—*continued*

C. Tukey weighting function; $\gamma = .4$

ω	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$
$K(0)(1+\delta(\omega, 0))$	.30000					.15000				
$K(\gamma)(1+\delta(\omega, 0))$	.28621					.14311				
					True values					
$T = 50$	.28635	.23179	.16049	.14380	.14385	.14350	.14347	.14340	.14339	.14338
$T = 200$	.28622	.14372	.14322	.14316	.14314	.14313	.14313	.14312	.14312	.14312
$T = 1000$	.28621	.14312	.14311	.14311	.14311	.14311	.14311	.14311	.14311	.14311

D. Tukey weighting function; $\gamma = .4$

ω	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$	$\frac{7\pi}{18}$	$\frac{8\pi}{18}$	$\frac{9\pi}{18}$
$K(0)(1+\delta(\omega, 0))$	.60000					.30000				
$K(\gamma)(1+\delta(\omega, 0))$	.54485					.27242				
					True values					
$T = 50$	.54498	.31166	.27468	.27336	.27299	.27284	.27276	.27272	.27270	.27269
$T = 200$	.54485	.27286	.27253	.27248	.27246	.27245	.27244	.27244	.27244	.27244
$T = 1000$	.54485	.27244	.27243	.27242	.27242	.27242	.27242	.27242	.27242	.27242

4. The exact value of the variance in the case of a normal white noise process. In order to compare the uses of the two asymptotic formulae as approximations to the true values in the finite case, we now calculate an exact expression for the variance of spectrum estimates in the case of a normal *white noise* process, i.e. where the spectrum $f(\omega)$ is constant, and the covariance function $R(v)$ vanishes except at $v = 0$. Here it is assumed that the mean value of the process is known and taken to be zero. The spectrum estimate formed from a discrete sample of length T is

$$f_T{}^*(\omega) = (2\pi)^{-1} \textstyle\sum_{v=-T}^{T} k(B_T v) R_T(v) \cos v\omega$$

where

$$R_T(v) = T^{-1} \textstyle\sum_{t=1}^{T-|v|} x(t)x(t+|v|).$$

Clearly,

$$E[R_T(v)] = R(v)(1-T^{-1}|v|), \qquad |v| < T \tag{4.1}$$
$$= R(0)\,\delta(v,0).$$

Therefore, since $k(0) = 1$, we have $E[f_T{}^*(\omega)] = (2\pi)^{-1}R(0)$. Next, if $0 \leqq u \leqq v$,

$$E[R_T(v)R_T(u)] = T^{-2}\textstyle\sum_{t=1}^{T-v}\sum_{s=1}^{T-u} E[x(t)x(t+v)x(s)x(s+u)]$$
$$= T^{-2}\textstyle\sum_{t=1}^{T-v}\sum_{s=1}^{T-u} \{R(v)R(u)+R(t-s)R(t-s+v-u)+R(t-s-u)R(t-s+v)\}$$

by the theorem in Isserlis (1918), since $x(t)$ is a normal process. But with $R(v) = \delta(v,0)\,R(0)$, this gives

$$T^{-2}R^2(0)\textstyle\sum_{t=1}^{T-v}\sum_{s=1}^{T-u}\{\delta(v,0)\delta(u,0)+\delta(t,s)\delta(u,v)+\delta(-u,v)\delta(t-s,v)\} \tag{4.2}$$
$$= T^{-2}R^2(0)\{T^2\delta(u,0)\delta(v,0)+(T-v)\delta(u,v)+T\delta(u,0)\delta(v,0)\}.$$

Now,

$$E[f_T{}^{*2}(\omega)] = \tfrac{1}{4}\pi^{-2}\textstyle\sum_{t=-T}^{T}\sum_{u=-T}^{T} g(v)g(u)E[R_T(v)R_T(u)]$$

where $g(v)$ is the even function, $g(v) = k(B_T v)\cos v\omega$. Writing this in terms of non-negative summation variables,

$$E[f_T{}^{*2}(\omega)] = \tfrac{1}{4}\pi^{-2}\{4\textstyle\sum_{v=1}^{T}\sum_{u=1}^{T} g(v)g(u)E[R_T(v)R_T(u)]$$
$$+4\textstyle\sum_{v=1}^{T} g(v)g(0)E[R_T(v)R_T(0)]+g^2(0)\cdot E[R_T{}^2(0)]\}$$
$$= \tfrac{1}{4}\pi^{-2}T^{-2}R^2(0)\{4\textstyle\sum_{v=1}^{T} g^2(u)(T-v)+0+g^2(0)(T^2+2T)\},$$

from (4.2). So, since $g(0) = 1$,

$$\operatorname{Var}[f_T{}^*(\omega)] = E[f_T{}^{*2}(\omega)]-E[f_T{}^*(\omega)]^2$$
$$= \tfrac{1}{4}\pi^{-2}T^{-2}R^2(0)\{4\textstyle\sum_{v=1}^{T}(T-v)k^2(B_T v)\cos^2 v\omega+2T\} \tag{4.3}$$
$$= \tfrac{1}{2}\pi^{-2}T^{-1}R^2(0)\textstyle\sum_{v=-T}^{T}(1-T^{-1}|v|)k^2(B_T v)\cos^2 v\omega.$$

Suppose now that $k(z)$ vanishes for $|z| > 1$; write $M_T = B_T^{-1}$ and $\Gamma = M_T T^{-1}$. Then

(4.4) $\mathrm{Var}(f_T^*(\omega)) = \frac{1}{2}\pi^{-2}R^2(0)\Gamma M_T^{-1}\sum_{v=-M_T}^{M_T}(1-\Gamma M_T^{-1}|v|)k^2(vM_T^{-1})\cos^2 v\omega.$

Clearly, if $M_T \to \infty$ with $\Gamma \to \gamma$, this tends to

$$\tfrac{1}{2}\pi^{-2}R^2(0)\gamma\int_{-\infty}^{\infty}(1-\gamma|z|)k^2(z)\,dz$$

if $\omega = 0$ or π, and to half this if $0 < \omega < \pi$, thus verifying the result of Section 2, since $f(\omega) = R(0)/2\pi$ in the white noise case.

5. Numerical comparisons. The exact expression for the variance (4.4) was computed for the Parzen and Tukey weighting functions with $T = 50$, 200, and 1000 and $\gamma = .2$ and .4, and compared with the asymptotic approximations obtained from (2.10) and the standard result (1.7). A value of $R(0) = 2\pi$, i.e. $f(\omega) = 1$, was used, so that the exact expression and the approximations for $\mathrm{Var}[f_T^*(\omega)]$ were respectively

(5.1) $$2T^{-1}\sum_{v=-M_T}^{M_T}(1-T^{-1}|v|)k^2(vM_T^{-1})\cos^2 v\omega$$

(5.2) $$\gamma K(\gamma)(1+\delta(\omega,0)+\delta(\omega,\pi))$$

(5.3) $$\gamma K(0)(1+\delta(\omega,0)+\delta(\omega,\pi))$$

in $0 \leqq \omega \leqq \pi$. In the white noise situation, both the exact values and the approximations are symmetrical with respect to $\omega = \pi/2$. Table 2(A–D) gives the values of these functions for $\omega = n\pi/18$, $n = 0, 1, \cdots, 9$. These tables demonstrate clearly the superiority of (2.10) over (1.7) as an approximation to the true values.

A more extended investigation of the exact behavior of the variance in finite cases with various different spectra may be found in Chapter 4 of Neave (1966) and in Neave (1968a).

REFERENCES

[1] Isserlis, L. (1918). On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika* **12** 134–139.

[2] Neave, H. R. (1966). A missing data problem in time series analysis. Ph.D. thesis, Univ. of Nottingham.

[3] Neave, H. R. (1968a). The variance of spectrum estimates. Technical Report No. 158, Department of Statistics, Univ. of Wisconsin.

[4] Neave, H. R. (1968b). The asymptotic variance of spectrum estimates using initially scarce data. Technical Report No. 153, Department of Statistics, Univ. of Wisconsin.

[5] Parzen, E. (1957). On consistent estimates of the spectrum of a stationary time series. *Ann. Math. Statist.* **28** 329–348.