

THE LAW OF THE ITERATED LOGARITHM FOR A MARKOV PROCESS

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1. Introduction. The purpose of this paper is to prove the law of the iterated logarithm for a sequence $\{f(x_n)\}$, where f is a real-valued function defined on the state space of a discrete Markov Process $\{x_n\}$ satisfying Doeblin's hypothesis [3].

Most of the known results concerning the law of the iterated logarithm are obtained for a sequence of independent random variables and the proofs mainly depend on (i) the rate of convergence in the central limit theorem and (ii) certain inequalities due to Kolmogorov/Lévy [5]. In Section 3 we obtain the rate of convergence of $n^{-\frac{1}{2}} \sum_{j=1}^n f(x_j)$ to the normal distribution. In Section 5 we obtain the rate of convergence of the maximum of the partial sums of the random variables $f(x_j)$ to the positive normal distribution and use this rate in the place of Kolmogorov/Lévy inequalities.

2. Preliminary assumptions and lemmas. Let X be a space of points ξ and let \mathcal{F}_X be a Borel field of subsets of X . Let $\{x_n\}$ be a Markov process with state space X and stationary transition probabilities:

$$(2.1) \quad P(\xi, A) = P(x_{n+1} \in A \mid x_n = \xi).$$

That is, $\{x_n\}$ is a sequence of measurable functions from some probability space (Ω, \mathcal{B}, P) to X such that (2.1) holds where the transition function $p(\xi, A)$ is a measurable function of ξ for fixed $A \in \mathcal{F}_X$ and is a probability measure on \mathcal{F}_X for fixed ξ . The initial distribution π is defined by $\pi(A) = P\{x_1 \in A\}$ and the n -step transition probabilities are given by $P^{(n)}(\xi, A) = P\{x_{n+1} \in A \mid x_1 = \xi\}$. Throughout the following discussion Doeblin's condition will be assumed. In fact, we shall assume the hypothesis (D_0) :

(a) Doeblin's condition is satisfied.

(b) There is only a single ergodic set and this contains no cyclically moving subsets.

It is known that if (D_0) holds then there exist positive constants γ and ρ , $\rho < 1$, and a unique stationary absolute distribution p such that $|p^{(n)}(\xi, A) - p(A)| < \gamma\rho^n$ for all $\xi \in X$ and $A \in \mathcal{F}_X$ and $n \geq 1$. Throughout the following discussion we shall make the assumption:

$$(2.2) \quad \pi = p.$$

Let C_1, C_2, \dots be absolute constants. We shall now obtain two lemmas which will be used in the later analysis. Let \mathcal{F}_m denote the σ -field generated by the random variables (rv's) x_1, \dots, x_m .

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LEMMA 2.1. *If f is measurable with respect to ${}_1\mathcal{F}_m$ and g is a bounded function measurable with respect to ${}_{k+m}\mathcal{F}_\infty$, $|g| \leq M$, then $|E\{g|f\} - E\{g\}| \leq 2M\gamma\rho^k$.*

PROOF. Since $E\{g|f\} = E\{E(g|x_1, \dots, x_m)|f\}$

$$|E\{g|f\} - E\{g\}| \leq E\{|E\{g|x_1, \dots, x_m\} - E\{g\}|\} |f\}.$$

The result follows from Lemma 7.2 page 224 of [3].

COROLLARY 2.1. *If $A \in {}_1\mathcal{F}_m$ and $B \in {}_{k+m}\mathcal{F}_\infty$ then $|P(B|A) - P(B)| \leq 2\gamma\rho^k$.*

LEMMA 2.2. *If $A \in {}_1\mathcal{F}_m$ and g is a function measurable with respect to ${}_{k+m}\mathcal{F}_\infty$ and if x is a possible value of the rv g then $|P(A|g=x) - P(A)| \leq 2\gamma\rho^k$.*

PROOF. Define for each integer m the events

$$H_m(x) = \{[x2^m]2^{-m} \leq g < ([x2^m] + 1)2^{-m}\}$$

where $[a]$ is the largest integer less than or equal to a . Notice that $P(H_m(x)) > 0$ for all m . It is known (page 335 of [5], that $P(A|g=x) = \lim_{m \rightarrow \infty} P(A|H_m(x))$. Then we have by Lemma 7.1 page 222 [3]

$$\begin{aligned} |P(A|g=x) - P(A)| &= \lim_{m \rightarrow \infty} |P(A|H_m(x)) - P(A)| \\ (2.3) \qquad \qquad \qquad &= \lim_{m \rightarrow \infty} |E(\chi_A \chi_{H_m}) - E(\chi_A)E(\chi_{H_m})| E^{-1}(\chi_{H_m}) \\ &\leq \lim_{m \rightarrow \infty} 2\gamma^{1/s} \rho^{k/s} E^{1/r}(\chi_A) E^{1/s}(\chi_{H_m}) E^{-1}(\chi_{H_m}) \end{aligned}$$

for $r, s > 1, (1/r) + (1/s) = 1$.

Take $s = 1 + (1/m)E(\chi_{H_m})$. Then $s(m, x) \rightarrow 1$ and $E^{1/s}(\chi_{H_m})E^{-1}(\chi_{H_m}) \rightarrow 1$ as $m \rightarrow \infty$. We therefore have from (2.3)

$$|P(A|g=x) - P(A)| \leq 2\gamma\rho^k.$$

3. Convergence of partial sums. Let f be a real-valued function measurable with respect to \mathcal{F}_X such that $E\{f(x_1)\} = 0$ and $E\{f^2(x_1)\} = \sigma^2$. In view of (2.2) we have for every k , $E\{f^2(x_k)\} = \sigma^2$. Without loss of generality σ may be taken to be 1 which we do. Then

$$\lim_{n \rightarrow \infty} E\{(n^{-\frac{1}{2}} \sum_{j=1}^n f(x_j))^2\} = \sigma_1^2$$

exists. If $\sigma_1^2 > 0$ and if

$$(3.1) \qquad \qquad \qquad E\{|f(x_1)|^{2+\delta}\} < \infty$$

for some $\delta > 0$ then it has been proved (Theorem 7.5 page 228 [3]) that

$$(3.2) \qquad \qquad \lim_{n \rightarrow \infty} P(S_n \leq x\sigma_1 n^{\frac{1}{2}}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x e^{-(\frac{t}{\sigma_1})^2} dt = \Phi(x)$$

where $S_n = \sum_{j=1}^n f(x_j)$. Throughout this paper we shall assume that $\sigma_1^2 > 0$ and that (3.1) holds for some $\delta \leq 1$.

The purpose of this section is to obtain an estimate of the difference between the distribution of $(S_n)/\sigma_1 n^{\frac{1}{2}}$ and the standard normal distribution.

Let $\alpha_n = [n^{\frac{1}{2}}]$ and $\beta_n = [n^{\frac{1}{2}}]$. Then $\mu_n = [n(\alpha_n + \beta_n)^{-1}] \sim \beta_n$. For notational convenience we shall ignore the suffix n and write $\alpha_n = \alpha$, $\beta_n = \beta$ and $\mu_n = \mu$. Define

$$(3.3) \quad \begin{aligned} y_m &= \sum_{j=(m-1)(\alpha+\beta)+1}^{(m-1)(\alpha+\beta)+\alpha} f(x_j) & m = 1, 2, \dots, \mu \\ y'_\mu &= \sum_{j=(m-1)(\alpha+\beta)+\alpha+1}^{m(\alpha+\beta)} f(x_j) & m = 1, 2, \dots, \mu \end{aligned}$$

$$(3.4) \quad \begin{aligned} y'_{m+1} &= \sum_{j=\mu(\alpha+\beta)+1}^n f(x_j). & \text{Write} \\ T_r &= \sum_{m=1}^r y_m \quad \text{and} \quad V_n = \sum_{m=1}^{\mu+1} y'_m. \end{aligned}$$

Under the assumption (2.2) y_m 's are identically distributed. Let $F(x)$ denote the distribution function of y_1 .

THEOREM 3.1. *There exists N_0 such that for $n \geq N_0$*

$$\sup_x |P(S_n \leq x\sigma_1 n^{\frac{1}{2}}) - \Phi(x)| \leq C_4 \max \{n^{-\delta/8}, n^{-1/12}\}.$$

PROOF. Let $\eta = \eta(n)$ be an arbitrary positive number.

$$(3.5) \quad \begin{aligned} P(S_n \leq x\sigma_1 n^{\frac{1}{2}}) &= P(T_\mu + V_n \leq x\sigma_1 n^{\frac{1}{2}}, |V_n| \leq \eta\sigma_1 n^{\frac{1}{2}}) \\ &\quad + P(T_\mu + V_n \leq x\sigma_1 n^{\frac{1}{2}}, |V_n| > \eta\sigma_1 n^{\frac{1}{2}}) \\ &\leq P(T_\mu \leq (x + \eta)\sigma_1 n^{\frac{1}{2}}) + P(|V_n| > \eta\sigma_1 n^{\frac{1}{2}}). \end{aligned}$$

Also

$$(3.6) \quad P(S_n \leq x\sigma_1 n^{\frac{1}{2}}) \geq P(T_\mu \leq (x - \eta)\sigma_1 n^{\frac{1}{2}}) - P(|V_n| > \eta\sigma_1 n^{\frac{1}{2}}).$$

Now consider

$$\begin{aligned} P(T_\mu \leq u) &= \int_{-\infty}^{\infty} P(T_\mu - y_1 \leq u - x_1 \mid y_1 = x_1) dF(x_1) \\ &= \int_{-\infty}^{\infty} P(T_\mu - y_1 \leq u - x_1) dF(x_1) + \theta_1(n), \quad \text{say.} \end{aligned}$$

By Corollary 2.1, $|\theta_1(n)| \leq 2\gamma\rho^\beta$. Also $P(T_\mu - y_1 \leq u - x_1) = \int_{-\infty}^{\infty} P(T_\mu - y_1 - y_2 \leq u - x_1 - x_2) dF(x_2) + \theta_2^*(n)$ so that

$$P(T_\mu \leq u) = \int_{-\infty}^{\infty} \{ \int_{-\infty}^{\infty} P(T_\mu - y_1 - y_2 \leq u - x_1 - x_2) dF(x_2) \} dF(x_1) + \theta_1(n) + \theta_2(n)$$

where $|\theta_2(n)| \leq \int_{-\infty}^{\infty} |\theta_2^*(n)| dF(x_1) \leq 2\gamma\rho^\beta$.

Proceeding as above we get

$$(3.7) \quad P(T_\mu \leq u) = P(Z_1 + \dots + Z_\mu \leq u) + \sum_{j=1}^{\mu-1} \theta_j(n)$$

where Z_1, \dots, Z_μ are independent random variables each distributed like y_1 and $|\theta_j(n)| \leq 2\gamma\rho^\beta$, $1 \leq j \leq \mu-1$. Also $E(Z_1 \alpha^{-\frac{1}{2}} \sigma_1^{-1})^2 \rightarrow 1$ as $n \rightarrow \infty$. It therefore follows that

$$\lim_{n \rightarrow \infty} P(Z_1 + \dots + Z_\mu \leq x\sigma_1 n^{\frac{1}{2}}) = \Phi(x).$$

In fact using Esseen's estimate [4] we get

$$(3.8) \quad \sup_x |P(Z_1 + \dots + Z_\mu \leq x\sigma_1 n^{\frac{1}{2}}) - \Phi(x)| \leq C_2 \mu^{-\delta/2} = C_2 n^{-\delta/8}$$

where C_2 does not depend on n .

From relations (3.5) to (3.8) we have

$$\begin{aligned}
 (3.9) \quad & \Phi(x - \eta) - C_2 n^{-\delta/8} + \sum_{j=1}^{\mu-1} \theta_j(n) - P(|V_n| > \eta \sigma_1 n^{\frac{1}{2}}) \\
 & \leq P(S_n \leq x \sigma_1 n^{\frac{1}{2}}) \\
 & \leq \Phi(x + \eta) + C_2 n^{-\delta/8} + \sum_{j=1}^{\mu-1} \theta_j(n) + P(|V_n| > \eta \sigma_1 n^{\frac{1}{2}}).
 \end{aligned}$$

But $|\Phi(x) - \Phi(x \pm \eta)| \leq \eta$. Following the discussion to prove (7.16), page 229 [3] we obtain $E(V_n^2) = O(n^{\frac{3}{2}})$. Applying Tchebyshev's inequality we get

$$P(|V_n| > \eta \sigma_1 n^{\frac{1}{2}}) < C_3 \eta^{-2} n^{-\frac{1}{2}}$$

where C_3 depends on σ_1 only.

We have then from (3.9)

$$\sup_x |P(S_n \leq x \sigma_1 n^{\frac{1}{2}}) - \Phi(x)| \leq \eta + C_2 n^{-\delta/8} + 2\gamma\mu\rho^\beta + C_3 \eta^{-2} n^{-\frac{1}{2}}.$$

Taking $\eta = \max \{n^{-\delta/8}, n^{-1/12}\}$ we get for n large, say, $\geq N_0$

$$\sup_x |P(S_n \leq x \sigma_1 n^{\frac{1}{2}}) - \Phi(x)| \leq C_4 \eta.$$

4. An approximation theorem for a multidimensional distribution. Set

$$\varepsilon_1 = 1/(3 + \delta), \quad \varepsilon_2 = \varepsilon_1 \delta/4, \quad \eta_1(n) = n^{-\varepsilon_2/2} (\log n)^{(1 + \delta/2)\varepsilon_1} \quad \text{and} \quad k(n) = [n^{\varepsilon_2} (\log n)^{\varepsilon_1}].$$

Define $\mu_i = [i\mu/k], i = 1, 2, \dots, k$.

In this section we approximate the distribution function of $(T_{\mu_1}, \dots, T_{\mu_k})$ with an appropriate k -dimensional normal distribution function. We follow the method of Chung [2].

Consider independent rv's ξ_1, \dots, ξ_k where ξ_j is distributed like $T_{\mu_j} - T_{\mu_{j-1}}, 1 \leq j \leq k. (T_{\mu_0} = 0).$

Denote $\zeta_i = \sum_{j=1}^i \xi_j$. Then

$$\begin{aligned}
 & P(T_{\mu_1} \leq x_1, \dots, T_{\mu_k} \leq x_k) \\
 & = \int_{-\infty}^{\infty} P(T_{\mu_1} \leq x_1, \dots, T_{\mu_{k-1}} \leq \min(x_{k-1}, x_k - u_k) \mid T_{\mu_k} - T_{\mu_{k-1}} = u_k) dP(\xi_k \leq u_k) \\
 & = \int_{-\infty}^{\infty} P(T_{\mu_1} \leq x_1, \dots, T_{\mu_{k-1}} \leq \min(x_{k-1}, x_k - u_k)) dP(\xi_k \leq u_k) + \Delta_1(n)
 \end{aligned}$$

where $|\Delta_1(n)| \leq 2\gamma\rho^\beta$ by Lemma 2.2.

Also

$$\begin{aligned}
 & P(T_{\mu_1} \leq x_1, \dots, T_{\mu_{k-1}} \leq \min(x_{k-1}, x_k - u_k)) \\
 & = \int_{-\infty}^{\infty} P(T_{\mu_1} \leq x_1, \dots, T_{\mu_{k-2}} \\
 & \leq \min(x_{k-2}, x_{k-1} - u_{k-1}, x_k - u_k - u_{k-1}) \mid T_{\mu_{k-1}} - T_{\mu_{k-2}} = u_{k-1}) \\
 & \quad \cdot dP(\xi_{k-1} \leq u_{k-1})
 \end{aligned}$$

so that

$$\begin{aligned}
 &P(T_{\mu_1} \leq x_1, \dots, T_{\mu_k} \leq x_k) \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(T_{\mu_1} \leq x_1, \dots, T_{\mu_{k-2}} \leq \min(x_{k-2}, x_{k-1} - u_{k-1}, x_k - u_k - u_{k-1})) \\
 &\quad \cdot dP(\xi_{k-1} \leq u_{k-1}) dP(\xi_k \leq u_k) + \Delta_1(n) + \Delta_2(n)
 \end{aligned}$$

where $|\Delta_2(n)| \leq 2\gamma\rho^\beta$.

Proceeding as above we arrive at

$$\begin{aligned}
 &P(T_{\mu_1} \leq x_1, \dots, T_{\mu_k} \leq x_k) \\
 (4.1) \quad &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P(\xi_1 \leq \min(x_1, x_2 - u_2, \dots, x_k - \sum_{j=2}^k u_j)) \\
 &\quad \cdot dP(\xi_2 \leq u_2) \dots dP(\xi_k \leq u_k) + \sum_{j=1}^{k-1} \Delta_j(n) \\
 &= P(\zeta_1 \leq x_1, \dots, \zeta_k \leq x_k) + \sum_{j=1}^{k-1} \Delta_j(n).
 \end{aligned}$$

Denote $F_j(x_1, \dots, x_j) = P(\zeta_1 \leq x_1, \dots, \zeta_j \leq x_j)$, $1 \leq j \leq k$.

Let Φ_j be the j dimensional normal distribution function with the same first and second order moments as F_j . Let Φ_j^* be the one dimensional normal distribution function with mean zero and variance $= E(\xi_j^2)$. Denote $F_j(x_1, \dots, x_j) - \Phi_j(x_1, \dots, x_j) = R_j(x_1, \dots, x_j)$ and $P(\xi_j \leq u) - \Phi_j^*(u) = R_j^*(u)$.

In view of (3.7) and (3.8) there exists a constant C_5 such that

$$(4.2) \quad \sup |R_j| \leq C_5 k^{\delta/2} n^{-\delta/8} \quad \text{and} \quad \sup |R_j^*| \leq C_5 k^{\delta/2} n^{-\delta/8} \quad 1 \leq j \leq k$$

for n large.

Consider

$$\begin{aligned}
 &F_{j+1}(x_1, \dots, x_{j+1}) \\
 &= \int_{-\infty}^{\infty} F_j(x_1, \dots, x_{j-1}, \min(x_j, x_{j+1} - u)) dP(\xi_{j+1} \leq u) \\
 &= \int_{-\infty}^{\infty} \{ \Phi_j(x_1, \dots, x_{j-1}, \min(x_j, x_{j+1} - u)) \\
 &\quad + R_j(x_1, \dots, x_{j-1}, \min(x_j, x_{j+1} - u)) \} dP(\xi_{j+1} \leq u) \\
 &= \int_{-\infty}^{\infty} \Phi_j(x_1, \dots, x_{j-1}, \min(x_j, x_{j+1} - u)) d\Phi_{j+1}^*(u) \\
 &\quad + \int_{-\infty}^{\infty} \Phi_j dR_{j+1}^* + \int_{-\infty}^{\infty} R_j dP(\xi_{j+1} \leq u).
 \end{aligned}$$

That is

$$(4.3) \quad R_{j+1}(x_1, \dots, x_{j+1}) = \int_{-\infty}^{\infty} \Phi_j dR_{j+1}^* + \int_{-\infty}^{\infty} R_j dP(\xi_{j+1} \leq u).$$

Now

$$\begin{aligned}
 &|\int_{-\infty}^{\infty} R_j dP(\xi_{j+1} \leq u)| \leq \sup |R_j|. \\
 &\int_{-\infty}^{\infty} \Phi_j(x_1, \dots, x_{j-1}, \min(x_j, x_{j+1} - u)) dR_{j+1}^*(u) \\
 &= \int_{-\infty}^{x_j+1-x_j} \Phi_j(x_1, \dots, x_j) dR_{j+1}^*(u) \\
 &\quad + \int_{x_j+1-x_j}^{\infty} \Phi_j(x_1, \dots, x_{j-1}, x_{j+1} - u) dR_{j+1}^*(u) \\
 &= \Phi_j(x_1, \dots, x_j) R_{j+1}^*(x_{j+1} - x_j) - \Phi_j(x_1, \dots, x_j) R_{j+1}^*(x_{j+1} - x_j) \\
 &\quad - \int_{x_j+1-x_j}^{\infty} R_{j+1}^*(u) d\Phi_j(x_1, \dots, x_{j+1} - u)
 \end{aligned}$$

on integration by parts. Hence $|\int_{-\infty}^{\infty} \Phi_j dR_{j+1}^*| \leq \sup |R_{j+1}^*|$. From (4.3) we therefore have

$$\sup |R_{j+1}| \leq \sup |R_j| + \sup |R_{j+1}^*|.$$

Using the relations at (4.2) we get by induction $\sup |R_k| \leq C_5 k^{1+\delta/2} n^{-\delta/8}$ for n large. From (4.1) and the above result we have

$$\begin{aligned} |P(T_{\mu_1} \leq x_1, \dots, T_{\mu_k} \leq x_k) - \Phi_k(x_1, \dots, x_k)| &\leq C_5 k^{1+\delta/2} n^{-\delta/8} + 2\gamma k \rho^\beta \\ &\leq C_6 k^{1+\delta/2} n^{-\delta/8}. \end{aligned}$$

We thus proved

LEMMA 4.1. *There exist constants C_6 and N_1 such that for all $n \geq N_1$*

$$\sup_{x_i, 1 \leq i \leq k} |F_k(x_1, \dots, x_k) - \Phi_k(x_1, \dots, x_k)| \leq C_6 \eta_1,$$

where η_1 is defined at the beginning of this section.

5. Rate of convergence of $\max_{1 \leq r \leq n} S_r$. Set $S_n^* = \max_{1 \leq r \leq n} S_r$ and $S_n^{**} = \max_{1 \leq j \leq \mu} S_{(\alpha+\beta)j}$. The limit distribution of S_n^* has been obtained by Billingsley [1].

We shall write $\alpha + \beta = \alpha_1$. Observe that

$$(5.1) \quad P(S_n^* \leq x \sigma_1 n^{\frac{1}{2}}) \leq P(S_n^{**} \leq x \sigma_1 n^{\frac{1}{2}}).$$

Let for each r , $\alpha_1(j(r) - 1) < r \leq \alpha_1 j(r)$. Define $D_r = \{S_{r-1}^* \leq x \sigma_1 n^{\frac{1}{2}}, S_r > x \sigma_1 n^{\frac{1}{2}}\}$ so that

$$(5.2) \quad \sum_{r=1}^n P(D_r) = P(S_n^* > x \sigma_1 n^{\frac{1}{2}}).$$

Write $D_r = D_r^{(1)} \cup D_r^{(2)}$ where $D_r^{(1)} = \{D_r \cap \{|S_{\alpha_1 j(r)} - S_r| \leq \eta_1 \sigma_1 n^{\frac{1}{2}}\}\}$ and $D_r^{(2)} = \{D_r \cap \{|S_{\alpha_1 j(r)} - S_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}\}\}$.

$$(5.3) \quad \sum_{r=1}^n P(D_r^{(1)}) \leq P(S_n^{**} > (x - \eta_1) \sigma_1 n^{\frac{1}{2}}).$$

In order to analyze $P(D_r^{(2)})$ we set $\delta_n = [n^{3\delta/(8+4\delta)}]$. Then if $\alpha_1 j(r) - r > \delta_n$

$$\begin{aligned} P(D_r^{(2)}) &\leq P(D_r \cap \{|S_{\alpha_1 j(r)} - S_{r+\delta_n}| > (\frac{1}{2})\eta_1 \sigma_1 n^{\frac{1}{2}}\}) \\ &\quad + P(|S_{r+\delta_n} - S_r| > (\frac{1}{2})\eta_1 \sigma_1 n^{\frac{1}{2}}) \\ &\leq P(D_r) \{P(|S_{\alpha_1 j(r)} - S_{r+\delta_n}| > (\frac{1}{2})\eta_1 \sigma_1 n^{\frac{1}{2}}) + 2\gamma \rho^{\delta_n}\} \\ &\quad + C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-(1+\delta/2)} \end{aligned}$$

by Corollary 2.1 and Tchebyshev's inequality. Therefore

$$\begin{aligned} P(D_r^{(2)}) &\leq P(D_r) \{C_8 \alpha_1^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-(1+\delta/2)} + 2\gamma \rho^{\delta_n}\} \\ &\quad + C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-(1+\delta/2)}. \end{aligned}$$

$$\sum_{r=1}^n P(D_r^{(2)}) \leq C_9 \eta_1^{-(2+\delta)} n^{-(2+\delta)/8} + 2\gamma \rho^{\delta_n} + C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-\delta/2}.$$

If $\alpha_1 j(r) - r < \delta_n$ also this inequality holds.

We have from (5.2), (5.3) and the above inequality

$$P(S_n^* > x\sigma_1 n^{\frac{1}{2}}) \leq P(S_n^{**} > (x - \eta_1)\sigma_1 n^{\frac{1}{2}}) + C_9 \eta_1^{-(2+\delta)} n^{-(2+\delta)/8} + 2\gamma\rho^{\delta n} + C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-\delta/2}.$$

This together with (5.1) gives

LEMMA 5.1. For η_1 as defined in Section 4 there exist constants C_{10} and N_2 such that for $n \geq N_2$

$$P(S_n^{**} \leq (x - \eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{10} \eta_1 \leq P(S_n^* \leq x\sigma_1 n^{\frac{1}{2}}) \leq P(S_n^{**} \leq x\sigma_1 n^{\frac{1}{2}}).$$

Denote $U_j = y_1' + \dots + y_j'$ and the event $\{|U_j| \leq \eta_1 \sigma_1 n^{\frac{1}{2}}\} = M_j$. Then

$$\begin{aligned} P(S_n^{**} \leq x\sigma_1 n^{\frac{1}{2}}) &= P(\{S_n^{**} \leq x\sigma_1 n^{\frac{1}{2}}\} \cap \{\bigcap_{j=1}^{\mu} M_j\}) \\ &\quad + P(\{S_n^{**} \leq x\sigma_1 n^{\frac{1}{2}}\} \cap \{\bigcap_{j=1}^{\mu} M_j\}') \\ (5.4) \qquad &\leq P(\max_{1 \leq r \leq \mu} T_r \leq (x + \eta_1)\sigma_1 n^{\frac{1}{2}}) + P(\bigcup_{j=1}^{\mu} M_j') \\ &\leq P(\max_{1 \leq r \leq \mu} T_r \leq (x + \eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{11} \eta_1^{-2} n^{-\frac{1}{2}}. \end{aligned}$$

Similarly

$$(5.5) \quad P(S_n^{**} \leq x\sigma_1 n^{\frac{1}{2}}) \geq P(\max_{1 \leq r \leq \mu} T_r \leq (x - \eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{11} \eta_1^{-2} n^{-\frac{1}{2}}.$$

From (5.4), (5.5) and Lemma 5.1 we have the following

LEMMA 5.2. For η_1 as defined in Section 4 there exist constants C_{12} and N_3 such that for $n \geq N_3$

$$\begin{aligned} P(\max_{1 \leq r \leq \mu} T_r \leq (x - 2\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{12} \eta_1 \\ \leq P(S_n^* \leq x\sigma_1 n^{\frac{1}{2}}) \\ \leq P(\max_{1 \leq r \leq \mu} T_r \leq (x + \eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{12} \eta_1. \end{aligned}$$

Set $T_{\mu}^* = \max_{1 \leq r \leq \mu} T_r$ and $T_{\mu}^{**} = \max_{1 \leq i \leq k} T_{\mu_i}$ where μ_i 's are as defined in Section 4.

LEMMA 5.3. We can find constants C_{16} and N_4 such that for $n \geq N_4$

$$P(T_{\mu}^{**} \leq (x - \eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{16} \eta_1 \leq P(T_{\mu}^* \leq x\sigma_1 n^{\frac{1}{2}}) \leq P(T_{\mu}^{**} \leq x\sigma_1 n^{\frac{1}{2}}).$$

PROOF. It is easily seen that

$$(5.6) \quad P(T_{\mu}^* \leq x\sigma_1 n^{\frac{1}{2}}) \leq P(T_{\mu}^{**} \leq x\sigma_1 n^{\frac{1}{2}}).$$

Define the events

$$E_r = \{T_{r-1}^* \leq x\sigma_1 n^{\frac{1}{2}}, T_r > x\sigma_1 n^{\frac{1}{2}}\}.$$

Then

$$(5.7) \quad \sum_{r=1}^{\mu} P(E_r) = P(T_{\mu}^* > x\sigma_1 n^{\frac{1}{2}}).$$

Suppose $\mu_{j(r)} < r \leq \mu_{j(r)+1}$. Then for any positive number $\eta_1 = \eta_1(n)$

$$(5.8) \quad E_r = \{E_r \cap \{|T_{\mu_{j(r)+1}} - T_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}\}\} \cup \{E_r \cap \{|T_{\mu_{j(r)+1}} - T_r| \leq \eta_1 \sigma_1 n^{\frac{1}{2}}\}\} \\ = E_r^{(1)} \cup E_r^{(2)}, \text{ say.}$$

$$(5.9) \quad \sum_{r=1}^{\mu} P(E_r^{(2)}) \leq P(T_{\mu}^{**} > (x - \eta_1)\sigma_1 n^{\frac{1}{2}}).$$

By Corollary 2.1

$$(5.10) \quad P(E_r^{(1)}) \leq P(E_r) \{P(|T_{\mu_{j(r)+1}} - T_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}) + 2\gamma\rho^{\beta}\}.$$

It is easily shown as in Theorem 3.1 that

$$(5.11) \quad P(|T_{\mu_{j(r)+1}} - T_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}) \leq P(|Z_1 + \dots + Z_m| > \eta_1 \sigma_1 n^{\frac{1}{2}}) + 4m\gamma\rho^{\beta},$$

where $m = \mu_{j(r)+1} - r$.

Let $B = B(n) = [n^{\frac{1}{2}}]$. If $m \leq B$ then by Tchebyshev's inequality and Lemma 7.4, page 225, [3] we have

$$(5.12) \quad P(|Z_1 + \dots + Z_m| > \eta_1 \sigma_1 n^{\frac{1}{2}}) \leq C_{13} \eta_1^{-(2+\delta)} n^{-(2+\delta)/8} B.$$

If $m > B$ using Lemma 7.4 and the Esseen's estimate we get

$$(5.13) \quad |P(|Z_1 + \dots + Z_m| > \eta_1 \sigma_1 n^{\frac{1}{2}}) - 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_v^{\infty} \exp(-(\frac{1}{2})t^2) dt| \leq C_{14} B^{-\delta/2}$$

where C_{14} depends only on σ_1 ; and $v = \eta_1 n^{\frac{1}{2}} m^{-\frac{1}{2}}$. Since $m \leq \mu/k$, $v > \eta_1 k^{\frac{1}{2}}$. Now

$$(5.14) \quad 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_v^{\infty} \exp(-(\frac{1}{2})t^2) dt \leq C_{15} v^{-1} e^{-(\frac{1}{2})v^2} \leq C_{15} \eta_1^{-1} k^{-\frac{1}{2}} e^{-(\frac{1}{2})\eta_1^2 k} \\ = C_{15} \eta_1^{-1} k^{-\frac{1}{2}} n^{-\frac{1}{2}}.$$

From the relations (5.7) to (5.14) we get

$$P(T_{\mu}^* > x\sigma_1 n^{\frac{1}{2}}) \leq P(T_{\mu}^{**} > (x - \eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{16} \eta_1.$$

This together with (5.6) gives the result.

From Lemmas 4.1, 5.2 and 5.3 we have with some constant C_{17}

$$(5.15) \quad \Phi_k((x - 3\eta_1)\sigma_1 n^{\frac{1}{2}}, \dots, (x - 3\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{17} \eta_1 \leq P(S_n^* \leq x\sigma_1 n^{\frac{1}{2}}) \\ \leq \Phi_k((x + \eta_1)\sigma_1 n^{\frac{1}{2}}, \dots, (x + \eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{17} \eta_1$$

for $n \geq \max(N_1, N_3 \text{ and } N_4)$.

If $\{x_n\}$ is a sequence of independent Bernoulli variables defined by $P(x_n = \pm 1) = \frac{1}{2}$ and $f(x) = x$ then it is well known that

$$(5.16) \quad |P(S_n^* \leq x\sigma_1 n^{\frac{1}{2}}) - I^*(x)| \leq C_{18} n^{-\frac{1}{2}} \quad \text{where}$$

$$(5.17) \quad I^*(x) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^x \exp(-(\frac{1}{2})t^2) dt.$$

We have $\sigma_1 = 1$ in this case. Applying the inequality (5.15) to the Bernoulli variables and using (5.16) we have

$$(5.18) \quad \Phi_k((x - 3\eta_1)\sigma_1 n^{\frac{1}{2}}, \dots, (x - 3\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{18} \eta_1 \\ \leq I^*(x) \leq \Phi_k((x + \eta_1)\sigma_1 n^{\frac{1}{2}}, \dots, (x + \eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{18} \eta_1.$$

Replacing x by $x + 4\eta$ and $x - 4\eta$ and using the fact that $|I^*(x) - I^*(x \pm 4\eta)| \leq C_{19}\eta_1$ we get from (5.15) and (5.18) the following

THEOREM 5.1. *There exist constants C_{20} and N_5 such that for $n \geq N_5$*

$$\sup_x |P(S_n^* \leq x\sigma_1 n^{\frac{1}{2}}) - I^*(x)| \leq C_{20}(\log n)^{\varepsilon_1(1+\delta/2)}n^{-\varepsilon_2/2}$$

where $\varepsilon_1 = 1/(3+\delta)$, $\varepsilon_2 = \varepsilon_1 \delta/4$ and $I^*(x)$ is defined at (5.17).

6. The law of the iterated logarithm.

THEOREM 6.1.

$$P\{\limsup \{(S_n)/(2\sigma_1^2 n \log \log n)^{\frac{1}{2}}\} = 1\} = 1.$$

PROOF. Write $\chi(n) = (2\sigma_1^2 n \log \log n)^{\frac{1}{2}}$.

From Theorem 3.1 we get for every b

$$|P(S_n \leq b\chi(n)) - \Phi(b(2 \log \log n)^{\frac{1}{2}})| \leq C_{21} \max(n^{-\delta/8}, n^{-1/12}).$$

Using the asymptotic relation for $1 - \Phi(x)$ we get from the above inequality

$$(6.1) \quad (\log n)^{-(1+\theta)b^2} < P(S_n > b\chi(n)) < (\log n)^{-b^2}$$

for any positive constants θ and b .

Corresponding to every $\tau < 1$ and integer k we can find an n_k such that $n_k \rightarrow \infty$ as $k \rightarrow \infty$ and $n_{k-1} < \tau^k \leq n_k$, $k = 1, 2, \dots$. We assume that $n_0 = 0$. Then

$$(6.2) \quad n_k \sim \tau^k \quad \text{and} \quad n_k - n_{k-1} \sim n_k(\tau - 1)/\tau.$$

We have from Theorem 5.1 for any $\xi > 0$

$$P(S_{n_k}^* > (1+\xi)\chi(n)) \leq 1 - I^*((1+\xi)(2 \log \log n_k)^{\frac{1}{2}}) + C_{20}(\log n_k)^{\varepsilon_1(1+\delta/2)}n_k^{-\varepsilon_2/2}.$$

For k large, say, $\geq K$, the right-hand side

$$\begin{aligned} &\leq C_{22}(2 \log \log n_k)^{-\frac{1}{2}}(\log n_k)^{-(1+\xi)^2} + C_{20}(\log n_k)^{\varepsilon_1(1+\delta/2)}n_k^{-\varepsilon_2/2} \\ &\leq C_{23} k^{-(1+\xi)^2} + C_{24} k^{\varepsilon_1(1+\delta/2)}\tau^{-k\varepsilon_2/2} \end{aligned}$$

so that

$$(6.3) \quad \sum_{k=K}^{\infty} P(S_{n_k}^* > (1+\xi)\chi(n_k)) < \infty.$$

Let ε be an arbitrary positive number. Consider

$$\begin{aligned} P(S_n > (1+\varepsilon)\chi(n) \text{ i.o.}) &\leq P\{\max_{n_{k-1} \leq n \leq n_k} S_n > (1+\varepsilon)\chi(n_{k-1}) \text{ i.o.}\} \\ &\leq P\{\max_{1 \leq n \leq n_k} S_n > (1+\varepsilon)\chi(n_{k-1}) \text{ i.o.}\} \end{aligned}$$

By (6.2) $\{\chi(n_k)\}/\{\chi(n_{k-1})\} \leq (2\tau - 1)^{\frac{1}{2}}$ for large k . Let τ be chosen such that $(1+\varepsilon)(2\tau - 1)^{-\frac{1}{2}} > 1 + \xi$. Then

$$(6.4) \quad P(S_n > (1+\varepsilon)\chi(n) \text{ i.o.}) \leq P(S_{n_k}^* > (1+\xi)\chi(n_k) \text{ i.o.}).$$

By the Borel–Cantelli lemma we get from (6.3) and (6.4)

$$(6.5) \quad P(S_n > (1 + \varepsilon)\chi(n) \text{ i.o.}) = 0$$

for any $\varepsilon > 0$.

Proof of the theorem will be complete if we show that $P(S_n > (1 - \varepsilon)\chi(n) \text{ i.o.}) = 1$ for any $\varepsilon > 0$.

Let us denote $\psi(n_k) = [2\sigma_1^2(n_k - n_{k-1}) \log \log(n_k - n_{k-1})]^{\frac{1}{2}}$. Set $m_k = [n_{k-1} + \tau^{2 \log k}]$. Consider for any positive $\xi < 1$

$$(6.6) \quad \begin{aligned} P(W_k) &= P(S_{n_k} - S_{m_k} > (1 - \xi)\psi(n_k)) \\ &\geq P(\{S_{n_k} > (1 - (\frac{1}{2})\xi)\psi(n_k)\} \cap \{S_{m_k} > (\frac{1}{2})\xi\psi(n_k)\}) \\ &\geq P(S_{n_k} > (1 - (\frac{1}{2})\xi)\psi(n_k)) - P(S_{m_k} > (\frac{1}{2})\xi\psi(n_k)). \end{aligned}$$

Now $\{\psi(n_k)\}/\{\chi(m_k)\} \rightarrow (\tau - 1)^{\frac{1}{2}}$ and $\{\psi(n_k)\}/\{\chi(n_k)\} \rightarrow ((\tau - 1)/\tau)^{\frac{1}{2}} < 1$. Using (6.1) we then have from (6.6) for any positive constant θ

$$\begin{aligned} P(W_k) &\geq (\log n_k)^{-(1+\theta)(1-(\frac{1}{2})\xi)^2} - (\log n_{k-1})^{-\xi^2(\tau-1)/5} \\ &\geq C_{2.5} \{k^{-(1+\theta)(1-(\frac{1}{2})\xi)^2} - k^{-\xi^2(\tau-1)/5}\} \\ &\geq (\frac{1}{2})C_{2.5} k^{-(1+\theta)(1-(\frac{1}{2})\xi)^2} \end{aligned}$$

for sufficiently large k and τ . The constant $C_{2.5}$ in the above inequality is independent of k . If we choose θ sufficiently small so that $(1 + \theta)(1 - (\frac{1}{2})\xi)^2 < 1$ we obtain

$$(6.7) \quad \sum_{k=K}^{\infty} P(W_k) = \infty.$$

By Corollary 2.1

$$|P(W_k | W_{k-1}, \dots, W_1) - P(W_k)| \leq 2\gamma\rho^{\tau^2 \log k}.$$

Since $\sum_{k=1}^{\infty} \rho^{\tau^2 \log k}$ converges, we get from (6.7) $\sum_{k=K}^{\infty} P(W_k | W_{k-1}, \dots, W_1) = \infty$. Then by Corollary 2 page 324 [3] we have

$$(6.8.) \quad P(W_k \text{ i.o.}) = 1$$

for any positive $\xi < 1$. Now as $k \rightarrow \infty$

$$(1 - \xi)\psi(n_k) - 2\chi(m_k) \sim \{(1 - \xi)(\tau - 1)^{\frac{1}{2}}\tau^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}}\}\chi(n_k).$$

If ε is an arbitrary fixed positive constant, we can choose positive numbers ξ and τ so that $(1 - \xi)(\tau - 1)^{\frac{1}{2}}\tau^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}} > 1 - \varepsilon$. Then

$$\begin{aligned} P(S_{n_k} > (1 - \varepsilon)\chi(n_k) \text{ i.o.}) \\ &\geq P(S_{n_k} > (1 - \xi)\psi(n_k) - 2\chi(m_k) \text{ i.o.}) \\ &\geq P(S_{n_k} - S_{m_k} > (1 - \xi)\psi(n_k) \text{ i.o.}) \end{aligned}$$

because from (6.5) $|S_n| \leq 2\chi(n)$ for $n \geq N_5(\omega)$ and all $\omega \in \Omega$ except for a set of probability measure zero. It now follows from (6.8) that

$$(6.9) \quad P(S_{n_k} > (1 - \varepsilon)\chi(n_k) \text{ i.o.}) = 1.$$

The assertion is an immediate consequence of (6.9).

NOTE. By standard arguments we relax the assumption (2.2) that the initial distribution is the stationary absolute probability distribution.

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