NOTE ON A PAPER BY H. G. TUCKER[1]

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0. Summary. The purpose of this note is to indicate a direct and natural way of proving theorems stated in [4] by using an explicit expression for the sequence of normalizing constants belonging to a distribution function attracted to a stable law. This results in a remark concerning a counter example given by Tucker and a slightly sharpened version of his Lemma 5.

1. Determination of normalizing constants. For a positive function f on the real line with $f(\infty) = \infty$ we define

$$(1) \qquad f^*(x) = \inf\{y \mid f(y) \geq x\} \qquad \text{for} \quad x > 0.$$

This is an extension of the concept of the inverse function. We mention the following property.

LEMMA 1. *Let ϕ_1 and ϕ_2 be measurable regularly varying functions* (see definition in [4]) *with exponent $\rho > 0$, then ϕ_1^* and ϕ_2^* are regularly varying with exponent ρ^{-1}. For any c with $0 \leq c \leq \infty$ we have*

$$(2) \qquad \phi_1(x)/\phi_2(x) \to c \qquad \text{for} \quad x \to \infty$$

if and only if

$$(3) \qquad \phi_1^*(x)/\phi_2^*(x) \to c^{-1/\rho} \qquad \text{for} \quad x \to \infty.$$

Following Tucker we write $F \in D(\alpha)$ when the distribution function F is in the domain of attraction of a stable law G_α of characteristic exponent α, i.e. if for suitably chosen constants $B_n > 0$ and A_n the n-fold convolutions F^{n*} of F satisfy $\lim_{n \to \infty} F^{n*}(B_n\{x + A_n\}) = G_\alpha(x)$ for every x. The numbers B_n are called *normalizing coefficients*.

LEMMA 2.[2]

(a) *If $F \in D(\alpha)$ ($0 < \alpha < 2$) then*

$$(4) \qquad B_n \sim c \inf\{x \mid 1 - F(x) + F(-x - 0) \leq 1/n\} \qquad \text{for} \quad n \to \infty.$$

(b) *If $F \in D(\alpha)$ ($0 < \alpha \leq 2$) then*

$$(5) \qquad B_n \sim c \inf\{x \mid x^{-2} \int_{-x}^{x} t^2 \, dF(t) \leq 1/n\} \qquad \text{for} \quad n \to \infty.$$

PROOF. As

$$(6) \qquad \phi(x) = \sup_{a < s \leq x} s^2 \{\int_{-s}^{s} t^2 \, dF(t)\}^{-1}$$

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which is a nondecreasing function for sufficiently large a and $x \geq a$, satisfies (slightly generalizing the argument in [3])

(7) $$\phi(x) \sim x^2 \left\{ \int_{-x}^{x} t^2 \, dF(t) \right\}^{-1} \qquad \text{for} \quad x \to \infty,$$

ϕ is regularly varying with exponent α. From this it follows

(8) $$\phi(x-0)/\phi(x+0) \to 1 \qquad \text{for} \quad x \to \infty.$$

It is easy to verify that

(9) $$\phi(\phi^*(x)-0) \leq x \leq \phi(\phi^*(x)+0).$$

Combining (8) and (9) we obtain $\phi(\phi^*(x)) \sim x$ for $x \to \infty$. From this it follows using (7)

(10) $$n\alpha_n^{-2} \int_{-\alpha_n}^{\alpha_n} t^2 \, dF(t) \to 1 \qquad \text{for} \quad n \to \infty, \text{ with}$$

$$\alpha_n = \inf \left\{ x \,\middle|\, x^{-2} \int_{-x}^{x} t^2 \, dF(t) \leq 1/n \right\}.$$

Relation (10) is identical with (8.14) page 304 of [1] and so the α_n are normalizing constants for F. As two sequences of normalizing coefficients are asymptotically equal except for a multiplicative constant we have proved (5). The first part of the lemma is an immediate consequence of Theorem 2 page 275 of [1] and Lemma 1.

2. Correspondence between F and $\{B_n\}$. Lemma 5 of [4] states that a sequence of positive numbers is a sequence of normalizing constants for an $F \in D(\alpha)$ $(0 < \alpha < 2)$ iff

(11) $$B_n \sim \phi(n) \qquad \text{for} \quad n \to \infty$$

where $\phi(x)$ is a regularly varying function with exponent α^{-1}. Clearly (11) implies

(12) $$B_n^{-1} B_{nm} \to m^{1/\alpha} \qquad \text{for} \quad n \to \infty \quad \text{and} \quad m = 1, 2, 3 \cdots.$$

Tucker gives an example of a sequence $\{B_n\}$ satisfying (12) and not (11). This example might be somewhat misleading as becomes apparent from the following observation.

A sequence of normalizing constants $\{B_n\}$ is always asymptotically equivalent to a monotone sequence of such coefficients (as is shown in Lemma 2). If we assume (12) for a sequence of positive numbers $\{B_n\}$ asymptotically equivalent to a monotone sequence $\{B_n'\}$ then (11) holds for $\phi(x) = B_{[x]}'$ (where ϕ is regularly varying with exponent α^{-1}) as can be seen from the next lemma. The sequence in Tucker's example is *not* asymptotically equivalent to a monotone sequence and this tends to confuse the point just made.

LEMMA 3. *If for a positive nondecreasing function ϕ defined on an interval (a, ∞) and a constant $\rho \geq 0$*

(13) $$\lim_{n \to \infty} \phi(nm)/\phi(n) = m^\rho \qquad \text{for} \quad m = 1, 2, 3, \cdots,$$

ϕ *is regularly varying with exponent ρ.*

PROOF. We first prove

(14) $$\phi(n+1)/\phi(n) \to 1 \qquad \text{for} \quad n \to \infty.$$

If (14) does not hold, we can select a sequence $\{k_r\}$ of positive integers such that $\lim_{r \to \infty} \phi(k_r+1)/\phi(k_r) = c > 1 \, (c \leq \infty)$. We choose m such that $1 \leq ((m+1)/m)^p < c$; take $n_r = [k_r m^{-1}]$, then by (13)

$$c > ((m+1)/m)^p = \lim_{r \to \infty} \phi(n_r(m+1))/\phi(n_r m)$$

$$= \lim_{r \to \infty} \prod_{k=n_r m}^{n_r(m+1)-1} \phi(k+1)/\phi(k) \geq \lim_{r \to \infty} \phi(k_r+1)/\phi(k_r) = c,$$

hence (14) is true.

Given $x > 0$ and $\varepsilon > 0$ we choose positive integers m and r such that

(15) $$x - \varepsilon < m/r < x < (m+1)/r < x + \varepsilon.$$

Defining for real $t > 0$, $n_t = [tr^{-1}]$ we have

(16) $$\phi(n_t m)/\phi((n_t+1)r) \leq \phi(tx)/\phi(t) \leq \phi((n_t+1)(m+1))/\phi(n_t r).$$

Combining (13), (14), (16) and (15) we find

$$(x-\varepsilon)^p \leq \liminf_{t \to \infty} \phi(tx)/\phi(t) \leq \limsup_{t \to \infty} \phi(tx)/\phi(t) \leq (x+\varepsilon)^p.$$

Hence we have $\lim_{t \to \infty} \phi(tx)/\phi(t) = x^p$.

REMARK. It suffices to require (13) for two integers m_1 and m_2 for which $\log m_1/\log m_2$ is irrational, e.g. $m_1 = 2$ and $m_2 = 3$. The proof is simpler when one requires (13) for all m.

Using the Lemmas 1, 2 and 3 and

$$\left\{ \frac{1}{1-F(x)} \right\}^{**} \sim \frac{1}{1-F(x)} \qquad \text{for} \quad x \to \infty,$$

we can restate Tucker's Lemma 3 in the following way.

LEMMA 4. (a) *Call two distribution functions F_1 and F_2 equivalent if for a c with $0 < c < \infty$*

$$1 - F_1(x) + F_1(-x-0) \sim c\{1 - F_2(x) + F_2(-x-0)\} \quad \text{for} \quad x \to \infty$$

and call two sequences of positive numbers $\{B_n\}$ and $\{B_n'\}$ equivalent if for a c with $0 < c < \infty$

$$B_n \sim cB_n' \qquad \text{for} \quad n \to \infty.$$

For each α with $0 < \alpha < 2$ there is a one-to-one correspondence between the equivalence classes of distribution functions F from $D(\alpha)$ and those equivalence classes of sequences of positive numbers $\{B_n\}$, which contain a nondecreasing sequence satisfying (12) (then every sequence in the equivalence class satisfies (12)). The correspondence is: $\{B_n\}$ is a sequence of normalizing constants for F.

(b) *Call two distribution functions F_1 and F_2 equivalent if for a c with $0 < c < \infty$*

$$\int_{-x}^{x} t^2 \, dF_1(t) \sim c \int_{-x}^{x} t^2 \, dF_2(t) \qquad\qquad for \quad x \to \infty$$

and call two sequences of positive numbers $\{B_n\}$ and $\{B_n'\}$ equivalent if for a c with $0 < c < \infty$

$$B_n \sim cB_n' \qquad\qquad or \quad n \to \infty.$$

There is a one-to-one correspondence between the equivalence classes of distribution functions F from D(2) and those equivalence classes of sequences of positive numbers $\{B_n\}$ which contain a nondecreasing sequence satisfying (12) with $\alpha^{-1} = 0$ (then every sequence in the equivalence class satisfies (12)). The correspondence is: $\{n^{\frac{1}{2}}B_n\}$ is a sequence of normalizing constants for F.

It is not difficult to give a direct proof of the statement about normalizing constants contained in Tucker's Theorem 2 based on Lemmas 1 and 2 in this note.

REFERENCES

[1] FELLER, W. (1966). An Introduction to Probability Theory and its Applications 2. Wiley, New York.
[2] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Theorems for Sums of Independent Random Variables*. Addison-Wesley, Reading.
[3] KARAMATA, J. (1930). Sur un mode de croissance régulière des fonctions. *Mathematica (Cluj)* 4 38–53.
[4] TUCKER, H. G. (1968). Convolutions of distributions attracted to stable laws. *Ann. Math. Statist.* 39 1381–1390.