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A LOCAL LIMIT THEOREM AND RECURRENCE CONDITIONS FOR SUMS OF INDEPENDENT NON-LATTICE RANDOM VARIABLES

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1. Introduction. A sequence of independent random variables $\{X_k\}_{k=1}^{\infty}$ with distribution functions $\{F_k(x)\}$ generates, in a natural way, a random walk whose "position" at "time" n is given by $S_n = \sum_{k=1}^{n} X_k$.

Since this paper will deal with the case in which not all of the X_k take on values in some fixed lattice, we say that the random walk is recurrent if

(1) for each $\Delta > 0$ and z,

$$P\{0 < S_n - z < \Delta, \quad \text{for infinitely many} \quad n\} = 1.$$

Here $P\{\ \}$ is the product probability measure on $\prod_{k=1}^{\infty} R_k$ generated by $\{F_k(x)\}$, the R_k's being copies of the real line. In effect, the random walk is recurrent if, with probability one, every interval is "visited" infinitely often. Any tail of the sequence of random variables generates another random walk where the role of S_n is played by $S_n' = S_{n+k} - S_k$. It is evident, since the random walk is recurrent if and only if every interval is visited at least once with probability one by every tail of the random walk, that (1) is equivalent to

(2) for each integer k, $\Delta > 0$, and z,

$$P\{0 < S_{n+k} - S_k - z < \Delta, \quad \text{for some} \quad n\} = 1.$$

In the case of identically distributed summands the criterion of Chung and Fuchs [1] states that the random walk is recurrent if and only if the characteristic function of the summands, $\varphi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_k(x)$, satisfies

(3) $$\limsup_{s \uparrow 1} \int_{-\alpha}^{\alpha} [1 - s\varphi(t)]^{-1}\, dt = \infty, \qquad \text{for some} \quad \alpha > 0.$$

Otherwise the random walk is transient, i.e. the expression in (1) is zero for all $\Delta > 0$ and z.

In the non-identically distributed case Orey [4] has given conditions under which the random walk must be either transient or recurrent. We exploit his results to obtain sufficient conditions for recurrence. The proofs consist of a modification of the standard renewal argument and employ estimates of the type found in local limit theorems. Under more restrictive conditions a local limit theorem for non-identically distributed summands is obtained. Local limit theorems in the identically distributed case have been found by Shepp [6] and with greater generality by Stone [7]. Rozanov [5] and Mitalauskas [3] have obtained local limit theorems for non-identically distributed lattice-valued random variables.

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2. Local limit theorem. Let $\{X_k\}$ be a sequence of independent random variables with

$$E(X_k) = 0, \qquad E(X_k^2) = \sigma_k^2, \quad \textit{and} \quad \sum_{k=1}^n \sigma_k^2 = B_n^2.$$

If the random variables satisfy the condition

$$(\alpha) \qquad \exists M > 0 \quad \text{and} \quad c > 0 \ni \sigma_k^{-2} \int_{|x|<M} x^2 \, dF_k(x) \geqq c, \qquad \forall_k$$

then by Chebyshev's inequality, if M is sufficiently large

$$(\alpha_1) \qquad \exists C' > 0 \ni P\{|X_k| < M\} \geqq C' \qquad \forall_k$$

and consequently

(α_2) $\exists$ a bounded sequence of numbers $\{a_k\} \ni$

$$\inf P\{|X_k - a_k| < \delta\} > 0 \qquad \forall \delta > 0.$$

Define

$$A(t, \varepsilon) = \{x \mid |x| < M, \ |xt - \pi m| \geqq \varepsilon \ \forall \text{ integer } m, \ |m| \leqq M\}$$

and suppose

(β) for each $t \neq 0$ $\exists$ $\varepsilon = \varepsilon(t) \ni$

$$1/\log B_n \sum_{k=1}^n P\{X_k - a_k \in A(t, \varepsilon)\} \to \infty$$

with $\{a_k\}$ satisfying (α_2).
The Lindeberg condition states

$$(\gamma) \qquad 1/B_n^2 \sum_{k=1}^n \int_{|x| > \epsilon B_n} x^2 \, dF_k(x) \to 0$$

for any $\varepsilon > 0$.

THEOREM 1. (α), (β), *and* (γ) *imply*

$$(2\pi)^{\frac{1}{2}} B_n P\{S_n \in (z, z+\Delta)\} - \Delta \exp\left[-\tfrac{1}{2}(z+\tfrac{1}{2}\Delta)^2/B_n^2\right] \to 0$$

for all z *and* $\Delta > 0$, *uniformly for bounded* Δ.

PROOF. Let $\varphi_k(t) = \int_{-\infty}^{\infty} e^{itx} \, dF_k(x)$; then by the inversion formula we have

$$(4) \quad (2\pi)^{\frac{1}{2}} B_n P\{S_n \in (z, z+\Delta)\} = \lim_{T\to\infty} B_n/(2\pi)^{\frac{1}{2}} \int_{-T}^{T} 2t^{-1} \sin \tfrac{1}{2}\Delta t \, e^{-itw} \prod_{k=1}^n \varphi_k(t) \, dt,$$

where $w = z + \frac{1}{2}\Delta$. We break up the integral on the right side of (4) viz.

$$(5) \qquad \begin{aligned} \int_{-\infty}^{\infty} &= \int_{0 \leqq |t| < A/B_n} + \int_{A/B_n \leqq |t| < B} + \int_{B < |t| < D} + \int_{D \leqq |t|} \\ &= I_1 \qquad\quad + I_2 \qquad\quad + I_3 \qquad + I_4. \end{aligned}$$

From (γ) we obtain $\prod_{k=1}^n \varphi_k(t/B_n) \to e^{-t^2/2}$ uniformly for t in a compact set.

Thus,

$$(6)\quad B_n/(2\pi)^{\frac{1}{2}}I_1-\Delta\exp(-\tfrac{1}{2}w^2/B_n{}^2)$$
$$= B_n/(2\pi)^{\frac{1}{2}}I_1-\Delta^1/(2\pi)^{\frac{1}{2}}\textstyle\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}t^2-itw/B_n)\,dt$$
$$= 1/(2\pi)^{\frac{1}{2}}\textstyle\int_{-A}^{A}[2(t/B_n)^{-1}\sin\tfrac{1}{2}\Delta t/B_n\prod_{k=1}^{n}\varphi_k(t/B_n)-\Delta e^{-t^2/2}]$$
$$\cdot\exp(-itw/B_n)\,dt+\Delta/(2\pi)^{\frac{1}{2}}\textstyle\int_{|t|>A}\exp(-t^2/2-itw/B_n)\,dt,$$

and for bounded Δ and fixed A the expression in brackets in the first integral will approach zero uniformly, while the second integral is made as small as desired by fixing A large.

We next show that $B_n I_2$ can be made arbitrarily small for all n by appropriate choice of the constants A and B.

Let

$$\rho_k(t)=\textstyle\int_{-M}^{M}e^{itx}\,dF_k(x)\quad\text{and}\quad h_k=\int_{-M}^{M}dF_k(x).$$

Then using the Taylor expansion of e^{itx} we have

$$|\rho_k(t)|^2=h_k{}^2-t^2\{h_k\textstyle\int_{-M}^{M}x^2\,dF_k(x)-[\int_{-M}^{M}x\,dF_k]^2\}+O(M^3t^3).$$

Note that

$$[\textstyle\int_{-M}^{M}x\,dF_k(x)]^2=[\int_{|x|\geqq M}x\,dF_k(x)]^2\leqq[M^{-1}\int_{|x|\geqq M}x^2\,dF_k(x)]^2\leqq M^{-2}\sigma_k{}^4.$$

From condition (α) the σ_k are bounded. Thus we may choose M sufficiently large so that

$$h_k>\tfrac{1}{2}\quad\text{and}\quad\sigma_k{}^2/M^2<\tfrac{1}{4}c.$$

Then for t sufficiently small, say $|t|<B$,

$$|\rho_k(t)|^2\leqq h_k{}^2(1-t^2c\sigma_k{}^2/(8h_k{}^2)).$$

Therefore

$$|\varphi_k(t)|\leqq(1-h_k)+h_k(1-rt^2\sigma_k{}^2/h_k)\leqq\exp(-rt^2\sigma_k{}^2),$$

where $r=c/16h_k$. Thus we may conclude that

$$\textstyle\prod_{k=1}^{n}|\varphi_k(t/B_n)|\leqq\exp(-rt^2)\qquad |t|<B$$

and consequently

$$(7)\qquad |B_nI_2|\leqq\Delta\textstyle\int_{A}^{BB_n}\exp(-rt^2)\,dt\leqq\Delta 2A^{-1}\exp(-rA^2).$$

In dealing with I_3 we shall assume the a_k of hypothesis (β) are identically zero. In effect, we translate the variable X_k by the constant a_k without altering the absolute value of its characteristic function: $|\varphi_k(t)|=|\varphi_k(t)\exp(-ia_k t)|$. Our method here is an adaptation of a technique due to Rozanov [5].

The characteristic function of X_k symmetrized is given by

$$|\varphi_k(t)|^2=\textstyle\int\int\cos t(x-y)\,dF_k(x)\,dF_k(y).$$

Since $|xt_0 - m\pi| \leqq |xt_0 - xt| + |yt| + |t(x-y) - m\pi|$ we have for $|t - t_0| < \varepsilon/4M$, $|t(x-y) - m\pi| < \frac{1}{4}\varepsilon$

$$\begin{aligned}|\varphi_k(t)|^2 - 1 &= \int\int (\cos t(x-y) - 1)\, dF_k(x)\, dF_k(y) \\ &\leqq \int_{|y| < \varepsilon/4D} \int_{|x| < M} (\cos t(x-y) - 1)\, dF_k(x)\, dF_k(y) \\ &\leqq \int_{|y| < \varepsilon/4D} dF_k(y) \cdot \int_{|xt_0 - m\pi| \geqq \varepsilon} (\cos xt_0 - 1)\, dF_k(x) \\ &= V_{t_0} \int_{A(t_0, \varepsilon)} (\cos xt_0 - 1)\, dF_k(x) \leqq -\tfrac{1}{4} V_{t_0} \varepsilon^2 P\{X_k \in A(t_0, \varepsilon)\},\end{aligned}$$

where $0 < V_{t_0} \leqq P\{|X_k| < \varepsilon/(4D)\}$, $\varepsilon = \varepsilon(t_0)$, and from (α_2) V_{t_0} is independent of k. Hence

$$\begin{aligned}|\varphi_k(t)| &\leqq \exp\left[\tfrac{1}{2}(|\varphi_k(t)|^2 - 1)\right] \\ &\leqq \exp\{-\tfrac{1}{4} V_{t_0} \varepsilon^2 P[X_k \in A(t_0, \varepsilon)]\}\end{aligned}$$

and from condition (β), $B_n \int_{|t - t_0| < \varepsilon/4M} \prod_{k=1}^{n} |\varphi_k(t)|\, dt \to 0$. This can be done for each $|t_0|$ in the closed interval $[B, D]$.
So, by compactness,

$$|B_n I_3| \to 0. \tag{8}$$

Let W be a random variable independent of the X_k's with characteristic function

$$\begin{aligned}\varphi(t) &= 1 - |t|/D \qquad |t| \leqq D \\ &= \quad 0 \qquad\quad |t| > D.\end{aligned}$$

By the preceding argument, and the fact that the introduction of φ will kill I_4, we have

$$(2\pi)^{\frac{1}{2}} B_n P\{0 < S_n + W - z < \Delta\} \to \Delta, \tag{9}$$

for any fixed $\Delta > 0$ and z. For given $\varepsilon > 0$, now choose D sufficiently large so that $P\{|W| \geqq \varepsilon\} < \varepsilon$. Then since

$$\begin{aligned}P\{|W| < \varepsilon\} &P\{0 < S_n + W - z - \varepsilon < \Delta - 2\varepsilon\} \\ &\leqq P\{0 < S_n - z < \Delta\} \\ &\leqq [P\{|W| < \varepsilon\}]^{-1} P\{0 < S_n + W - z + \varepsilon < \Delta + 2\varepsilon\},\end{aligned}$$

it follows from (9) that

$$\begin{aligned}(1 - \varepsilon)(\Delta - 2\varepsilon) &\leqq \liminf (2\pi)^{\frac{1}{2}} B_n P\{0 < S_n - z < \Delta\} \\ &\leqq \limsup (2\pi)^{\frac{1}{2}} B_n P\{0 < S_n - z < \Delta\} \leqq (1 - \varepsilon)^{-1}(\Delta + 2\varepsilon),\end{aligned}$$

proving the theorem.

Condition (β) can be replaced with stronger, but less lugubrious hypotheses. Note the following corollaries:

COROLLARY 1. *In Theorem* 1, (β) *may be replaced by*

(β_1) $\exists\, T \ni F_k$ *has a density on* $(-M, M)$ *bounded in absolute value by* T.

COROLLARY 2. *In Theorem* 1, *condition* (β) *may be replaced by*

(β_2) $\exists\, d_1$, d_2 *and* d_3 *rationally independent, such that for all* $\varepsilon > 0$,

$$\inf_k P\{|X_k - d_i| < \varepsilon\} > 0 \qquad i = 1, 2, 3.$$

3. Recurrence theorem. The main result of this section is based on a lemma of general utility for establishing the recurrence of a random walk. Essentially, it is a modification of a standard renewal argument (cf., for instance Feller [2], page 313). A further application of the lemma is made in Section 4.

Let $\Delta_z = (z, z+\Delta)$ and $\gamma_z = (z, z+\gamma)$. For $n \geqq k$, define ${}_kU_n(z) = P\{S_n - S_k \in \Delta_z\}$, and $f_n(\gamma_z) = P\{S_n \in \gamma_z, S_m \notin \Delta_z, \text{ all } m < n\}$.
Then for fixed $\Delta > 0$ and z,

$$(10) \qquad {}_0U_m(z) = \sum_{k=1}^{m} \int_0^{\Delta} {}_kU_m(-\gamma)\, d_\gamma f_k(\gamma_z)$$

where the integral is the ordinary Stietjes integral with respect to $f_k(\gamma_z)$ over Δ_0. Summing (10) for $1 \leqq m \leqq n$ and dividing by the left-hand side, we have

$$(11) \qquad 1 = \sum_{k=1}^{n} \int_0^{\Delta} \{\sum_{m=k}^{n} {}_kU_m(-\gamma)[\sum_{m=1}^{n} {}_0U_m(z)]^{-1}\}\, d_\gamma f_k(\gamma_z)$$

which we define to be equal to $\sum_{k=1}^{n} \int_0^{\Delta} a_{nk}(\gamma)\, d_\gamma f_k(\gamma_z)$.
If now, for each γ, k,

$$(12) \qquad \lim_{n\to\infty} a_{nk}(\gamma) = 1; \qquad \text{and}$$

for all n and k, and bounded γ,

$$(13) \qquad a_{nk}(\gamma) < L < \infty$$

then by the dominated convergence theorem,

$$\sum_{k=1}^{\infty} f_k(\Delta_z) = 1,$$

which is equivalent to (2) with $k = 0$. Note that the argument remains valid if (12) and (13) hold only for some sub-sequence $\{n_i\}$ of values of n. We have proved

LEMMA. *If* $\{X_k\}_{k=1}^{\infty}$ *is such that, for all tails of the sequence, and for all* z *and* $\Delta > 0$, *conditions* (12) *and* (13) *hold, then the random walk generated by the sequence is recurrent.*

Orey has given conditions [4] which imply that $h_k(z) = P\{0 < S_{n+k} - S_k - z < \Delta$ infinitely often, for all $\Delta > 0\}$ is identically one or identically zero for all k and z. These same conditions imply that

$$h_k(z, \Delta) = P\{0 < S_{n+k} - S_k - z < \Delta \quad \text{infinitely often}\}$$

is identically one or identically zero for all k, z and $\Delta > 0$. In the next three paragraphs we give a short proof of this fact.

It is clear that $\{h_k(z, \Delta)\}$ is a solution to the system of equations

$$d_k(z) = \int d_{k+1}(z+y)\, dF_{k+1}(y), \qquad k \geqq 0$$

and that if

$$r_k(z) = \int h_k(z+y, \Delta) g(y)\, dy$$

where $g(y)$ is a bounded L_1 function, then $r_k(z)$ is bounded and continuous and $\{r_k(z)\}$ satisfies the above system of equations. If now one of the conditions a_1, a_2, or a_3 of Orey's Theorem 3.1 (our conditions δ_1, δ_2, and δ_3 below) are satisfied then $r_k(z) =$ constant, all k. Since $g(y)$ is an arbitrary bounded L_1 function, this implies that $h_k(z, \Delta) = c$, all k, and almost all z.

If $\Delta_1 > \Delta$ and $|x_1 - x_2| < \Delta_1 - \Delta$ then

$$\{y/|y-x_1| < \Delta_1\} \supset \{y/|y-x_2| < \Delta\}.$$

Therefore, if $h_k(y, \Delta) = c > 0$, a.e. y, then for all y and $\Delta_1 > \Delta$, $h_k(y, \Delta_1) \geqq c$. A simple argument shows that this entails that $h_k(z, \Delta_1) = 1$: Since $h_k(z, \Delta_1) \geqq c$, we can find an N_1 such that

$$P\{|S_{n+k} - S_k - z| > \Delta_1 \quad \text{for all} \quad n < N_1\} < 1 - \tfrac{1}{2}c.$$

Suppose $y = S_{N_1+k} - S_k$. We can find an $N_2 = N_2(y)$ such that

$$P\{|S_{n+N_1+k} - S_{N_1+k} + y - z| > \Delta_1 \quad \text{for all} \quad n < N_2\} < 1 - \tfrac{1}{2}c.$$

Continuing, we find that for any k,

$$P\{|S_{n+k} - S_k - z| > \Delta_1 \quad \text{for all} \quad n > 0\} = 0$$

which is tantamount to our assertion.

If now some $\Delta > 0$, $h_k(y, \Delta) = 0$ a.e., then, since

$$\{y/0 < y - x < n\Delta\} \subset \textstyle\bigcup_{i=1}^{n-1} \{y/0 < y - x_i < \Delta\}$$

implies that $h_k(y, n\Delta) < \sum h_k(x_i, \Delta)$ when the x_i are suitably chosen, we have, under Orey's conditions, established the dichotomy mentioned above, viz. $h_k(z, \Delta)$ is either identically zero for all x, k and Δ, or is identically one.

The full statement of this result is as follows:

Let $\{a_k\}$ be a sequence of real numbers such that for all $\varepsilon > 0$, $\inf P\{|X_k - a_k| < \varepsilon\} > 0$.

Let $\Gamma\{a_k\} = \{x \mid \sum_{k=1}^{\infty} P\{|X_k - a_k - x| < \varepsilon\} = \infty, \text{ for all } \varepsilon > 0\}$.

Let Γ^* be the closure of the group generated by Γ. If either

(δ_1) $\Gamma^* =$ reals, or

(δ_2) for some $d > 0$, $\Gamma^* = \{nd\}_{n=1}^{\infty}$, and $\sum X_k$ is not essentially convergent modulo d, or

(δ_3) $\Gamma^* = \{0\}$ and $\sum X_k$ is not essentially convergent,

then $\{X_k\}$ generates a random walk which is either transient or recurrent.

The hypothesis of the previous statement will be referred to as condition (δ^1). Note that it is much weaker than (β) of Theorem 1, which stipulates not merely the divergence, but the rate of divergence of a series.

(Orey also gives a weaker version of the hypothesis, a version which we will not elaborate.)

THEOREM 2. *If* $\{X_k\}$ *is a sequence of independent random variables satisfying*

(α^1) $\exists\, M > 0$ *and* $c > 0 \ni \sigma_k^{-2} \int_{|x|<M} x^2\, dF_k(x) \geqq c,\ \forall_k$

(β^1) $\liminf \sigma_k > 0$

(γ^1) *letting* $B_{nk}^2 = B_{n+k}^2 - B_k^2$, *for any* $\varepsilon > 0$,

$$B_{nk}^{-2} \sum_{i=k+1}^{n+k} \int_{|x|>\varepsilon B_{nk}} x^2\, dF_k(x) \to 0, \quad \textit{uniformly in } k$$

and condition (δ^1), *then* $\{X_k\}$ *generate a recurrent random walk.*

PROOF. The argument of Theorem 1 shows that (α^1) and (γ^1) imply that, if Δ is larger than some fixed constant and y is in a bounded set, then for n sufficiently large,

$$0 < l_1 \leqq B_{nk}\, {}_kU_{n+k}(\Delta_y) \leqq l_2 < \infty$$

where l_1 and l_2 do not depend on k.

By (α^1) and (β^1), for n sufficiently large

$$0 < l_3 \leqq B_{nk}\, n^{-\frac{1}{2}} \leqq l_4 < \infty,$$

independently of k. Hence for n sufficiently large, and all k

$$(l_1/l_4) n^{-\frac{1}{2}} \leqq {}_kU_{n+k}(\Delta_y) \leqq (l_2/l_3) n^{-\frac{1}{2}},$$

which implies that $a_{nk}(\gamma)$ is uniformly bounded for $0 < \gamma < \Delta$.

To prove the first hypothesis of the lemma, (12), note that

$$(14) \quad B_n[{}_0U_n(\Delta_z) - {}_kU_n(\Delta_y)] = \lim_{T\to\infty} B_n/2\pi \int_{-T}^{T} \rho(t) \prod_{i=k+1}^{n} \varphi_i(t)\, 2t^{-1} \sin \tfrac{1}{2}\Delta t\, dt$$

where
$$\rho(t) = \prod_{i=1}^{k} \varphi_i(t)\, e^{-izt} - e^{-i\gamma t}.$$

Since $|\rho(t)| < 2$, and $|\rho(t/B_n)| = O(B_n^{-1})$ uniformly for bounded z, γ and t, the estimates of Theorem 1 show that for any $\varepsilon > 0$, the upper limit of the left-hand side of (14) can be bounded by

$$\limsup_{n\to\infty} B_n/2\pi \int_{0 \leqq |t| \leqq A/B_n} O(B_n^{-1}) \left|\prod_{i=k+1}^{n} \varphi_i(t)\, 2t^{-1} \sin \tfrac{1}{2}\Delta t\right| dt + \varepsilon$$

by choosing A sufficiently large.

Therefore, given $\varepsilon > 0$, there is a constant $c_1 \ni$

$$\left|{}_0U_n(\Delta_z) - {}_kU_n(\Delta_y)\right| \leqq c_1 B_n^{-2} + \varepsilon B_n^{-1},$$

proving (12).

Thus, for arbitrary z and for Δ larger than some fixed constant, (2) is verified. This means that the random walk cannot be transient, and hence by condition (γ^1) and Orey's result, the random walk is recurrent.

We note that analogous theorems can be proved whenever the random variables $\{X_k\}$ follow a local limit theorem associated with a stable law of index $\alpha \geqq 1$.

4. A further application of the recurrence lemma. In this section we consider $\{X_k\}$ where $X_k = \sum_{i=n_{k-1}}^{n_k - 1} Y_i$, $0 = n_0 < n_1 < n_2 \cdots < n_k < \cdots$ and $\{Y_k\}$ is a sequence of independent identically distributed non-lattice random variables. In effect, we consider a random walk with stationary increments which is "observed" only at a certain sub-sequence of times $\{n_k\}$. The results depend on a local limit theorem obtained by Stone ([7], Corollary 3). The special case of his theorem which we require is as follows: If $\{Y_k\}$ is a sequence of non-lattice random variables in the domain of attraction of a stable law of index α, with density function $p(x)$, in the sense that

$$P\{n^{-1/\alpha}\textstyle\sum_{k=1}^{n} Y_k < x\} \to \int_{-\infty}^{x} p(t)\,dt, \qquad \textit{all } x,$$

then for all z and $\Delta > 0$,

$$n^{1/\alpha}\,{}_0U_n(\Delta_z) \to \Delta p(0).$$

Using this result and the recurrence lemma of the previous section, the following theorem can be obtained:

THEOREM 3. *If $\{Y_k\}$ is a sequence of independent identically distributed random variables, Y_k is in the domain of attraction of a stable law of index $\alpha \geqq 1$, with density function $p(x)$ and if $\{n_k\}$ is an increasing sub-sequence of integers satisfying either*

$$(15) \qquad \limsup_{k\to\infty} n_k^{-1}k > 0 \qquad \textit{or}$$

$$(16) \qquad \liminf_{k\to\infty} \inf_{m>0} n_m^{-1}(n_{m+k}-n_k) > 0 \qquad \textit{and}$$

$$\sum (n_k)^{-1/\alpha} = \infty$$

then the random walk generated by $\{X_k\}$, $X_k = \sum_{i=n_{k-1}}^{n_k-1} Y_i$, is recurrent. In particular, if $\alpha > 1$, the random walk is recurrent if $n_k = k^{\alpha}$.

PROOF. Since (15) entails that $\sum(n_k)^{-1/\alpha} = \infty$, and since, defining $a_{nk}(\gamma)$ and ${}_kU_n(z)$ as before, $(n)^{1/\alpha}[{}_0U_n(z) - {}_kU_n(\gamma)] \to 0$, it is clear that in either case, Stone's result implies that $\lim_{n\to\infty} a_{nk}(\gamma) = 1$, for all k and γ.

If (15) is satisfied, let $\{n_{k'}\}$ be a sub-sequence of $\{n_k\}$ such that $\inf_{k'} k'/n_{k'} = \delta > 0$. Then, by Stone's result, if $n = k' > k$,

$$\begin{aligned} a_{nk}(\gamma) &= O\{(\textstyle\sum_{m=k+1}^{k'} (n_m - n_k)^{-1/\alpha})(\sum_{m=1}^{k'} (n_m)^{-1/\alpha})^{-1}\} \\ &= O\{(n_{k'})^{1-1/\alpha}(k' n_{k'}^{-1/\alpha})^{-1}\} \\ &= O\{1/\delta\}, \end{aligned}$$

where the bound is independent of k. Thus (13) is established for the sub-sequence $\{k'\}$. If (16) is satisfied, the verification of (13) is even simpler.

The conclusion of the theorem follows now from the recurrence lemma.

REFERENCES

[1] CHUNG, K. L. and FUCHS, W. H. J. (1951). On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.* **6**.

[2] FELLER, W. (1957). *An Introduction to Probability Theory and its Applications*, **1**. Wiley, New York.

[3] MITALAUSKAS, A. A. (1962). Local limit theorems for stable limit distributions. *Theor. Prob. Appl.* **7** 180–184.

[4] OREY, S (1966). Tail events for sums of independent random variables. *J. Math. Mech.* **15** 937–951.
[5] ROZANOV, Y. A. (1957). On a local limit theorem for lattice distributions. *Theor. Prob. Appl.* **2** 260–265.
[6] SHEPP, L. (1964). A local limit theorem. *Ann. Math. Statist.* **35** 419–423.
[7] Stone, C. (1965). A local limit theorem for non-lattice multidimensional distribution functions. *Ann. Math. Statist.* **36** 546–551.