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SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS FOR HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENT[1]

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1. Introduction and summary. Many distributions in multivariate analysis can be expressed in a form involving hypergeometric functions ${}_pF_q$ of matrix argument e.g. the noncentral Wishart (${}_0F_1$) and the noncentral multivariate $F({}_1F_1)$. For an exposition of distributions in this form see James [9]. The hypergeometric function ${}_pF_q$ has been defined by Constantine [1] as the power series representation

$$(1.1) \qquad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; R) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(R)}{k!}$$

where $a_1, \cdots, a_p, b_1, \cdots, b_q$ are real or complex constants,

$$(a)_\kappa = \prod_{i=1}^m (a - \tfrac{1}{2}(i-1))_{k_i}, \qquad (a)_n = a(a+1)\cdots(a+n-1)$$

and $C_\kappa(R)$ is the zonal polynomial of the $m \times m$ symmetric matrix R corresponding to the partition $\kappa = (k_1, k_2, \cdots, k_m)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_m$, of the integer k into not more than m parts. The functions defined by (1.1) are identical with the hypergeometric functions defined by Herz [5] by means of Laplace and inverse Laplace transforms. For a detailed discussion of hypergeometric functions and zonal polynomials, the reader is referred to the papers [1] of Constantine and [7], [8], [9] of James.

From a practical point of view, however, the series (1.1) may not be of great value. Although computer programs have been developed for calculating zonal polynomials up to quite high order, the series (1.1) may converge very slowly. It appears that some asymptotic expansions for such functions must be obtained. It is well known that asymptotic expansions for a function can in many cases be derived using a differential equation satisfied by the function (see e.g. Erdélyi [4]), and so, with this in mind, a study of differential equations satisfied by certain hypergeometric functions certainly seems justified.

In this paper a conjecture due to A. G. Constantine is verified i.e. it is shown that the function

$$(1.2) \qquad {}_2F_1(a, b; c; R) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(R)}{k!}$$

satisfies the system of partial differential equations

$$(1.3) \quad R_i(1-R_i)\,\partial^2 F/\partial R_i^2 + \{c - \tfrac{1}{2}(m-1) - (a+b+1-\tfrac{1}{2}(m-1))R_i + \tfrac{1}{2}\textstyle\sum_{j=1, j\neq i}^m [R_i(1-R_i)/(R_i - R_j)]\}\,\partial F/\partial R_i - \tfrac{1}{2}\textstyle\sum_{j=1, j\neq i}^m [R_j(1-R_j)/(R_i-R_j)]\,\partial F/\partial R_j = abF \qquad (i = 1, 2, \cdots, m)$$

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where $R_1, R_2, \cdots, R_m$ are the latent roots of the complex symmetric $m \times m$ matrix R. When $m = 1$, the system (1.3) clearly reduces to the classical hypergeometric equation.

It appears difficult to establish this conjecture directly, and the method used has necessitated a section devoted to a summary of the argument involved (Section 3). The main result in the paper is summarized in Theorem 3.1 of this section. Section 4 contains proofs referred to in Section 3. Using the fact that $C_\kappa(R)$ satisfies the partial differential equation (James [10])

$$(1.4)\quad \sum_{i=1}^m R_i^2\, \partial^2 y/\partial R_i^2 + \sum_{i=1}^m \sum_{j=1, j\neq i}^m [R_i^2/(R_i - R_j)]\, \partial y/\partial R_i = \sum_{i=1}^m k_i(k_i + m - i - 1)y,$$

James and Constantine [11] have obtained the effects of certain differential operators on $C_\kappa(R)$. These results are given in Section 2 and are used in many proofs in Section 4. Section 5 is probably of most interest statistically, for here systems of partial differential equations similar to (1.3) are given for ${}_1F_1(a; c; R)$ and ${}_0F_1(c; R)$. These two functions occur often in multivariate distributions. The differential equations for ${}_1F_1(a; c; R)$ have been used by Constantine [3] to obtain an asymptotic expansion for the noncentral likelihood ratio criterion, and by the author [12] to obtain asymptotic distributions of Hotelling's generalized T_0^2 statistic, Pillai's $V^{(m)}$ criterion, and for the largest latent root of the covariance matrix. The system for ${}_0F_1(c; R)$ is a generalization of that given by James [6] for ${}_0F_1(m/2; R)$.

2. Notation and preliminary results. In the ensuing sections, use will be made of the following definitions and results. Denote by $\binom{\kappa}{\sigma}$ the coefficient of $C_\sigma(R)/C_\sigma(I)$ in the "binomial" expansion

$$(2.1)\quad C_\kappa(I+R)/C_\kappa(I) = \sum_{s=0}^k \sum_\kappa \binom{\kappa}{\sigma} C_\sigma(R)/C_\sigma(I).$$

These coefficients have been tabulated to $k = 4$ by Constantine [2] and to $k = 8$ by Pillai and Jouris [13]. We introduce the following differential operators

$$(2.2)\quad E = \sum_{i=1}^m R_i\, \partial/\partial R_i$$

$$(2.3)\quad D^* = \sum_{i=1}^m R_i^2\, \partial^2/\partial R_i^2 + \sum_{i=1}^m \sum_{j=1, j\neq i}^m [R_i^2/(R_i - R_j)]\partial/\partial R_i,$$

$$(2.4)\quad \varepsilon = \sum_{i=1}^m \partial/\partial R_i \qquad \text{and}$$

$$(2.5)\quad \delta^* = \sum_{i=1}^m R_i\partial^2/\partial R_i^2 + \sum_{i=1}^m \sum_{j=1, j\neq i}^m [R_i/(R_i - R_j)]\partial/\partial R_i.$$

Now James [10] has shown that

$$(2.6)\quad EC_\kappa(R) = kC_\kappa(R) \qquad \text{and}$$

$$(2.7)\quad D^*C_\kappa(R) = [\rho_\kappa + k(m-1)]C_\kappa(R) \qquad \text{where}$$

$$\rho_\kappa = \sum_{i=1}^m k_i(k_i - i).$$

(2.6) follows from the fact that $C_\kappa(R)$ is an eigenfunction of Euler's operator E, and (2.7) from the fact that $C_\kappa(R)$ is an eigenfunction of the Laplace-Beltrami operator.

Corresponding to the partition κ, let $\kappa_i = (k_1, k_2, \cdots, k_i+1, \cdots, k_m)$ and $\kappa^{(i)} = (k_1, k_2, \cdots, k_i-1, \cdots, k_m)$ wherever they are admissible i.e. so long as the parts are in decreasing order. Then James and Constantine [11] have shown that

$$(2.8) \qquad \varepsilon C_\kappa(R)/C_\kappa(I) = \textstyle\sum_{i=1}^m \binom{\kappa}{\kappa^{(i)}} C_{\kappa^{(i)}}(R)/C_{\kappa^{(i)}}(I) \qquad \text{and}$$

$$(2.9) \qquad \delta^* C_\kappa(R)/C_\kappa(I) = \textstyle\sum_{i=1}^m \binom{\kappa}{\kappa^{(i)}}(k_i - 1 + \frac{1}{2}(m-i)) C_{\kappa^{(i)}}(R)/C_{\kappa^{(i)}}(I).$$

The summations in (2.8) and (2.9) are over all i such that $\kappa^{(i)}$ is admissible. This convention will be adopted in all future summations involving κ_i and $\kappa^{(i)}$.

(2.8) is proved from first principles in the following way:

$$\begin{aligned}\varepsilon C_\kappa(R)/C_\kappa(I) &= \textstyle\sum_{i=1}^m [\partial C_\kappa(R)/\partial R_i]/C_\kappa(I)\\ &= \lim_{\lambda\to 0}[C_\kappa(R+\lambda I) - C_\kappa(R)]/\lambda C_\kappa(I)\\ &= \lim_{\lambda\to 0}[\textstyle\sum_{i=1}^m \binom{\kappa}{\kappa^{(i)}} C_{\kappa^{(i)}}(R)/C_{\kappa^{(i)}}(I) + \text{terms of higher degree in } \lambda]\\ &= \textstyle\sum_{i=1}^m \binom{\kappa}{\kappa^{(i)}} C_{\kappa^{(i)}}(R)/C_{\kappa^{(i)}}(I).\end{aligned}$$

(2.9) is easily shown by noting that $\delta^* = \frac{1}{2}(\varepsilon D^* - D^*\varepsilon)$ and applying the operators ε and D^* to $C_\kappa(R)/C_\kappa(I)$.

One of the most important results in this paper, and one which will be needed later is summarized in

THEOREM 2.1. *Each of the m partial differential equations in the system* (1.3) *has the same unique solution F subject to the conditions*

(a) *F is a symmetric function of* $R_1, R_2, \cdots, R_m$, *and*

(b) *F is analytic about* $(0, 0, \cdots, 0)$, *and* $F(0) = 1$.

PROOF. It will be seen later that it is sufficient to consider the first differential equation ($i = 1$) i.e.

$$(2.10) \qquad \begin{aligned} R_1(1-R_1)\,\partial^2 F/\partial R_1^2 &+ \{c - \tfrac{1}{2}(m-1) - (a+b+1-\tfrac{1}{2}(m-1))R_1\\ &+ \tfrac{1}{2}\textstyle\sum_{j=2}^m [R_1(1-R_1)/(R_1-R_j)]\}\,\partial F/\partial R_1\\ &- \tfrac{1}{2}\textstyle\sum_{j=2}^m [R_j(1-R_j)/(R_1-R_j)]\,\partial F/\partial R_j = abF.\end{aligned}$$

We use the same method and notation employed by James [6] i.e. we transform (2.11) to a partial differential equation in terms of the elementary symmetric functions $r_1, r_2, \cdots, r_m$ of $R_1, R_2, \cdots, R_m$. Let $r_j^{(i)}$ for $j = 1, 2, \cdots, m-1$ denote the jth elementary symmetric function formed from the variables $R_1, R_2, \cdots, R_m$ omitting R_i. Defining r_0 and $r_0^{(i)}$ to be 1, we clearly have

$$(2.11) \qquad r_j = R_i r_{j-1}^{(i)} + r_j^{(i)} \qquad j = 1, 2, \cdots, m-1.$$

Using (2.11) and the relations

$$\partial/\partial R_i = \textstyle\sum_{\nu=1}^m r_{\nu-1}^{(i)}\,\partial/\partial r_\nu$$

$$\partial^2/\partial R_1^2 = \textstyle\sum_{\mu,\nu=1}^m r_{\mu-1}^{(1)} r_{\nu-1}^{(1)}\,\partial^2/\partial r_\mu \partial r_\nu,$$

it is readily verified that (2.10) becomes

$$(2.12)\quad \textstyle\sum_{\mu,\,\nu=1}^{m}(\partial^2 F/\partial r_\mu\,\partial r_\nu)\{\sum_{j=1}^{m} a_{\mu\nu}^{(j)}(r_{j-1}^{(1)}-r_j+r_j^{(1)})\}$$
$$+\textstyle\sum_{j=1}^{m}\{(c-\frac{1}{2}(j-1))r_{j-1}^{(1)}+(a+b+1-\frac{1}{2}j)r_j^{(1)}-(a+b+1)r_j\}\,\partial F/\partial r$$
$$-abF = 0$$

where (James [6] page 372) $a_{\mu\nu}^{(j)} = a_{\nu\mu}^{(j)}$, and for $\mu \leqq \nu$

$$\begin{aligned} a_{\mu\nu}^{(j)} &= r_{\mu+\nu-j} && \text{for } 1 \leqq j \leqq \mu \\ &= 0 && \text{for } \mu < j \leqq \nu \\ &= -r_{\mu+\nu-j} && \text{for } \nu < j \leqq \mu+\nu \\ &= 0 && \text{for } \mu+\nu < j \qquad (j = 1, \cdots, m) \end{aligned}$$

i.e. the matrix

$$A^{(j)} = (a_{\mu\nu}^{(j)}) =$$

$$\left[\begin{array}{cccccccccccc}
0 & & \cdots & & 0 & 0 & -1 & & & & & \\
 & & & & 0 & -1 & -r_1 & & & & & \\
\vdots & & & & -1 & -r_1 & -r_2 & & & & & \\
 & & & & \vdots & \vdots & \vdots & & & & & \\
0 & 0 & -1 & \cdots & -r_{j-6} & -r_{j-5} & -r_{j-4} & & 0 & & & \\
0 & -1 & -r_1 & \cdots & -r_{j-5} & -r_{j-4} & -r_{j-3} & & & & & \\
-1 & -r_1 & -r_2 & \cdots & -r_{j-4} & -r_{j-3} & -r_{j-2} & & & & & \\
 & & & & & & & r_j & r_{j+1} & \cdots & & r_m \\
 & & & & & & & r_{j+1} & r_{j+2} & \cdots & r_m & 0 \\
 & & & & 0 & & & \vdots & & & & \vdots \\
 & & & & & & & r_m & 0 & \cdots & & 0
\end{array}\right].$$

Clearly any solution of (2.12) satisfies condition (a). In (2.12) we can equate coefficients of $r_{j-1}^{(1)}$ to zero for $j = 1, \cdots, m$ (James [6] Lemma page 371) to obtain the system of partial differential equations

$$(2.13)\quad \textstyle\sum_{\mu,\,\nu=1}^{m}(a_{\mu\nu}^{(j)})\,\partial^2 F/\partial r_\mu\,\partial r_\nu+\sum_{\mu,\,\nu=1}^{m}(a_{\mu\nu}^{(j-1)})\,\partial^2 F/\partial r_\mu\,\partial r_\nu+(c-\frac{1}{2}(j-1))\,\partial F/\partial r_j$$
$$+(a+b+1-\tfrac{1}{2}(j-1))\,\partial F/\partial r_{j-1}-\delta_{1j}\{\textstyle\sum_{\mu,\,\nu=1}^{m}\ \partial^2 F/\partial r_\mu\,\partial r_\nu \sum_{i=1}^{m} a_{\mu\nu}^{(i)} r_i$$
$$+(a+b+1)\textstyle\sum_{i=1}^{m} r_i\,\partial F/\partial r_i+abF\} = 0 \qquad (j = 1, 2, \cdots, m).$$

As in James [6] the first two terms in (2.13) can be expressed as traces of matrices. Now, put

$$F(r_1, r_2, \cdots, r_m) = \textstyle\sum_{j_1, \cdots, j_m=0}^{\infty} \alpha_{j_1 \cdots j_m} r_1^{j_1} \cdots r_m^{j_m} \tag{2.14}$$

with $\alpha_{00\cdots 0} = 1$.

Introduce dictionary ordering for the coefficients $\alpha_{j_1 \cdots j_m}$ on the basis of the subscripts arranged in the order $j_m j_{m-1} \cdots j_2 j_1$. Substituting (2.14) in the differential equation with $j = m$ gives a recurrence relation which expresses $\alpha_{j_1 \cdots j_m}$ in terms of coefficients whose last subscript is less than j_m, and by iteration we can thus express $\alpha_{j_1 \cdots j_m}$ in terms of coefficients whose last index is zero. Putting $r_m = 0$ in the differential equation with $j = m-1$, we can then express coefficients of the form $\alpha_{j_1 \cdots j_{m-1}0}$ in terms of coefficients of the form $\alpha_{t_1 \cdots t_{m-2}00}$. Repeating this method we can express all coefficients in terms of $\alpha_{00\cdots 0}$ which we put equal to 1.

For example, the differential equation with $j = 3$ and $r_4 = \cdots = r_m = 0$, is

$$-2\partial^2 F/\partial r_1 \partial r_2 - r_1 \partial^2 F/\partial r_2{}^2 + r_3 \partial^2 F/\partial r_3{}^2 - \partial^2 F/\partial r_1{}^2 + r_2 \partial^2 F/\partial r_2{}^2$$
$$+ 2r_3 \partial^2 F/\partial r_2 \partial r_3 + (c-1)\partial F/\partial r_3 + (a+b)\partial F/\partial r_2 = 0.$$

On substituting

$$F(r_1, r_2, r_3) = \textstyle\sum_{j_1, j_2, j_3=0}^{\infty} \alpha_{j_1 j_2 j_3 0 \cdots 0} r_1^{j_1} r_2^{j_2} r_3^{j_3}$$

in this differential equation and equating coefficients of $r_1^{j_1} r_2^{j_2} r_3^{j_3}$, we get the recurrence relation

$$\begin{aligned}
\alpha_{j_1 j_2 j_3 0 \cdots 0} = \{ & 2(j_1+1)(j_2+1)\alpha_{j_1+1\, j_2+1\, j_3-1\, 0 \cdots 0} \\
& + (j_2+1)(j_2+2)\alpha_{j_1-1\, j_2+2\, j_3-1\, 0 \cdots 0} \\
& + (j_1+1)(j_1+2)\alpha_{j_1+2\, j_2\, j_3-1\, 0 \cdots 0} \\
& - (j_2+1)(j_2+2j_3+a+b-2)\alpha_{j_1\, j_2+1\, j_3-1\, 0 \cdots 0}\}/(j_3(j_3+c-2))
\end{aligned}$$

which can be iterated to express $\alpha_{j_1 j_2 j_3 0 \cdots 0}$ in terms of coefficients of the form $\alpha_{t_1 t_2 00 \cdots 0}$.

Clearly then, all the coefficients $\alpha_{j_1 \cdots j_m}$ in (2.14) are uniquely determined by the recurrence relations, and Condition (b) is satisfied. Hence the differential equation (2.10) has a unique solution F subject to the Conditions (a) and (b). But it is easily seen that each of the m differential equations in the system (1.3) gives rise to the same system of equations (2.13). (The proof is the same—we just equate coefficients of $r_{j-1}^{(i)}$ to zero for $i = 2, \cdots, m$). Hence each of the differential equations in (1.3) has the same unique solution subject to (a) and (b). This completes the proof of Theorem 2.1.

Finally we note that coefficients in the system (2.13) of differential equations do not involve m explicitly, so that the coefficients $\alpha_{j_1 \cdots j_m}$ obtained from the recurrence relations will be functions of a, b, c, and j_i but will be independent of m. In fact, since $r_h \equiv 0$ for $h > m$, the system (2.13) could formally be extended to hold for all $i = 1, 2, \cdots$ ad infinitum, and the upper limit m on the summations can be dropped. The coefficients α in (2.14) are thus defined for any number of subscripts $j_1, \cdots, j_h$, and are completely independent of m. Now, the series (2.14) could be rearranged as a series of zonal polynomials

$$F = \sum_{k=0}^{\infty} \sum_{\kappa} a_{\kappa} C_{\kappa}(R). \tag{2.15}$$

Since the zonal polynomials when expressed in terms of the elementary symmetric functions do not explicitly depend on m, the coefficients a_{κ} in (2.15) will also be functions of a, b, c, and κ but not m. Again, since $C_{\kappa} \equiv 0$ for any partition into more than m non-zero parts, the a_{κ} can be defined for partitions of any number of parts, and are therefore completely independent of m.

Summarizing, we have

COROLLARY. *The solution F in Theorem* 2.1 *can be obtained as a series of zonal polynomials $F = \sum_{k=0}^{\infty} \sum_{\kappa} a_{\kappa} C_{\kappa}(R)$ with coefficients a_{κ} independent of m.*

3. Summary of the argument. For simplicity, let us denote the m differential equations in (1.3) by the obvious notation

$$\Delta_i F = abF \qquad (i = 1, 2, \cdots, m). \tag{3.1}$$

In Theorem 2.1 we proved that, subject to the condition (a) and (b), that each differential equation in the system (3.1) has a unique solution $F(R)$. This solution can be expressed as a multiple power series with the elementary symmetric functions as variables, and the coefficients in this series can be obtained recursively by means of the differential equations (2.13). Unfortunately, we do not have an expression for ${}_2F_1(a, b; c; R)$ as a series of this type, but we do have an explicit expression as a series of zonal polynomials given by (1.2). This suggests substituting a series of the form

$$F(R) = \sum_{k=0}^{\infty} \sum_{\kappa} \gamma_{\kappa} C_{\kappa}(R), \qquad \gamma_{(0)} = 1 \tag{3.2}$$

in (3.1) to derive recurrence relations for the γ_{κ}, and showing that the coefficients in (1.2) satisfy the recurrence relations. Owing to the non-symmetry in the differential operators Δ_i, it appears difficult to do this directly, and a rather more roundabout method must be adopted. In outline, the method is as follows.

It is easily seen that the solution $F(R)$ of the system (3.1) must also be a solution of the differential equation formed by summing the differential equations in the system i.e. F is a solution of the differential equation

$$\Delta y = \sum_{i=1}^{m} \Delta_i y = maby. \tag{3.3}$$

But the operator Δ splits up into a linear combination of the operators D^*, E, δ^*, and ε defined previously. In fact (3.3) is

$$\delta^* y + (c - \tfrac{1}{2}(m-1))\varepsilon y - D^* y - (a+b+1-\tfrac{1}{2}(m-1))Ey = maby. \tag{3.4}$$

We can then apply each of these operators term-by-term to the series (3.2), using the results in Section 2. It is readily verified that comparing coefficients of $C_\kappa(R)$ on both sides of (3.4) gives rise to the following recurrence relations for the γ_κ

$$(3.5)\quad \sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(c+k_i-\tfrac{1}{2}(i-1))C_{\kappa_i}(I)\gamma_{\kappa_i} = (mab+\rho_\kappa+ka+kb+\tfrac{1}{2}k(m+1))C_\kappa(I)\gamma_\kappa.$$

The condition $F(0) = 1$ (i.e. $\gamma_{(0)} = 1$) is not sufficient to solve these relations uniquely, since there are more coefficients of degree $k+1$ than of degree k. For example, with $\gamma_{(0)} = 1$, we have $\gamma_{(1)} = ab/c$, but

$$2(c+1)C_{(2)}(I)\gamma_{(2)}+2(c-\tfrac{1}{2})C_{(1,1)}(I)\gamma_{(1,1)} = mab(mab+a+b+\tfrac{1}{2}(m+1))/c$$

which is only one equation for the two unknowns $\gamma_{(2)}$ and $\gamma_{(1,1)}$. There are, in fact, an infinity of solutions of (3.5), and from these we have to select the correct one i.e. the one corresponding to the solution of the system (3.1).

However, the further condition that

(c) the coefficients γ_κ be independent of m, is sufficient to give a unique solution of (3.5). As we saw in the corollary to Theorem 2.1, this condition is also satisfied by the solution of the system (3.1), so that the function defined by this solution of (3.5) must agree with the solution F of the system (3.1). (See Theorem 4.1 below).

To show that ${}_2F_1(a, b; c; R)$ is the solution of (3.1), it is then only necessary to prove that the coefficients in (1.2) satisfy Condition (c) and the recurrence relation (3.5). Clearly (c) is satisfied i.e. the coefficients $(a)_\kappa(b)_\kappa/(c)_\kappa k!$ are independent of m. In Theorem 4.2 below, these coefficients are shown to satisfy the recurrence relation (3.5) as required.

We can summarize our results in the following

THEOREM 3.1. *${}_2F_1(a, b; c; R)$ is the unique solution of each of the m differential equations*

$$\begin{aligned}&[R_i(1-R_i)]\,\partial^2F/\partial R_i^2+\{c-\tfrac{1}{2}(m-1)-(a+b+1-\tfrac{1}{2}(m-1))R_i\\&\quad+\tfrac{1}{2}\textstyle\sum_{j=1,\,j\neq i}^{m} R_i(1-R_i)/(R_i-R_j)\}\partial F/\partial R_i\\&\quad-\tfrac{1}{2}\textstyle\sum_{j=1,\,j\neq i}^{m}[R_j(1-R_j)/(R_i-R_j)]\,\partial F/\partial R_j = abF \qquad (i = 1,2,\cdots,m)\end{aligned}$$

subject to the conditions

(a) *F is a symmetric function of $R_1, R_2, \cdots, R_m$, and*

(b) *F is analytic about $R = 0$, and $F(0) = 1$.*

4. Results and proofs. In this section, results referred to in Section 3 are proved wherever necessary.

We have seen that in order to show that the unique solution of the system (3.1) is in fact the unique solution of the differential equation (3.4), it is only necessary to prove that the recurrence relations (3.5) yield a unique solution for the γ_κ, when the γ_κ are assumed to be independent of m. This is done in the following

THEOREM 4.1. *Under the condition that the γ_κ are independent of m, the recurrence relations* (3.5) *have a unique solution.*

PROOF. Since

$$C_{\kappa_i}(I)/C_\kappa(I) = (m+2k_i-i+1)\chi_{[2\kappa_i]}(1)/(2k+1)\chi_{[2\kappa]}(1) \tag{4.1}$$

where $\chi_{[2\kappa]}(1)$ is the dimension of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols (see James [9] page 478), we obtain, on substituting for $C_{\kappa_i}(I)$ from (4.1) into (3.5), the identity

$$\sum_i \binom{\kappa_i}{\kappa}(c+k_i-\tfrac{1}{2}(i-1))(m+2k_i-i+1)\chi_{[2\kappa_i]}(1)\gamma_{\kappa_i} = (2k+1)(mab+\rho_\kappa+ka+kb+\tfrac{1}{2}k(m+1))\chi_{[2\kappa]}(1)\gamma_\kappa. \tag{4.2}$$

Now assume that the γ_κ are independent of m. Then, coefficients of m on both sides of (4.1) may be equated to give

$$\sum_i \binom{\kappa_i}{\kappa}(c+k_i-\tfrac{1}{2}(i-1))\chi_{[2\kappa_i]}(1)\gamma_{\kappa_i} = (2k+1)(ab+\tfrac{1}{2}k)\chi_{[2\kappa]}(1)\gamma_\kappa, \tag{4.3}$$

while equating the constant terms in (4.2) gives

$$\sum_i \binom{\kappa_i}{\kappa}(c+k_i-(i-1)/2)(2k_i-i+1)\chi_{[2\kappa_i]}(1)\gamma_{\kappa_i} = (2k+1)(\rho_\kappa+ka+kb+\tfrac{1}{2}k)\chi_{[2\kappa]}(1)\gamma_\kappa. \tag{4.4}$$

Let $N(k)$ be the number of partitions of k. Then (4.3) and (4.4) constitute $2N(k)$ equations in the $N(k+1)$ unknowns $\gamma_{(k+1)}, \gamma_{(k,1)}, \cdots \gamma_{(1,1,\cdots,1)}$. Since $2N(k) \geqq N(k+1)$, there are more than enough equations to determine the coefficients correctly. For example, with $\gamma_{(0)} = 1$, we have $\gamma_{(1)} = ab/c$, and (4.3) gives

$$2(c+1)\gamma_{(2)}+4(c-\tfrac{1}{2})\gamma_{(1,1)} = 3(ab+\tfrac{1}{2})ab/c,$$

while (4.4) gives

$$4(c+1)\gamma_{(2)}-4(c-\tfrac{1}{2})\gamma_{(1,1)} = 3(a+b+\tfrac{1}{2})ab/c.$$

These two equations in the unknowns $\gamma_{(2)}$ and $\gamma_{(1,1)}$ yield the solution $\gamma_{(2)} = (a)_{(2)}(b)_{(2)}/(c)_{(2)}/2!$ and $\gamma_{(1,1)} = (a)_{(1,1)}(b)_{(1,1)}/(c)_{(1,1)}2!$. Thus the recurrence relations (3.5) yield a unique solution for the γ_κ as required. □

Finally we show that the coefficients $(a)_\kappa(b)_\kappa/(c)_\kappa k!$ satisfy the recurrence relations (3.5). We require the following lemmas.

LEMMA 4.1.

$$s_1 \operatorname{etr}(R) = \sum_{k=0}^\infty \sum_\kappa kC_\kappa(R)/k! \tag{4.5}$$

$$s_2 \operatorname{etr}(R) = \sum_{k=0}^\infty \sum_\kappa \rho_\kappa C_\kappa(R)/k! \tag{4.6}$$

$$s_1 s_2 \operatorname{etr}(R) = \sum_{k=0}^\infty \sum_\kappa (k-2)\rho_\kappa C_\kappa(R)/k! \tag{4.7}$$

where $s_i = R_1{}^i + \cdots + R_m{}^i$ *and* $\operatorname{etr}(R) = e^{\operatorname{tr}(R)}$.

PROOF.

$$s_1 \operatorname{etr}(R) = \sum_{k=0}^\infty s_1^{k+1}/k! = \sum_{k=0}^\infty ks_1{}^k/k! = \sum_{k=0}^\infty \sum_\kappa kC_\kappa(R)/k!$$

which proves (4.5).

Applying D^* to both sides of

(4.8) $$\operatorname{etr}(R) = \sum_{\kappa=0}^{\infty} \sum_{\kappa} C_\kappa(R)/k!$$

and equating coefficients of $C_\kappa(R)$ gives (4.6). Applying E to both sides of (4.6) and equating coefficients of $C_\kappa(R)$ using (4.6) gives (4.7).

LEMMA 4.2.

(4.9) $$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa} C_{\kappa_i}(I) = m(k+1)C_\kappa(I)$$

(4.10) $$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(k_i - \tfrac{1}{2}(i-1))C_{\kappa_i}(I) = k(k+1)C_\kappa(I).$$

PROOF. Applying ε to both sides of (4.8) and equating coefficients of $C_\kappa(R)$ gives (4.9). Applying δ^* to both sides of (4.5) and equating coefficients of $C_\kappa(R)$ using (4.5) gives (4.10). We are now in a position to prove the following

THEOREM 4.2. *${}_2F_1(a, b;\ c;\ R)$ is a solution of the differential equation* (3.4).

PROOF. It is sufficient to show that

(4.11) $$\gamma_\kappa = (a)_\kappa (b)_\kappa / (c)_\kappa\, k!$$

is a solution of the recurrence relations (3.5). Substituting (4.11) into (3.5), the problem reduces to showing that

(4.12) $$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(a + k_i - \tfrac{1}{2}(i-1))(b + k_i - \tfrac{1}{2}(i-1))C_{\kappa_i}(I) \\ = (k+1)(mab + \rho_\kappa + ka + kb + \tfrac{1}{2}k(m+1))C_\kappa(I).$$

Using (4.9) and (4.10) in (4.12), it remains to prove that

(4.13) $$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(k_i - \tfrac{1}{2}(i-1))^2 C_{\kappa_i}(I) = (k+1)(\rho_\kappa + \tfrac{1}{2}k(m+1))C_\kappa(I).$$

Applying δ^* to both sides of (4.6) and equating coefficients of $C_\kappa(R)$ using (4.5), (4.6), and (4.7) gives

(4.14) $$\sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(k_i - \tfrac{1}{2}(m-i))\rho_{\kappa_i} C_{\kappa_i}(I) = (k+1)[2mk + \rho_\kappa(k+2+\tfrac{1}{2}m(m-1))]C_\kappa(I).$$

Putting $\rho_{\kappa_i} = \rho_\kappa + 2k_i - i + 1$ and using (4.9) and (4.10) in (4.14) gives (4.13), as required. □

5. Differential equations for ${}_1F_1(a; c; R)$ and ${}_0F_1(c; R)$. We conclude by giving systems of partial differential equations satisfied by ${}_1F_1(a;\ c;\ R)$ and ${}_0F_1(c;\ R)$.

THEOREM 5.1. *${}_1F_1(a;\ c;\ R)$ is the unique solution of each of the m differential equations*

(5.1) $$R_i\, \partial^2 F/\partial R_i^2 + \{c - \tfrac{1}{2}(m-1) - R_i + \tfrac{1}{2}\textstyle\sum_{j=1,\, j\neq i}^{m} R_i/(R_i - R_j)\}\, \partial F/\partial R_i \\ - \tfrac{1}{2}\textstyle\sum_{j=1,\, j\neq i}^{m} [R_j/(R_i - R_j)]\, \partial F/\partial R_j = aF, \qquad (i = 1, 2, \cdots, m)$$

and ${}_0F_1(c;\ R)$ is the unique solution of each of the m differential equations

(5.2) $$(R_i)\, \partial^2 F/\partial R_i^2 + \{c - \tfrac{1}{2}(m-1) + \tfrac{1}{2}\textstyle\sum_{j=1,\, j\neq i}^{m} R_i/(R_i - R_j)\}\, \partial F/\partial R_i \\ - \tfrac{1}{2}\textstyle\sum_{j=1,\, j\neq i}^{m} [R_j/(R_i - R_j)]\, \partial F/\partial R_j = F, \qquad (i = 1, 2, \cdots, m)$$

subject to the conditions that

(a) *F is symmetric in $R_1, R_2, \cdots, R_m$, and*

(b) *F is analytic about $R = 0$, and $F(0) = 1$.*

PROOF. We can prove this theorem with the same method used for the proof of Theorem 3.1. For instance, the sum of the m differential equations in (5.2) is

$$(5.3) \qquad (c-\tfrac{1}{2}(m-1))\varepsilon F+\delta^* F = mF.$$

The recurrence relations corresponding to (5.3) analogous to (3.6) are

$$(5.4) \qquad \textstyle\sum_{i=1}^m \binom{\kappa_i}{\kappa}(c+k_i-\frac{1}{2}(i-1))C_{\kappa_i}(I)\gamma_{\kappa_i} = mC_\kappa(I)\gamma_\kappa.$$

On substituting $\gamma_\kappa = 1/(c)_\kappa k!$ into (5.4), the problem then reduces to proving that (4.9) is true. This is done in Lemma 4.2. Similarly, (4.10) shows that $(a)_\kappa/(c)_\kappa k!$ satisfies the recurrence relation obtained from the sum of the differential equations in (5.1). An alternative proof is also available using a limiting procedure. From Theorem 3.1, ${}_2F_1(a, b; c; 1/bR)$ satisfies the system

$$(5.5) \quad R_i(1-b^{-1}R_i)\,\partial^2 F/\partial R_i^{\,2}+\{c-\tfrac{1}{2}(m-1)-(a+b+1-\tfrac{1}{2}(m-1))b^{-1}R_i$$
$$+\tfrac{1}{2}\textstyle\sum_{j=1,\,j\neq i}^m R_i(1-b^{-1}R_i)/(R_i-R_j)\}\,\partial F/\partial R_i$$
$$-\tfrac{1}{2}\textstyle\sum_{j=1,\,j\neq i}^m [R_j(1-b^{-1}R_j)/(R_i-R_j)]\,\partial F/\partial R_j = aF \quad (i = 1, 2, \cdots, m).$$

Letting $b \to \infty$, ${}_2F_1(a, b; c; b^{-1}R) \to {}_1F_1(a; c; R)$ and the system (5.5) tends to the system (5.1). Similarly, since ${}_1F_1(a; c; a^{-1}R) \to {}_0F_1(c; R)$ as $a \to \infty$, the system (5.2) can be obtained as the limit of the system (5.1).

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