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CHARACTERISTIC FUNCTIONS, MOMENTS, AND THE CENTRAL LIMIT THEOREM

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0. Summary. In [2], Lindeberg conditions of order $\nu \geqq 2$ are defined and shown to be NSC for convergence of νth absolute moments in the Central Limit Theorem when $\nu = 2k, k = 2, 3, \cdots$. Section 4 contains the extension of that result to the case of all $\nu > 2$, the proof depending on some of the theorems, given in Section 2, relating the existence of moments to the integrability of the characteristic function near the origin. The proofs of the results of Section 2 are deferred to Section 3 and depend, in turn, on known results listed in Section 1.

Throughout, we use the notations $\Re l x$, $\Im m x$ for the real, imaginary (respectively) parts of x, and $[x]$ to mean the largest integer strictly less than x.

1. Introduction. There are various known forms of the limited expansion in powers of t, of the ch.f. $\phi(t)$ of a rv X. One expression is (Pitman [4] Theorem 4)

$$\phi(t) = 1 + \sum_{j=1}^{n-1} \frac{EX^j (it)^j}{j!} + \frac{(it)^n}{n!} \{EX^{+n} \phi_1(t) + (-)^n EX^{-n} \phi_2(t)\} \tag{1}$$

where $E|X|^n < \infty$ for some positive integer n and ϕ_1, ϕ_2 are ch.f.'s of positive rv's related to X^+, X^-; while another is (Loève [3] page 199)

$$\phi(t) = 1 + \sum_{j=1}^{m} \frac{EX^j (it)^j}{j!} + \alpha_m(t), \tag{2}$$

where $\nu > 0$, $E|X|^\nu < \infty$, $m = [\nu]$, and $\alpha_m(t) = C_\nu \theta_\nu(t) E|X|^\nu |t|^\nu$, C_ν being a real constant depending only on ν, and θ_ν a complex-valued function with $|\theta_\nu| \leqq 1$, all t.

Integral-ordered moments of X are displayed in these expansions, but it seems less well-known that absolute moments of non-integral orders are also available from $\phi(t)$, via a lemma of von Bahr ([6] Lemma 4) which, in slightly more generality, is

LEMMA 1.

$$E|X|^\nu \Re l K_\nu = \int_0^\infty \Re l \alpha_m(t) t^{-(\nu+1)}\, dt; \tag{3}$$

if ν *is not an odd integer and* $E|X|^\nu < \infty$, *then*

$$\Im m K_\nu (E(X^+)^\nu - E(X^-)^\nu) = \int_0^\infty \Im m \alpha_m(t) t^{-(\nu+1)}\, dt; \tag{4}$$

if $X \geqq 0$ a.e, *then*

$$K_\nu EX^\nu = \int_0^\infty \alpha_m(t) t^{-(\nu+1)}\, dt; \tag{5}$$

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where $\alpha_m(t) = Ef_m(tX)$,

$$f_m(u) = e^{iu} - \sum_{j=0}^{m} (iu)^j/j!, \tag{6}$$

and where $K_\nu = \pi[2\Gamma(\nu+1)]^{-1}\{-(\sin\frac{1}{2}\nu\pi)^{-1} + i(\cos\frac{1}{2}\nu\pi)^{-1}\}$.

PROOF. Von Bahr's method ([6]) is first used to establish (5), noting that the real and imaginary parts of $f_m(u)$, and hence of $\alpha_m(t)$, are of constant sign. Equations (3) and (4) follow by addition and subtraction of versions of (5) for X^+, X^-.

We note that

(i) if $E|X|^m < \infty$, $\alpha_m(t)$ is given by (2);

(ii) (3) and (5) hold even if a real or imaginary part is infinite, because the integrands have constant signs;

(iii) in particular (3) and the real part of (5) are infiite if ν is an even integer, while the imaginary part of (5) is infinite if ν is an odd integer.

Information relating the existence of $E|X|^\nu$ to the integrability of the end-terms of the limited expansions is now available by using equations (1) to (6). The results obtained are related to previous comparisons of asymptotic behavior of the distribution function $F(x)$ with the local behavior of $\phi(t)$, given by Pitman ([4], [5], using functions of regular growth) and Boas ([1], using Lipschitz behavior).

2. Results on local integrability of characteristic functions.

THEOREM 1. (i) ν *is not an even integer and* $E|X|^\nu < \infty$ iff

$$\int_0^\varepsilon \Re\,\alpha_m(t)t^{-(\nu+1)}\,dt < \infty \quad \textit{for some} \quad \varepsilon > 0.$$

(ii) *If* $X \geqq 0$ a.e., *then in addition to* (i), ν *is not an odd integer and* $EX^\nu < \infty$ iff

$$\int_0^\varepsilon \Im\,\alpha_m(t)t^{-(\nu+1)}\,dt < \infty \quad \textit{for some} \quad \varepsilon > 0.$$

THEOREM 2. *Let* $E|X|^\nu < \infty$. *Then, if* ν *is not an even integer*, $\Re\,\alpha_m(t) = o(|t|^\nu)$ *as* $t \to 0$. *If also* $X \geqq 0$ a.e., *then*

$$\alpha_m(t) = o(|t|^\nu) \quad \textit{for non-integral } \nu,$$

$$\Re\,\alpha_m(t) = o(|t|^\nu) \quad \textit{for odd integers } \nu, \qquad \textit{and}$$

$$\Im\,\alpha_m(t) = o(|t|^\nu) \quad \textit{for even integers } \nu; \qquad \textit{as } t \to 0.$$

THEOREM 3. *Let* $X \geqq 0$ a.e., $EX^n < \infty$ *for some* $n = 1, 2, \cdots$, *and let* X_n *be a* rv *with* ch.f. $\phi_1(t)$, *the* ch.f. *associated with* $\phi(t)$, *appearing as* ϕ_1 *in* (1). *For all* $\delta > 0$, $EX^{n+\delta} < \infty$ iff $EX_n^{\ \delta} < \infty$, *and*

$$EX^{n+\delta} = EX_n^{\ \delta} \cdot EX^n / nB(n, 1+\delta).$$

REMARKS. If X is not $\geqq 0$ a.e., there is a more complicated but similar version of the theorem.

Strictly speaking, X_n is not a new rv, but the *same* rv as X on a new probability space, i.e., if X is a rv on $(\Omega, \mathscr{F}, P)$, then by X_n we mean the same real function X, on $(\Omega, \mathscr{F}, P_n)$, where the relationship between the distribution functions of P and P_n is constructed by Pitman [4].

THEOREM 4. *Let* $\{X_n\}$ *be a sequence of* rv's *with* ch.f.'s

$$\phi_n(t) = 1 + \sum_{j=1}^{m} \frac{(it)^j E X_n^{\ j}}{j!} + \alpha_n(t),$$

where $E|X_n|^\nu < \infty$, $n = 1, 2, \cdots$, *and* ν *is not an even integer.* $\{|X_n|^\nu, n = 1, 2, \cdots\}$ *is uniformly integrable* (u.i.) iff

$$\text{(7)} \qquad \int_0^\varepsilon \Re\mathrm{l}\, \alpha_n(t) t^{-(\nu+1)} dt \to_{\varepsilon \to 0} 0 \quad \textit{uniformly in } n.$$

If $X_n \geqq 0$ a.e., $n = 1, 2, \cdots$, *then in* (7), $\Re\mathrm{l}\, \alpha_n(t)$ *can be replaced by* $\alpha_n(t)$ *for non-integral* ν, *or by* $\Im\mathrm{m}\, \alpha_n(t)$ *for even integers* ν.

3. Proofs.

PROOF of THEOREM 1. The theorem comes directly from Lemma 1 by noting that the convergence of the integrals involved depends only on the integrals over a neighborhood of zero, since $\alpha_m(t) = E(f_m(tX))$, and because of (6).

PROOF of THEOREM 2. Let $\lambda = [\nu/2]$ and $X \geqq 0$ a.e., with df $F(x)$. For $\lambda = 1, 2, \cdots$, it is easily shown from (6) that

$$\text{(8)} \qquad 0 \leqq (-)^{\lambda+1} \Re\mathrm{l} f_m(u) \leqq u^{2\lambda+2}/(2\lambda+2)!, \qquad \text{and}$$

$$\text{(9)} \qquad 0 \leqq (-)^{\lambda+1} \Re\mathrm{l} f_m(u) \leqq u^{2\lambda}/(2\lambda)!, \qquad \text{for all } u \geqq 0.$$

Let $K^2 = (2\lambda+1)(2\lambda+2)$, $x^\nu dF(x) = dG(x)$, with $\int_0^\infty dG(x) = EX^\nu < \infty$. Then

$$\begin{aligned}(-)^{\lambda+1} \Re\mathrm{l}\, \alpha_m(t) &= (-)^{\lambda+1} E(\Re\mathrm{l} f_m(tX)) \\ &\leqq \int_{[Xt<K]} (Xt)^{2\lambda+2}/(2\lambda+2)! + \int_{[Xt \geqq K]} (Xt)^{2\lambda}/(2\lambda)! \\ &\leqq \frac{t^\nu}{(2\lambda)!} \left\{ \frac{t^{2\lambda+2-\nu}}{K^2} \int_0^{K/t} x^{2\lambda+2-\nu} dG(x) + K^{2\lambda-\nu} \int_{K/t}^\infty dG(x) \right\}.\end{aligned}$$

Both terms within the brackets are $o(1)$ as $t \to 0$, the first by the version of Kronecker's lemma for integrals, when ν is not an even integer. Therefore

$$\text{(10)} \qquad \Re\mathrm{l}\, \alpha_m(t) = o(|t|^\nu) \quad \text{as} \quad t \to 0 \quad \text{for} \quad \lambda = 1, 2, \cdots,$$

and ν not an even integer. For $\lambda = 0$, (8) holds but (9) is replaced by $0 \leqq -\Re\mathrm{l} f_0(u) \leqq 2$; the proof that (10) holds for $\lambda = 0$ follows in identical fashion.

It can be shown similarly that $\Im\mathrm{m}\, \alpha_m(t) = o(|t|^\nu)$ as $t \to 0$ if ν is not an odd integer. The remainder of the theorem follows by simple computations with versions of the above X^+ and X^-.

PROOF OF THEOREM 3. With $\nu = n + \delta$, we use (5) (choosing $\Re\mathrm{l}$ or $\Im\mathrm{m}$ parts where appropriate) and apply (2) for the ch.f. $\phi_1(t)$.

PROOF OF THEOREM 4. Dropping the subscript n,

$$\begin{aligned} \infty &> (-)^{\lambda+1} \int_0^\varepsilon \Re \alpha_m(t) t^{-(\nu+1)} \, dt \\ &= (-)^{\lambda+1} \int_0^\varepsilon E(\Re f_m(tX)) t^{-(\nu+1)} \, dt \\ &= E|X|^\nu A(\varepsilon |X|) \qquad \text{(using Fubini's theorem)} \end{aligned} \tag{11}$$

where $A(x) = (-)^{\lambda+1} \int_0^x \Re f_m(u) u^{-(\nu+1)} \, du$.

Thus $A(x)$ is continuous and increasing with $A(0) = 0$, $A(\infty) = (-)^{\lambda+1} K_\nu$ (see von Bahr [6] Lemma 4, or Pitman [5], page 427). Therefore

$$E(|X|^\nu A(\varepsilon |X|)) \geqq \int_{[|X| \geqq y]} |X|^\nu A(\varepsilon y). \tag{12}$$

For $\{X_n\}$, let $E(|X_n|^\nu A(\varepsilon |X_n|)) \to_{\varepsilon \to 0} 0$ uniformly in n.

From (12),

$$\int_{[|X_n| \geqq y]} |X_n|^\nu \leqq (A(\varepsilon y))^{-1} E(|X_n|^\nu A(\varepsilon |X_n|)), \quad \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad \text{uniformly in} \quad n,$$

by letting $\varepsilon y =$ constant.

Therefore $\{|X_n|^\nu\}$ is u.i. Conversely, let $\{|X_n|^\nu\}$ be u.i. and choose y large enough for

$$\int_{[|X_n| \geqq y]} |X_n|^\nu \leqq \delta, \qquad \text{all } n.$$

Then

$$\begin{aligned} E|X_n|^\nu A(\varepsilon |X_n|) &\leqq \int_{[|X_n| < y]} A(\varepsilon y) |X_n|^\nu + (-)^{\lambda+1} K_\nu \int_{[|X_n| \geqq y]} |X_n|^\nu \\ &\leqq y^\nu A(\varepsilon y) + (-)^{\lambda+1} K_\nu \cdot \delta, \end{aligned}$$

which is small, for small ε, uniformly in n.

The proofs of the case $X \geqq 0$ a.e. are similar, with the usual modifications.

4. Convergence of moments in the Central Limit Theorem. Consider a sequence of independent rv's $\{X_n\}$ with $EX_n = 0$, $S_n = X_1 + \cdots + X_n$, $s_n^2 = ES_n^2$, $n = 1, 2, \cdots$. L_ν, a Lindeberg condition of order $\nu \geqq 2$ is said to hold if $s_n < \infty$, $n = 1, 2, \cdots$, and

$$\sum_{j=1}^n \int_{[|X_j| \geqq \varepsilon s_n]} |X_j|^\nu = o(s_n^\nu) \quad \text{as} \quad n \to \infty, \qquad \text{for all} \quad \varepsilon > 0. \tag{13}$$

L_2 is the classical Lindeberg condition. It was shown in [2] that for $\nu > 2$, (13) is equivalent to

$$\sum_{j=1}^n E|X_j|^\nu = o(s_n^\nu) \quad \text{as} \quad n \to \infty; \tag{14}$$

that $L_a \Rightarrow L_b$ for $2 \leqq b \leqq a$; and that for $\nu = 2k$, $k = 2, 3, \cdots$, L_ν is NSC for the Central Limit Theorem and the convergence of $E|S_n/s_n|^\nu$ to the νth absolute moment of a $N(0, 1)$ distribution. This result is now extended from even integral values of ν to all $\nu > 2$.

THEOREM 5. *For all $\nu > 2$, L_ν is* NSC *for the Central Limit Theorem and*

$$E|S_n/s_n|^\nu \to_{n \to \infty} m_\nu = \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} \{|x|^\nu e^{-\frac{1}{2}x^2} \, dx\}.$$

To prove the theorem we first need an equivalence for L_ν given by

LEMMA 2. *Let $U(x)$ be any continuous positive bounded increasing function on $0 \leqq x < \infty$ with $U(\infty) = K$, and $\nu > 2$. Then L_ν holds* iff *L_2 holds and*

$$V(n, \varepsilon) = s_n^{-\nu} \textstyle\sum_{j=1}^n E(|X_j|^\nu U(\varepsilon |X_j|/s_n)) \to_{\varepsilon \to 0} 0 \tag{15}$$

uniformly in $n = 1, 2, \cdots$.

PROOF. If L_ν holds, then L_2 holds and

$$V(n, \varepsilon) \leqq K s_n^{-\nu} \textstyle\sum_{j=1}^n E|X_j|^\nu < \delta \quad \text{for} \quad n \geqq n(\delta), \qquad \text{from (14).}$$

For each n, $V(n, \varepsilon) \downarrow 0$ as $\varepsilon \to 0$. Choose $\varepsilon \leqq \varepsilon(\delta)$ so that $V(n, \varepsilon) < \delta$ for $n = 1, 2, \cdots n(\delta) - 1$. Therefore $V(n, \varepsilon) < \delta$ for $\varepsilon \leqq \varepsilon(\delta)$ and all $n = 1, 2, \cdots$, proving uniform convergence. Conversely, if (15) and L_2 hold, then

$$K s_n^{-\nu} \textstyle\sum_{j=1}^n E|X_j|^\nu = s_n^{-\nu} E \sum_{j=1}^n |X_j|^\nu \{U(\varepsilon |X_j/s_n) + (K - U(\varepsilon |X_j|/s_n))\}$$
$$\leqq \delta + s_n^{-\nu} \textstyle\sum_{j=1}^n \{K \int_{[|X_j| < bs_n]} |X_j|^\nu + K \int_{[bs_n \leqq |X_j| < as_n/\varepsilon]} |X_j|^\nu$$
$$+ \alpha \textstyle\int_{[|X_j| \geqq as_n/\varepsilon]} |X_j|^\nu\}$$

where $\varepsilon < \varepsilon(\delta)$ and $U(a) = K - \alpha$, with a, α fixed;

$$\leqq \delta + K \cdot s_n^{-\nu} (bs_n)^{\nu-2} \textstyle\sum_{j=1}^n EX_j^2 + K s_n^{-\nu} (a/\varepsilon)^{\nu-2} s_n^{\nu-2} \sum_{j=1}^n \int_{[|X_j| \geqq bs_n]} X_j^2$$
$$+ \alpha s_n^{-\nu} \textstyle\sum_{j=1}^n E|X_j|^\nu.$$

Therefore

$$(K - \alpha) s_n^{-\nu} \textstyle\sum_{j=1}^n E|X_j|^\nu \leqq \delta + K b^{\nu-2} + K(a/\varepsilon)^{\nu-2} s_n^{-2} \sum_{j=1}^n \int_{[|X_j| \geqq bs_n]} X_j^2$$

which is made arbitrarily small by taking δ, then b small, and by L_2.

PROOF of THEOREM 5. The case $\nu = 2k$, $k = 2, 3, \cdots$ is covered by Theorem 1.1 of [2], so we assume that ν is not an even integer. Let $\lambda = [\nu/2]$. We can assume that $L_{2\lambda}$ holds, since either L_ν holds, implying $L_{2\lambda}$, or in the converse, the CLT holds and $E|S_n/s_n|^\nu \to m_\nu$, whence $E(S_n/s_n)^{2\lambda} \to m_{2\lambda}$, implying $L_{2\lambda}$ by Theorem 1.1 of [2]. We assume *further* that L_m holds ($m = [\nu]$); there is no extra assumption if $m = 2\lambda$, but if $m = 2\lambda + 1$, we must first follow the whole proof through for the case $\nu = 2\lambda + 1$, $m = 2\lambda$, establishing the theorem for odd integers, then return to the present position, able to assume, as reasoned above, that L_m holds. Recalling that L_2 is NSC for the Central Limit Theorem, L_2, and $E|S_n/s_n|^\nu \to_{n \to \infty} m_\nu \Leftrightarrow L_2$, and $|S_n/s_n|^\nu$ is uniformly integrable (well known) $\Leftrightarrow L_2$, and $W(n, \varepsilon) \to_{\varepsilon \to 0} 0$ uniformly in n, by Theorem 4 (see (11)) where

$$W(n, \varepsilon) = E(|S_n/s_n|^\nu A(\varepsilon |S_n|/s_n)).$$

But $L_\nu \Leftrightarrow L_2$, and $V(n, \varepsilon) \to_{\varepsilon \to 0} 0$ uniformly in n, by Lemma 2. Therefore, to prove the theorem it is sufficient to show the equivalence of the uniform convergence to 0, as $\varepsilon \to 0$, of $V(n, \varepsilon)$ and $W(n, \varepsilon)$, since we already assume L_m which implies L_2. Recall, from the proof of Theorem 4, that

$$W(n, \varepsilon) = (-)^{\lambda+1} \textstyle\int_0^\varepsilon \mathfrak{Rl}\, \Lambda_n(t) t^{-(\nu+1)}\, dt, \tag{16}$$

where the ch.f. of S_n/s_n is

$$(17)\quad f_n(t) = 1+\sum_{r=2}^{m}\frac{(it)^r E(S_n/s_n)^r}{r!}+\Lambda_n(t) \qquad \text{(cf. (2))}$$

$$= \textstyle\prod_{j=1}^{n}\phi_j(t/s_n)$$

$$(18)\quad = \prod_{j=1}^{n}\left\{1+\sum_{r=2}^{m-1}\frac{(it)^r EX_j^{\,r}}{s_n^{\,r} r!}+\frac{(it)^m}{m!}(EX_j^{\,+m}\phi_{j1}(t/s_n)+(-)^m EX_j^{\,-m}\phi_{j2}(t/s_n))\right\} \qquad \text{(cf. (1))}$$

$$(19)\quad = \prod_{j=1}^{n}\left\{1+\sum_{r=2}^{m}\frac{(it)^r EX_j^{\,r}}{s_n^{\,r} r!}+\theta_j(t/s_n)\right\} \qquad \text{(cf. (2))}$$

and where ϕ_j is the ch.f. of X_j. By inspecting (17), (18) and (19) we can write

$$(20)\quad \Re\mathrm{l}\,\Lambda_n(t) = \textstyle\sum_{j=1}^{n}\Re\mathrm{l}\,\theta_j(t/s_n)+B_n(t),$$

where $B_n(t)$ is the sum of those terms in the product (18) which are $(2\lambda+2)$th or higher even powers of (it). A bound for $B_n(t)$ is obtainable by noting that

(i) $t^{2\lambda+2+2j}$ can be replaced by $t^{2\lambda+2}$ since $0 \leqq t \leqq \varepsilon < 1$,
(ii) such terms are products of *at least two* non-unity terms in (18); we get

$$B_n(t) \leqq t^{2\lambda+2}\left\{\prod_{j=1}^{n}\left(\exp\sum_{r=2}^{m}\frac{E|X_j|^r}{s_n^{\,r}\cdot r!}\right)-1-\sum_{j=1}^{n}\sum_{r=2}^{m}\frac{E|X_j|^r}{s_n^{\,r}\cdot r!}\right\}$$

$$(21)\quad \leqq \frac{t^{2\lambda+2}}{2!}\left(\sum_{j=1}^{n}\sum_{r=2}^{m}\frac{E|X_j|^r}{s_n^{\,r}\cdot r!}\right)^2$$

$$= \tfrac{1}{2}t^{2\lambda+2}(\tfrac{1}{2}+o(1))^2, \qquad \text{as} \quad n\to\infty$$

since $L_3, \cdots L_m$ are assumed to hold. Applying (21) in (20), then in turn in (16), we obtain

$$W(n,\varepsilon) = (-)^{\lambda+1}s_n^{\,-\nu}\textstyle\sum_{j=1}^{n}\int_0^{\varepsilon/s_n}\Re\mathrm{l}\,\theta_j(y)\cdot y^{-(\nu+1)}\,dy+O(\varepsilon^{2\lambda+2-\nu})$$

$$= s_n^{\,-\nu}\textstyle\sum_{j=1}^{n}E(|X_j|^\nu A(\varepsilon|X_j|/s_n))+O(\varepsilon^{2\lambda+2-\nu}),$$

using (11) (in the proof of Theorem 4), $= V(n,\varepsilon)+O(\varepsilon^{2\lambda+2-\nu})$. This demonstrates the required equivalence of uniform convergences of W and V as $\varepsilon\to 0$; and completes the proof.

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