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ANALYSIS OF ROOM'S SQUARE DESIGN[1]

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1. Summary. A combinatorial arrangement was proposed by T. G. Room (1955). Archbold and Johnson (1958) gave a method of construction for these arrangements (which they called Room's squares) and suggested an application to experimental design. The purpose of the present communication is to rectify an error in the analysis given by Archbold and Johnson in the above paper.

2. Layout and analysis. A Room's square of side $s = 2n-1$ is an arrangement of $2n$ symbols in s rows and s columns such that (i) each row (column) contains every symbol with $(n-1)$ blank cells in each row (column) and a pair of symbols in each of the remaining n cells, (ii) every unordered pair of symbols occurs once in a cell.

It may be relevant to note that Room's squares have been obtained for *all* odd s excepting the following cases: (i) $s > 50$ with $s = 2^{2^m}+1$, (ii) $s > 50, s = 3m$ where $3 \nmid m$, (iii) $s > 50$, $s = 5m$ where $5 \nmid m$, (iv) $s = 3$, (v) $s = 5$ (Stanton and Horton (1969), Stanton and Mullin (1968), Mullin and Nemeth (1969a, 1969b)). Cases (i), (ii) and (iii) appear to be undecided. For (iv) and (v) non-existence was proved by Room (1955).

Archbold and Johnson suggested the following application for $s = 7$. There are two non-interacting factors, one is called varieties and is at s levels; the other is called treatment and has $2n$ levels. The Room's square gives the design where rows denote the blocks, the columns denote the varieties and the symbols denote the treatments. Thus the design has the following properties:

(A) Every treatment occurs exactly once with every variety and exactly once in each block.

(B) With the Room's square chosen by Archbold and Johnson for $s = 7$, the design for varieties and blocks is a symmetrical balanced incomplete block design. This is not true for all values of s, as the necessary conditions for existence of a balanced design are not satisfied for all values of s.

In what follows we shall restrict our discussion to the above case considered by Archbold and Johnson with indications for the analysis for higher values of s. One possibility is to regard this as an incomplete block design for varieties with main plots split into two sub-plots each for treatments. The model is

$$x_{tij} = A + B_t + \alpha_i + \beta_j + u_{ti} + z_{tij} \tag{2.1}$$

where x_{tij} denotes the observation in the tth block for variety i and treatment j,

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B_t, α_i and β_j are block, variety and treatment effects respectively satisfying $\sum B_t = \sum \beta_j = \sum \alpha_i = 0$. u_{ti} and z_{tij} are independent random variables with

(2.2) $$E(u_{ti}) = 0, \quad V(u_{ti}) = \sigma_\beta^2 \qquad E(z_{tij}) = 0, \quad V(z_{tij}) = \sigma^2.$$

Let $\sigma_1^2 = \sigma^2 + 2\sigma_\alpha^2$.

The analysis of Archbold and Johnson ignores the fact that the main plots are *not* full replicates of treatments, and hence differences between main plots also contain information on treatment differences. Correct analysis of variance turns out to be as follows:

Source	df
Blocks	$s-1$
Varieties	$s-1$
Treatments	s
Error (i)	$(n-3)s+1$
Main plots	$ns-1$
Treatments	s
Error (ii)	$(n-1)s$
Total	$2ns-1$

To obtain the first part of the above analysis of variance we note that the normal equations for treatment parameters are

(2.3) $$T_j = s\beta_j + \textstyle\sum_{j' \neq j} \beta_{j'} \qquad j = 1, 2, \cdots, s+1,$$

where T_j denotes the sum of main plot yields for all main plots in which treatment j is applied. (Blocks and Varieties parameters are not involved because of the orthogonality). Normal equations for blocks and varieties do not involve treatment parameters and can be obtained by the usual methods. Denoting by $\hat{\alpha}_i$ and $\hat{\beta}_j$ the solutions to the normal equations one gets

(2.4) $$V(\hat{\beta}_j - \hat{\beta}_{j'}) = 2\sigma_1^2/(n-1).$$

Variance for $\hat{\alpha}_i - \hat{\alpha}_{i'}$ averaged over all paired comparisons is σ_1^2/rE where E is the efficiency factor for the variety-block design. For the design used by Archbold and Johnson this turns out to be $2\sigma_1^2/s$ and since the design is balanced variance of any estimated varietal difference is $2\sigma_1^2/s$.

Further, denoting by β_j^* the solutions to the normal equations in the sub-plot analysis one gets

(2.5) $$V(\beta_j^* - \beta_{j'}^*) = 2\sigma^2/n.$$

In practice, one can combine the two estimates for $\beta_j - \beta_{j'}$ using the same process as is used in the recovery of inter-block information in balanced incomplete block designs.

The procedure suggested by Graybill and Deal (1959) will be fruitful when $(n-3)s+1 \geqq 9$ in which case n must exceed 4. When $n \geqq 4$ (i.e. all non-trivial cases) the procedures of Stein (1966) or Shah (1964) can be applied to combine the two estimates. With each of the procedures mentioned above the combined estimate would be

$$\bar{\beta}_j(\hat{\rho})-\bar{\beta}_{j'}(\hat{\rho}) = \frac{(n-1)(\hat{\beta}_j-\hat{\beta}_{j'})+n\hat{\rho}(\beta_j{}^*-\beta_{j'}^*)}{(n-1)+n\hat{\rho}} \tag{2.6}$$

where $\hat{\rho}$ is an estimate of $\rho = \sigma_1{}^2/\sigma^2$.

Estimates of ρ by the three methods are as follows

Graybill and Deal $\hat{\rho} = \dfrac{\text{Error m.s. (i)}}{\text{Error m.s. (ii)}}$

Stein $\hat{\rho} = \dfrac{(n-1)(ns-s+2)\sum_j(\hat{\beta}_j-\beta_j{}^*)^2}{(s-2)\text{ Error s.s. (ii)}} - \dfrac{n-1}{n}$

Shah $\hat{\rho} = \dfrac{(n-1)(ns-s)\sum(\hat{\beta}_j-\beta_j{}^*)^2}{s\text{ Error s.s. (ii)}} - \dfrac{n-1}{n}$

It follows from Shah (1969) that under the model used here in each of the above cases estimates of treatment differences are improved when $\hat{\rho}$ is truncated at unity.

It is of some interest to note that when $n = 4$ estimate of $\hat{\rho}$ based on the two error mean squares is *not* precise enough to yield estimates uniformly better than $\beta_j{}^*-\beta_{j'}^*$, but one based on $\sum(\beta_j{}^*-\hat{\beta}_j)^2$ and error mean square (ii) has this property.

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REFERENCES

[1] ARCHBOLD, J. W. and JOHNSON, N. L. (1958). A construction for Room's squares and an application in experimental design. *Ann. Math. Statist.* **29** 219–225.

[2] GRAYBILL, F. A. and DEAL, R. B. (1959). Combining unbiased estimators. *Biometrics* **15** 543–550.

[3] MULLIN, R. C. and NEMETH, E. (1969). On furnishing Room squares. To appear.

[4] MULLIN, R. C. and NEMETH, E. (1969). An existence theorem for Room squares. To appear.

[5] ROOM, T. G. (1955). A new type of magic square. *Math. Gaz.* **39** 307.

[6] SHAH, K. R. (1964). Use of inter-block information to obtain uniformly better estimators. *Ann. Math. Statist.* **35** 1064–1078.

[7] SHAH, K. R. (1969). Use of truncated estimator of variance ratio in recovery of inter-block information. Submitted to *Ann. Math. Statist.*

[8] STANTON, R.G. and MULLIN, R. C. (1968). Construction of Room squares. *Ann. Math. Statist.* **39** 1540–1548.

[9] STANTON, R. G. and HORTON, J. D. Composition of Room squares. To appear.

[10] STEIN, S. (1966). An approach to the recovery of inter-block information in balanced incomplete block designs. *Research Papers in Statistics* (ed. F. N. David). Wiley, New York, 351–366.