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A WEAK CONVERGENCE THEOREM FOR RANDOM SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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0. Summary. A limit theorem for random sums of independent random elements of $D[0, 1]$ is proved. The theorem extends and simplifies the proof of a result obtained by R. Pyke in [3]. In a personal communication Professor Pyke remarked to me that the method employed is also the right approach to the problem studied in [4], by adequately defining sequences of random elements of $D[0, 1]$.

1. Introduction. Let D denote the set of all real functions on $[0, 1]$ which are right continuous on $[0, 1)$ and have left limits on $(0, 1]$. It is well known that there exists a metric $\mathfrak{d}$ on D under which this set becomes a complete separable metric space. For details of the definition of D and $\mathfrak{d}$, and basic properties of the Skorohod topology, the reader is referred to Chapter 3 of [1]. Let $\mathscr{D}$ denote the sigma field generated by the Skorohod topology. For $x \in D$ define $\|x\| = \sup_{0 \leqq t \leqq 1} |x(t)|$. Let $\mathscr{R}$ be the Borel sets of the real line R, and let $\mathscr{R}^{[0,1]}$ denote the product sigma field of $R^{[0,1]}$. In what follows all the random variables under consideration are defined on a fixed probability space $(\Omega, \mathscr{A}, P)$. If X is a random element of D, $\mathscr{L}(X)$ will denote its distribution. The symbol $\rightarrow_\omega$ indicates weak-star convergence. Let $\{\tau_n\}_{n=1,2\cdots}$ be a sequence of positive integer valued random variables, satisfying the following condition:

CONDITION 1. There exist sequences $\{b_n\}_{n=1,2\cdots}$ $\{c_n\}_{n=1,2\cdots}$ of positive integers such that $1 \leqq b_n < c_n$, $b_n \rightarrow \infty$, $c_n/b_n \rightarrow 1$ as $n \rightarrow \infty$, and $P[\tau_n \notin (b_n, c_n)] \rightarrow 0$ as $n \rightarrow \infty$.

It is not difficult to show that this condition is satisfied, if and only if, there exist a sequence $\{a_n\}_{n=1,2\cdots}$, $a_n > 0$, $a_n \rightarrow \infty$, such that $\tau_n/a_n \rightarrow_P 1$. The main result of the paper is the following.

THEOREM. *Let $\{X_n\}_{n=1,2\cdots}$ be a sequence of random elements of D which are independent and identically distributed. Let $\{\tau_n\}_{n=1,2\cdots}$ be a sequence of positive random variables satisfying Condition* 1. *Then $\mathscr{L}(S_n/n^{\frac{1}{2}}) \rightarrow_\omega \mu$ implies $\mathscr{L}(S_{\tau_n}/\tau_n^{\frac{1}{2}}) \rightarrow_\omega \mu$ where $S_n = \sum_{i=1}^n X_i$ and $S_{\tau_n} = \sum_{i=1}^{\tau_n} X_i$.*

NOTE. The theorem is stated for a sequence of positive integer random variables. If we have a positive integer valued stochastic process $\{\tau_t : t \geqq 0\}$, and $\tau_t/a_t \rightarrow 1$, $0 < a_t$, $a_t \rightarrow \infty$ as $t \rightarrow \infty$, the proof remains the same, with the obvious modifications in the notation.

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2. Proof of the theorem. We first state and indicate the proofs of several propositions.

PROPOSITION 2.1. *$\mathscr{R}^{[0,1]} \cap D = \mathscr{D}$ where $\mathscr{R}^{[0,1]} \cap D = \{A \cap D : A \in \mathscr{R}^{[0,1]}\}$.*

PROOF. The result follows from Theorem 14.5 of [1] and comments preceding that theorem.

COROLLARY 2.1. *The application $(x, y) \to x+y$ from $(D \times D, \mathscr{D} \times \mathscr{D})$ into $(D, \mathscr{D})$ is measurable.*

PROOF. The corollary follows from Proposition 2.1 and the fact that for all t, $0 \leqq t \leqq 1$, the application $(x, y) \to x(t)+y(t)$ is measurable. The following inequality is well known for real random variables; ([2] Section 17, page 246). The proof for random elements of D is the same provided it is checked that addition is a measurable operation. This is given to us by Corollary 2.1.

PROPOSITION 2.2 *Let $\{X_i\}_{i=1,2\cdots n}$ be independent random elements of D, $S_k = \sum_{i=1}^{k} X_i\, k = 1, 2 \cdots n$. Then for all $t > 0$*

$$P[\max_{1 \leqq k \leqq n} \|S_k\| \geqq 2t] \leqq \frac{P[\|S_n\| \geqq t]}{1-\max_{1 \leqq i < n} P[\|S_n - S_i\| > t]}.$$

PROPOSITION 2.3. *For all $r \geqq 0$, the sets $\{x : \|x\| \leqq r\}$ and $\{x : \|x\| \geqq r\}$ are closed in the Skorohod topology. That is, the application $x \to \|x\|$ is continuous in the Skorohod topology.*

PROOF. The result follows easily using the fact that $\|x\| = \|x \circ \lambda\|$, where λ is an increasing homeomorphism of [0, 1] onto [0, 1] and $\circ$ indicates composition.

COROLLARY 2.2. *Let $\mathscr{P}$ be a family of probabilities on $(D, \mathscr{D})$ which is tight. Then $\lim_{r \to \infty} \sup_{\mu \in \mathscr{P}} \mu\{x : \|x\| \geqq r\} = 0$.*

Now to the proof of the theorem. First we compare $S_{\tau_n}/\tau_n^{\frac{1}{2}}$ and $S_{\tau_n}/b_n^{\frac{1}{2}}$.
For all $\varepsilon > 0$ we have

$$\begin{aligned} &P[\|S_{\tau_n}/\tau_n^{\frac{1}{2}} - S_{\tau_n}/b_n^{\frac{1}{2}}\| \geqq \varepsilon] \\ &\quad = P[\|S_{\tau_n}/b_n^{\frac{1}{2}}\|\, |(b_n/\tau_n)^{\frac{1}{2}} - 1| \geqq \varepsilon] \\ &\quad \leqq P[\|S_{\tau_n}/b_n^{\frac{1}{2}}\|\, |(b_n/\tau_n)^{\frac{1}{2}} - 1| \geqq \varepsilon,\ \tau_n \in (b_n, c_n)] + P[\tau_n \notin (b_n, c_n)] \\ &\quad \leqq P[\|S_{\tau_n}/b_n^{\frac{1}{2}}\| \geqq \varepsilon/|(b_n/c_n)^{\frac{1}{2}} - 1|] + P[\tau_n \notin (b_n, c_n)]. \end{aligned}$$

Since for all x and $y \in D$ $\mathfrak{d}(x, y) \leqq \|x-y\|$, this inequality together with Condition 1 and Corollary 2.2 shows that it is enough to prove that $\mathscr{L}(S_{\tau_n}/b_n^{\frac{1}{2}}) \to_{\omega} \mu$.
Now for all $\varepsilon > 0$, we have

$$\begin{aligned} &P[\|S_{\tau_n}/b_n^{\frac{1}{2}} - S_{b_n}/b_n^{\frac{1}{2}}\| \geqq \varepsilon] \\ &\quad \leqq P[\max_{b_n < k < c_n} \|S_k - S_{b_n}\| \geqq \varepsilon b_n^{\frac{1}{2}}] + P[\tau_n \notin (b_n, c_n)] \\ &\quad \leqq \frac{P[\|S_{c_n} - S_{b_n}\| \geqq \frac{1}{2}\varepsilon b_n^{\frac{1}{2}}]}{1-\max_{b_n \leqq k < c_n} P[\|S_{c_n} - S_k\| \geqq \frac{1}{2}\varepsilon b_n^{\frac{1}{2}}]} + P[\tau_n \notin (b_n, c_n)]. \end{aligned}$$

This last inequality follows from Proposition 2.2. Take d_n such that

$$P[\|S_{c_n} - S_{d_n}\| \geqq \tfrac{1}{2}\varepsilon b_n^{\frac{1}{2}}] = \max_{b_n \leqq k < c_n} P[\|S_{c_n} - S_k\| \geqq \tfrac{1}{2}\varepsilon b_n^{\frac{1}{2}}].$$

Now for all n, $b_n \leqq d_n < c_n$, and therefore $d_n \to \infty$ and $d_n/b_n \to 1$. If $\{a_n\}_{n=1,2\cdots}$ stands for any of the sequences $\{b_n\}_{n=1,2\cdots}$ or $\{d_n\}_{n=1,2\cdots}$ the result will follow if we show that

$$P[\|S_{c_n} - S_{a_n}\| \geqq \tfrac{1}{2}\varepsilon b_n^{\frac{1}{2}}] \to 0.$$

This is equivalent to (since the random elements are identically distributed)

$$P[\|S_{c_n - a_n}/(c_n - a_n)^{\frac{1}{2}}\| \geqq \tfrac{1}{2}\varepsilon[b_n/(c_n - a_n)]^{\frac{1}{2}}].$$

Since $\{\mathscr{L}(S_{c_n - a_n}/(c_n - a_n)^{\frac{1}{2}})\}_{n=1,2,\cdots}$ is tight and $b_n/(c_n - a_n) \to \infty$ the result follows from Corollary 2.2.

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