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THE EXIT DISTRIBUTION OF AN INTERVAL FOR COMPLETELY ASYMMETRIC STABLE PROCESSES[1]

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1. Main results. Let X_t be the stable process on the line R having exponent $\alpha \neq 1$ and log characteristic function

$$\log E \exp\left[i\theta(X_t - X_0)\right] = -t|\theta|^\alpha \left[1 - i \operatorname{sgn}(\theta) \tan(\tfrac{1}{2}\pi\alpha)\right] \tag{1.1}$$

We will assume that X_t is a version of the process that is a standard Markov process. Let $a < b$ and let $\tau = \inf\{t > 0 : X_t \notin (a, b)\}$ be the first exit time from the open interval (a, b). Our primary purpose in this note is to explicitly compute the distribution of X_τ as well as the related Green's function of $R-(a, b)$.

The results we obtain here are new for $\alpha > 1$. For $\alpha < 1$ the distribution of X_τ was first computed by Dynkin [2] and by a different method by Ikeda and Watanabe [3]. For the sake of completeness we will show how the potential theoretic methods used here also yield a very easy derivation for the case $\alpha < 1$. The results we obtain here should be compared with those of Blumenthal, Getoor, and Ray [1] for the isotropic case.

THEOREM 1. *Let $\mu_x(dy) = P_x(X_\tau \in dy)$. If $\alpha < 1$, then μ_x is the unit mass at x if $x \notin [a, b)$, while for $x \in [a, b)$*

$$\begin{aligned} \mu_x(dy) &= (\sin \pi\alpha/\pi)\left[(b-x)/(y-b)\right]^\alpha (y-x)^{-1}, && y > b \\ &= 0, \quad \textit{elsewhere}. \end{aligned} \tag{1.2}$$

On the other hand if $\alpha > 1$ and $x \in (a, b)$

$$\mu_x(\{a\}) = \left[(b-x)/(b-a)\right]^{\alpha-1} \tag{1.3}$$

$$\begin{aligned} \mu_x(dy) &= \pi^{-1} \sin\left[(\alpha-1)\pi\right]\left[(b-x)/(y-b)\right]^{\alpha-1} \cdot \\ &\qquad (y-x)^{-1}\left[(x-a)/(y-a)\right], && y > b \\ &= 0, && y \notin \{a\} \cup [b, \infty). \end{aligned} \tag{1.4}$$

For $x \notin (a, b)$, $\mu_x(dy)$ is the unit mass at x.

Let B be a Borel subset of (a, b). The Green's function of $R-(a, b)$ is the function $G(x, y)$ such that

$$E_x \int_0^\tau 1_B(X_t)\, dt = \int_B G(x, y)\, dy,$$

where 1_B is the indicator function of B.

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THEOREM 2. *For* $\alpha < 1$,

(1.5) $$G(x,y) = (\Gamma(\alpha))^{-1}\cos(\tfrac{1}{2}\pi\alpha)(y-x)^{\alpha-1} \qquad a \leqq x < y < b$$
$$= 0, \qquad \textit{elsewhere.}$$

For $\alpha > 1$, $G(x,y) = 0$ *if* x *or* $y \notin (a,b)$. *If* $x, y \in (a,b)$ *then*

(1.6) $$G(x,y) = (\Gamma(\alpha))^{-1}\sin(\tfrac{1}{2}\pi(\alpha-1)).$$
$$\{(y-a)^{\alpha-1}[(b-x)/(b-a)]^{\alpha-1} - (y-x)^{\alpha-1}\}, \qquad a < x < y < b$$
$$= (\Gamma(\alpha))^{-1}\sin(\tfrac{1}{2}\pi(\alpha-1))(y-a)^{\alpha-1}[(b-x)/(b-a)]^{\alpha-1},$$
$$a < y < x < b.$$

From $G(x,y)$ one may, in principle, compute all of the moments of τ by means of the formula

(1.7) $$E_x\tau^n = n!\int_a^b \cdots \int_a^b G(x,x_1)\cdots G(x_{n-1},x_n)dx_1\cdots dx_n.$$

In particular for $x \in (a,b)$,

(1.8) $$E_x\tau = (\Gamma(\alpha+1))^{-1}\cos(\tfrac{1}{2}\pi\alpha)(b-x)^{\alpha}, \qquad \alpha < 1$$
$$= (\Gamma(\alpha+1))^{-1}\sin(\tfrac{1}{2}\pi(\alpha-1))(x-a)(b-x)^{\alpha}, \qquad \alpha > 1.$$

Let $\tau' = \inf\{t > 0 : X_t > b\}$. By letting $a \to -\infty$ in (1.2) and (1.4) we obtain the density of the distribution of $X_{\tau'}$. Thus for $x < b$ and $y > b$

(1.9) $$P_x(X_{\tau'} \in dy) = \pi^{-1}\sin(\pi\alpha)[(b-x)/(y-b)]^{\alpha}(y-x)^{-1}, \qquad \alpha < 1$$
$$= \pi^{-1}\sin\pi(\alpha-1)[(b-x)/(y-b)]^{\alpha-1}(y-x)^{-1}, \qquad \alpha > 1.$$

The Levy measure $M(d\xi)$ of the process X_t is concentrated on $(0,\infty)$. Consequently, the only jumps the process can have must be in the positive direction. For $\alpha < 1$, $X_t - X_0$ is strictly increasing while for $\alpha > 1$, $X_t - X_0$ decreases only in a continuous manner.

Let $T = \inf\{t > 0 : X_t \in (a,b)\}$ be the first entrance time of (a,b). From the above facts it is easy to see that the distribution of X_T is as follows.

COROLLARY 1. *Let* $H(x,dy) = P_x(X_T \in dy;\ T < \infty)$. *Then for* $\alpha < 1$ $H(x,dy)$ *is the unit mass at* x *if* $x \in [a,b)$, *and*

(1.10) $$H(x,dy) = \pi^{-1}\sin(\pi\alpha)[(a-x)/(y-a)]^{\alpha}(y-x)^{-1}dy, \qquad x < a, a \leqq y \leqq b$$
$$= 0 \qquad x > b.$$

For $\alpha > 1$, $H(x,dy)$ *is the unit mass at* x *for* $x \in [a,b]$. *Let* $I_b(dy)$ *be the unit mass at* b. *For* $\alpha > 1$, *and let* $a \leqq y \leqq b$. *Then*

(1.11) $$H(x,dy) = \pi^{-1}\sin\pi(\alpha-1)[(a-x)/(y-a)]^{\alpha-1}(y-x)^{-1}dy$$
$$+ P_x(X_{\tau'} > b)I_b(dy), \qquad x < a,$$
$$= I_b(dy), \qquad x > b.$$

For $\alpha > 1$ the distribution of $X_{T'}$ was derived in [4] by an argument similar to that used to derive that of X_τ. However, the formula given there was more complicated than that in (1.11). [However, the change of variable $(t-x)/(y-t) \to s$ in (3.1) of [4] will yield the same formula as (1.11).]

2. Proofs.

Let $p(t,x)$ be the density of $X_t - X_0$. For $\lambda > 0$ set $p^\lambda(x) = \int_0^\infty e^{-\lambda t} p(t,x)\,dt$ and set $H^\lambda(x,dy) = E_x(e^{-\lambda\tau}, X_\tau \in dy)$. Then for any Borel set B the first passage relation yields

$$\int_B [p^\lambda(y-x) - \int_{(a,b)^c} H^\lambda(x,dz)p^\lambda(y-z)]\,dy = \int_0^\infty e^{-\lambda t} P_x(\tau_a > t, X_t \in B)\,dt. \tag{2.1}$$

It follows that the measure on the right hand side has an upper semi-continuous density $G^\lambda(x,y)$ satisfying the relation

$$p^\lambda(y-x) = \int_{(a,b)^c} H^\lambda(x,dz)p^\lambda(y-z) = G^\lambda(x,y). \tag{2.2}$$

It is here that we must separate the case $\alpha < 1$ (transient case) from the case $\alpha > 1$ (recurrent case).

For $\alpha < 1$ it is known (See [5], Eq. (1.9)), where however the factor $(1+h^2)^{-1}$ was omitted from the right hand side) that $p^\lambda(x) \uparrow g(x)$, $\lambda \downarrow 0$ where

$$g(x) = (\Gamma(\alpha))^{-1}\cos(\tfrac{1}{2}\pi\alpha)x^{\alpha-1}, \qquad x > 0$$
$$= 0 \qquad x < 0.$$

Since

$$\int_{(a,b)^c} H^\lambda(x,dz)p^\lambda(y-z) = E_x[e^{-\lambda\tau}p^\lambda(y-X_\tau)] \uparrow E_x[g(y-X_\tau)]$$

we see that $G^\lambda(x,y) \to G(x,y) < \infty$ and

$$g(y-x) - \int_{(a,b)^c} \mu_x(dz)g(y-z) = G(x,y).$$

Now as the process X_t can only move to the right, $\mu_x(dz)$ is concentrated on $[b,\infty)$. From this, and the fact that $g(x) = 0$ for $x \leqq 0$, it follows that $G(x,y) = g(y-x)$ for $a \leqq x < y < b$ and $G(x,y) = 0$ elsewhere. Thus we obtain the integral equation

$$(y-x)^{\alpha-1} = \int_{[b,y]} \mu_x(dz)(y-z)^{\alpha-1}, \qquad a \leqq x < b,\ y > b.$$

This equation has a unique solution given by

$$\begin{aligned}\Gamma(\alpha)\Gamma(1-\alpha)\mu_x([0,t]) &= \int_b^t (y-x)^{\alpha-1}(t-y)^{-\alpha}\,dy \\ &= \Gamma(\alpha)\Gamma(1-\alpha) - \int_a^b [(y-x)/(t-y)]^{\alpha-1}(y-x)^{-1}\,dy \\ &= \Gamma(\alpha)\Gamma(1-\alpha) - \int_0^{(b-x)/(t-b)} s^{\alpha-1}(1+s)^{-1}\,ds.\end{aligned}$$

Thus

$$\begin{aligned}\mu_x(dt) &= [\Gamma(\alpha)\Gamma(1-\alpha)]^{-1}[(b-x)/(t-b)]^\alpha(t-x)^{-1}\,dt \\ &= \pi^{-1}\sin(\alpha\pi)[(b-x)/(t-b)]^\alpha(t-x)^{-1}\,dt, \qquad t > b,\ a < x < b.\end{aligned}$$

We now turn our attention to the case $\alpha > 1$ which is more complicated. Let $A^\lambda(x) = p^\lambda(0) - p^\lambda(x)$. Then we may rewrite (2.1) as

$$(2.3) \qquad A^\lambda(y-x) - \int_{(a,b)^c} H^\lambda(x,dz) A^\lambda(y-z) = -G^\lambda(x,y) + \lambda p^\lambda(0) \int_0^\infty e^{-\lambda t} P_x(\tau > t)\, dt.$$

Let

$$A(x) = (\Gamma(\alpha))^{-1} \sin(\tfrac{1}{2}\pi(\alpha-1)) x^{\alpha-1}, \qquad x > 0$$
$$= 0 \qquad x \leqq 0.$$

It was shown in [4] that $A^\lambda(x) \to A(x)$ uniformly on compacts, and that $G^\lambda(x,y) \uparrow G(x,y) < \infty$ as $\lambda \downarrow 0$. It was also shown that $G(x,y) = 0$ if x or $y \notin [a,b]$. Since $\lambda p^\lambda(0) = \Gamma(1-1/\alpha)\lambda^{1/\alpha} \to 0$ as $\lambda \downarrow 0$ and $E_x\tau < \infty$ it follows from (2.3) that

$$A(y-x) - \lim_{\lambda \downarrow 0} \int_{(a,b)^c} H^\lambda(x,dz) A^\lambda(y-z) = -G(x,y).$$

The Levy measure of X_t is $M(d\xi) = \xi^{-(\alpha+1)}\, d\xi$, $\xi > 0$ and $M(d\xi) = 0, \xi < 0$. Thus X_t can jump only to the right and must move continuously to the left. Hence both $H^\lambda(x,dz)$ and $\mu_x(dz)$ are concentrated on $\{a\} \cup [b,\infty)$. Since $A^\lambda(x)$ and $A(x)$ are continuous and $H^\lambda(x,dz)$ converge weakly to $\mu_x(dz)$ it follows from (2.3) that for any $r > b$

$$(2.4) \qquad A(y-x) - \mu_x(\{a\})A(y-a) - \int_{[b,r]} \mu_x(dz) A(y-z) = -G(x,y) + \lim_{\lambda \downarrow 0} \int_{(r,\infty)} H^\lambda(x,dz) A^\lambda(y-z).$$

Simple computations show that there are constants K_1 and K_2 such that for all $\lambda \geqq 0$ and $x \in R$, $|A^\lambda(x)| \leqq K_1 + K_2|x| = \varphi(x)$. By Theorem 1 of [3] we then obtain that

$$\int_{(r,\infty)} H^\lambda(x,dz)\, |A^\lambda(y-z)| = \int_a^b G^\lambda(x,w)\, dw \int_r^\infty M(dz-w)\, |A^\lambda(y-z)|$$
$$\leqq \int_a^b G(x,w)\, dw \int_r^\infty |z-w|^{-(\alpha+1)} \varphi(y-z)\, dz$$
$$= O(r^{1-\alpha}).$$

Thus $\lim_{r\uparrow\infty} \lim_{\lambda\downarrow 0} \int_{(r,\infty)} H^\lambda(x,dz)\, |A^\lambda(y-z)| = 0$, and consequently from (2.4) we obtain

$$(2.5) \qquad A(y-x) - \mu_x(\{a\})A(y-a) - \int_{[b,\infty)} \mu_x(dz) A(y-z) = -G(x,y).$$

In particular for $y > b$ we obtain an integral equation for $\mu_x(dy)$.

$$(2.6) \qquad (y-x)^{\alpha-1} - \mu_x(\{a\})(y-a)^{\alpha-1} = \int_b^y \mu_x(dz)(y-z)^{\alpha-1}, \qquad y > b,\ x \in (a,b).$$

This equation has a unique solution. To solve it let $\alpha - 1 = \beta$ and $f_x(z)\, dz = \mu_x(dz)$, $z \geqq b$. Then

$$
\begin{aligned}
&\Gamma(\beta)\Gamma(1-\beta)\textstyle\int_b^t f_b(z)\,dz\\
&\quad= \textstyle\int_b^t [(y-x)^{\beta-1}-\mu_x(\{a\})(y-a)^{\beta-1}](t-y)^{-\beta}dy\\
&\quad= [1-\mu_x(\{a\})]\Gamma(\beta)\Gamma(1-\beta)\\
&\qquad -\textstyle\int_x^b [(y-x)/(t-y)]^{\beta}(y-x)^{-1}dy+\mu_x(\{a\})\int_a^b[(y-a)/(t-y)]^{\beta}(y-a)^{-1}dy\\
&\quad= [1-\mu_x(\{a\}]\Gamma(\beta)\Gamma(1-\beta)\\
&\qquad -\textstyle\int_0^{(b-x)/(t-b)} s^{\beta-1}(1+s)^{-1}ds+\mu_x(\{a\})\int_0^{(b-a)/(t-b)} s^{\beta-1}(1+s)^{-1}\,ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(2.7)\qquad f_x(t) = \pi^{-1}\sin(\pi\beta)[((b-x)/(t-b))^{\beta}(t-x)^{-1}\\
-\mu_x(\{a\})((b-a)/(t-b))^{\beta}(t-a)^{-1}].
\end{aligned}
$$

However $\mu_x(\{a\})$ is the same as the probability that the two point set $\{a,b\}$ is first hit at a. From results in [4] we obtain that

$$
(2.8)\qquad \mu_x(\{a\}) = [(b-x)/(b-a)]^{\beta}.
$$

Substituting this into (2.7) yields

$$
(2.9)\quad f_x(t) = \pi^{-1}\sin(\pi\beta)[(b-x)/(t-b)]^{\beta}(x-a)[(t-x)(t-a)]^{-1}, \qquad t > b,\ a < x < b.
$$

This establishes Theorem 1.

To establish Theorem 2, note that (2.5) shows that for $\alpha > 1$

$$
\begin{aligned}
G(x,y) &= \mu_x(\{a\})A(y-a)-A(y-x), && a < x < y < b\\
&= \mu_x(\{a\})A(y-a), && a < y < x < b;
\end{aligned}
$$

and the theorem follows.

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