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AN ADDENDUM TO "STOCHASTIC APPROXIMATION AND NONLINEAR REGRESSION"

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The purpose of this note is to point out a strengthening of results on asymptotic normality given in Albert and Gardner [1]. In brief, probability one convergence of the method of differential corrections can be used to replace global limit requirements by local ones. The authors wish to thank Professor Vaclav Fabian for bringing this point to their attention. The approach was evidently first taken by Hodges and Lehmann [2] in their study of the large sample distribution of the Robbins–Monro procedure.

In what follows the undefined quantities are those in [1]. We will be discussing only Theorem 5.2, but it will be clear that corresponding alterations should also be made in certain theorems which precede it.

Consider, then, the estimate $t_n = t_n^{(2)}$ of Theorem 5.2 with a gain constant

$$A_2 > \tfrac{1}{2}.$$

The assumptions imply those of Theorem 2.1 or, more precisely, its corollary which is Theorem 2.3, Condition 3. Consequently, $t_n \to \theta$ a.s. as $n \to \infty$. Let $\delta > 0$ be given. By Egorov's theorem there is an N, depending on δ but not on points in the sample space, such that $|t_n - \theta| < \delta$ for all $n \geqq N$ with probability at least $1-\delta$. For $n \geqq N$ define new regression functions F_n' with derivatives

$$\begin{aligned} \dot{F}_n'(x) &= \dot{F}_n(\theta-\delta) \quad \text{for} \quad x < \theta - \delta \\ &= \dot{F}_n(x) \quad \text{for} \quad x \in J_\delta = [\theta-\delta, \theta+\delta] \\ &= \dot{F}_n(\theta+\delta) \quad \text{for} \quad x > \theta+\delta \end{aligned}$$

where x is further restricted to belong to the original interval J. For $n < N$, set $F_n' = F_n$ for all $x \in J$. Define new estimates $\{t_n'\}$ in terms of $\{F_n'\}$. Then

$$P\{t_n = t_n' \text{ for all } n \geqq N\} > 1 - \delta.$$

Now the primed problem obeys all the conditions of Theorem 5.2 with

$$b_n' = \inf_{x \in J_\delta} |\dot{F}_n'(x)| = b_n \inf_{x \in J_\delta} g_n(x).$$

We have uniformly in $x \in J_\delta$

$$g_n'(x) = \frac{b_n}{b_n'} g_n(x) \to \frac{g(x)}{\inf_{x \in J_\delta} g(x)} = g'(x) \qquad (n \to \infty)$$

with the appropriate replacement of x by the boundary points when x is outside the interval. By the conclusion of Theorem 5.2, $S_n(t_n' - \theta)$ is asymptotically normal,

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for appropriate S_n, provided A_2 exceeds $c'/2$, where $c' = \sup_{x \in J_\delta} g'(x)$. But such will be the case because c' goes to unity as $\delta \to 0$. Letting

$$z_n = S_n(t_n - \theta),$$

with or without a prime, and letting $\Phi(x)$ be the $N(0, 1)$ cdf, we have

$$\begin{aligned}|P\{z_n \leqq x\} - \Phi(x)| &\leqq |P\{z_n' \leqq x\} - \Phi(x)| + |P\{z_n \leqq x\} - P\{z_n' \leqq x\}| \\ &\leqq \delta + P\{t_n \neq t_n'\} \\ &\leqq 2\delta\end{aligned}$$

for all sufficiently large n. This proves that the "adaptive gains" in Theorem 5.2 are asymptotically normal for any $A_2 > \frac{1}{2}$.

It should be noted that in the above we have weakened the Gain 2 proviso in the conclusion of Theorem 5.2 in replacing $c^2/2$ by $c/2$. Such can be shown to be permissible. Furthermore, due to an oversight, the global Gain 3 proviso $A_3 > c^2/2$ should be replaced by the local condition $A_3 > g(\theta_0^{(3)})/2$.

REFERENCES

[1] Albert, A. E. and Gardner, L. A., Jr. (1968). *Stochastic Approximation and Nonlinear Regression.* MIT Press, Cambridge, Mass.

[2] Hodges, J. L. Jr., and Lehmann, E. L. (1956). Two approximations to the Robbins-Monro process. *Proc. Third Berkeley Symp. Math. Statist. Prob.* **1** 95–104. Univ. of California Press.