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# WEAK CONVERGENCE OF PROBABILITY MEASURES ON THE FUNCTION SPACE $C[0, \infty)$[1]

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**1. The space** $C[0, \infty)$. Let $C \equiv C[0, \infty)$ be the set of all continuous functions on $[0, \infty)$ with values in a complete separable metric space $(E, m)$. Stone (1961, 1963) has obtained simple criteria for weak convergence of sequences of probability measures on $\mathscr{C}$, the $\sigma$-field generated by the open subsets of $C$, when $C$ is endowed with the topology of uniform convergence on compacta, cf. [4] page 229. We shall obtain further properties of $(C, \mathscr{C})$ by defining a metric $\rho$ on $C$ which induces this same topology.

For any two functions $x$ and $y$ in $C$, let $\rho : C \times C \to R$ be defined as

$$\rho(x, y) = \sum_{j=1}^{\infty} 2^{-j} \rho_j(x, y)/[1 + \rho_j(x, y)],$$

where $\rho_j(x, y) = \sup_{0 \leqq t \leqq j} m[x(t), y(t)]$.

THEOREM 1. *The function space $(C, \rho)$ is a complete separable metric space in which $\lim_{n\to\infty} \rho(x_n, x) = 0$ if and only if $\lim_{n\to\infty} \rho_j(x_n, x) = 0$ for all $j \geqq 1$.*

COROLLARY 1. *The metric topology in $(C, \rho)$ is the topology of uniform convergence on compacta.*

Since the proofs of Theorem 1 and Corollary 1 are straightforward, we omit them.

Let $\mathscr{M}_p(C)$ be the set of all probability measures on $\mathscr{C}$. A net of probability measures $\{P_\alpha\}$ in $\mathscr{M}_p(C)$ is said to converge weakly to a probability measure $P$ in $\mathscr{M}_p(C)$ if

$$\lim_\alpha \int_C f \, dP_\alpha = \int_C f \, dP$$

for every bounded continuous real-valued function $f$ on $C$, and we write $P_\alpha \Rightarrow P$. Since $(C, \rho)$ is a complete separable metric space, cf. [5] II. 6,

COROLLARY 2. *The space $\mathscr{M}_p(C)$ with the topology of weak convergence is metrizable as a complete separable metric space.*

The metric defined by Prohorov (1956) is one such metric, cf. [1] page 237.

We now wish to characterize the $\sigma$-field $\mathscr{C}$. For each $t \geqq 0$, let $\pi_t : C \to E$ be the coordinate projection, defined for any $x \in C$ by $\pi_t(x) = x(t)$. Let $E$ be a measurable space with the $\sigma$-field generated by the open subsets and let $E^k$ be the $k$-fold product of $E$ with itself endowed with the product topology and the corresponding $\sigma$-field

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generated by the open subsets of $E^k$. Finally, let $\sigma(\pi_t)$ be the smallest $\sigma$-field of subsets of $C$ with respect to which all coordinate projections are measurable.

THEOREM 2. *The $\sigma$-fields $\mathscr{C}$ and $\sigma(\pi_t)$ coincide.*

PROOF. Follow the argument for $C[0, 1]$ of [5] page 212.

COROLLARY 3. *If $P_1$ and $P_2$ are two probability measures on $(C, \mathscr{C})$, then a necessary and sufficient condition for $P_1 = P_2$ is that $P_1 \pi_{t_1, \cdots, t_k}^{-1} = P_2 \pi_{t_1, \cdots, t_k}^{-1}$ for all $k$ and all $t_1, \cdots, t_k \in [0, \infty)$, where $P\pi_{t_1, \cdots, t_k}^{-1}$ is a measure on $E^k$ induced by $P$ through the map $\pi_{t_1, \cdots, t_k}: C \to E^k$, defined for any $x \in C$ by $\pi_{t_1, \cdots, t_k}(x) = [x(t_1), \cdots, x(t_k)]$.*

PROOF. Follow the argument for $C[0, 1]$ of [5] page 213.

COROLLARY 4. *The Wiener measure $W$ exists on $(C, \mathscr{C})$ with $E = R$.*

PROOF. Use the standard construction on $(C, \sigma(\pi_t))$ given in [3] page 12. It is also possible to use [1] pages 62, 96.

Recall that a subset $\Pi$ of $\mathscr{M}_p(C)$ is tight if for any positive $\varepsilon$ there exists a compact set $K \subset C$ such that $P(K) > 1 - \varepsilon$ for all $P \in \Pi$.

THEOREM 3. *If $P_n$ $(n = 1, 2, \cdots)$ and $P$ are probability measures on $(C, \mathscr{C})$, then $P_n \Rightarrow P$ if and only if:*

(i) *the finite-dimensional distributions of $P_n$ converge weakly to those of $P$, and*
(ii) *the sequence $\{P_n\}$ is tight.*

PROOF. The argument used for $C[0, 1]$ with $E = R$ applies, cf. [1] pages 35, 54, 241.

We now want to relate weak convergence of probability measures on $C[0, \infty)$ to weak convergence of associated probability measures on $C_j \equiv C[0, j]$. Define the metric $\rho_j$ on $C_j$ by setting $\rho_j(x, y) = \sup_{0 \leqq t \leqq j} m[x(t), y(t)]$ for any functions $x$ and $y$ in $C_j$. Let $\mathscr{C}_j$ be the $\sigma$-field generated by the open subsets of $C_j$. Let $r_j : C[0, \infty) \to C[0, j]$ be the simple projection or restriction to $[0, j]$; that is, for any $x \in C$, let $r_j(x)(t) = x(t)$, $0 \leqq t \leqq j$. Since $r_j$ is continuous and thus measurable, we can use $r_j$ to induce measures on $(C_j, \mathscr{C}_j)$. For each $j \geqq 1$ and any probability measure $P$ on $\mathscr{C}$, define $\mathrm{P}r_j^{-1}$ on $\mathscr{C}_j$ by setting $\mathrm{P}r_j^{-1}(A) = P(r_j^{-1}(A))$ for each $A \in \mathscr{C}_j$. By the continuous mapping theorem [1] Theorem 5.1, if $P_n \Rightarrow P$, then, for all $j \geqq 1$, $P_n r_j^{-1} \Rightarrow \mathrm{P}r_j^{-1}$. We want to establish an implication in the other direction. For this purpose, let $w_x^j : (0, j] \to [0, \infty)$ be the modulus of continuity of a function $x$ in $C_j$, defined by $w_x^j(\delta) = \sup_{0 \leqq s, t \leqq j, |s-t| < \delta} m[x(t), x(s)], 0 < \delta \leqq j$, cf. [1] page 54.

LEMMA 1. *A subset $A$ of $C$ has compact closure if and only if*

(i) *$\{x(t), x \in A\}$ has compact closure in $E$ for each $t \geqq 0$, and*
(ii) *for all $j \geqq 1$, $\lim_{\delta \to 0} \sup_{r_j(x) \in r_j(A)} w_x^j(\delta) = 0$.*

PROOF. This is just one version of the classical Arzelà-Ascoli Theorem, cf. [4] Theorem 7.18.

THEOREM 4. *Let $\{P_n\}$ be a sequence of probability measures on $C[0, \infty)$. The sequence $\{P_n\}$ is tight if and only if these two conditions hold:*

(i) *For each $t \geqq 0$ and each positive $\eta$, there exists a compact set $K$ in $E$ such that*

$$P_n\{x \in C : x(t) \in K_t\} > 1-\eta, \qquad n \geqq 1.$$

(ii) *For each $j \geqq 1$ and positive $\varepsilon$ and $\eta$, there exists a $\delta$, with $0 < \delta < 1$, and an integer $n_0$ such that*

$$P_n\{x \in C : w_x^{\ j}(\delta) \geqq \varepsilon\} \leqq \eta, \qquad n \geqq n_0.$$

PROOF. This proof will follow [1] Theorem 8.2. Suppose $\{P_n\}$ is tight. Given $j$, $\varepsilon$, and $\eta$, choose a compact set $K$ in $C[0, \infty)$ such that $P_n(K) > 1-\eta$ for all $n$. For each $t \geqq 0$, let the compact set $K_t$ in $E$ be $\pi_t(K)$. Since $\{x \in K\} \subseteq \{x : x(t) \in \pi_t(K)\}$, condition (i) holds. For small enough $\delta$, $K \subset \{x : w_x^{\ j}(\delta) \geqq \varepsilon\}$. Hence, condition (ii) holds. This proves the necessity of (i) and (ii).

Since each individual probability measure $P$ is tight, cf. [1] Theorem 1.4, it suffices to consider $n_0 = 1$ in (ii) when proving sufficiency. Assume that $\{P_n\}$ satisfies (i) and (ii) with $n_0 = 1$. Let the sequence $\{t_i\}$ be an enumeration of a countable dense subset of $[0, \infty)$. It is easy to show, in the presence of (ii), that $\{x(t), x \in A\}$ has compact closure in $E$ for all $t \geqq 0$ if and only if it has compact closure for all $t$ in $\{t_i\}$. Given $\eta > 0$, choose compact sets $K_{t_i}$ in $E$ so that, if $B_i = \{x \in C : x(t_i) \in K_{t_i}\}$ then $P_n(B_i) \geqq 1-\eta 2^{-(i+2)}$ for all $n \geqq 1$. Then choose $\delta_{k_j}$ so that, if $B_{k_j} = \{x \in C : w_x^{\ j}(\delta_{k_j}) < 1/k\}$, then $P_n(B_{k_j}) \geqq 1-\eta 2^{-(j+k+2)}$ for all $n$. If $K$ is the closure of $\bigcap_{i=1}^{\infty} B_i \cap \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} B_{k_j}$, then $P_n(K) \geqq 1-\eta$ for all $n$. By Lemma 1, $K$ is compact. Hence $\{P_n\}$ is tight.

COROLLARY 5. *The sequence $\{P_n\}$ is tight if and only if the sequence $\{P_n r_j^{-1}\}$ is tight for each $j \geqq 1$.*

PROOF. Conditions (i) and (ii) of Theorem 4 can be expressed in terms of $\{P_n r_j^{-1}\}$. Combining Theorem 3 and Corollary 5 gives

THEOREM 5. *If $P_n(n = 1, 2, \cdots)$ and $P$ are probability measures on $(C, \mathscr{C})$, then $P_n \Rightarrow P$ if and only if $P_n r_j^{-1} \Rightarrow \mathrm{P} r_j^{-1}$ for all $j \geqq 1$.*

Finally, we obtain the same conditions given by Stone (1963):

THEOREM 6. *If $P_n(n = 1, 2, \cdots)$ and $P$ are probability measures on $(C, \mathscr{C})$, then $P_n \Rightarrow P$ if and only if*

(i) *the finite-dimensional distributions of $P_n$ converge weakly to the finite-dimensional distributions of $P$ as $n \to \infty$; and*

(ii) *for every $\varepsilon > 0$ and $j \geqq 1$,*

$$\lim_{n\to\infty,\ \delta\to 0} P_n\{x \in C : w_x^{\ j}(\delta) > \varepsilon\} = 0.$$

PROOF. It is only necessary to observe that condition (i) above implies condition (i) of Theorem 4. Since the finite-dimensional distributions converge weakly, they are tight in $E^k$. Hence, for each $t$ and $\eta$, the appropriate compact set $K_t$ can be constructed.

**2. Product spaces.** We shall now change our notation slightly in order to treat product spaces. In particular, let $(C[0, \infty), E)$ represent $C[0, \infty)$ with range $E$ and let $(C[0, \infty), E)^k$ be the $k$-fold product of $(C[0, \infty), E)$ with itself. On all our product spaces $S_1 \times \cdots \times S_k = S$ generate the product topology with the metric $d$, defined for any $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k)$ in $S$ by $d(x, y) = \max_{1 \leqq i \leqq k} d_i(x_i, y_i)$, where $d_i$ is the metric on $S_i$. Let $\rho_1$ be the metric so generated on $(C[0, \infty), E)^k$ and let $\rho_2$ be the metric so generated on $(C[0, \infty), E^k)$.

THEOREM 7. *The sets* $(C[0, \infty), E)^k$ *and* $(C[0, \infty), E^k)$ *have the same elements and the metrics* $\rho_1$ *and* $\rho_2$ *are uniformly equivalent.*

PROOF. Let $A^T$ denote the set of all functions with domain $T$ and range $A$. Then $(A \times B)^T = A^T \times B^T$ is an elementary identity. Since $x \in (A \times B)^T$ is continuous if and only if the projections $\pi_A(x) \in A^T$ and $\pi_B(x) \in B^T$ are both continuous, cf. [4] page 91, $(C[0, \infty), E)^k = (C, [0, \infty), E^k)$ as sets of functions.

The uniform equivalence of $\rho_1$ and $\rho_2$ is a straightforward but tedious verification. Hence, we shall only exhibit the proof in one direction. Suppose $\rho_1(x, y) < \varepsilon$. Find the integer $J$ such that $2^{-(J+1)} < \varepsilon \leqq 2^{-J}$. Since $\rho_1(x, y) = \max_{1 \leqq i \leqq k} \rho(x_i, y_i)$, for all $i$,

$$\rho(x_i, y_i) = \textstyle\sum_{j=1}^{\infty} 2^{-j} \rho_j(x_i, y_i)/[1 + \rho_j(x_i, y_i)] < \varepsilon,$$

where $\rho_j(x_i, y_i) = \sup_{0 \leqq t \leqq j} m[x_i(t), y_i(t)]$. Since $\sum_{j=J+3}^{\infty} 2^{-j} < \varepsilon/2$,

$$\textstyle\sum_{j=1}^{J+2} 2^{-j} \rho_j(x_i, y_i)/[1 + \rho_j(x_i, y_i)] < \varepsilon/2$$

and $\rho_j(x_i, y_i)/[1 + \rho_j(x_i, y_i)] < 2^{j-1}\varepsilon$ for $j = 1, \cdots, J+2$. Since $2^{j-1}\varepsilon < 1$ for $j \leqq J$,

$$\rho_j(x_i, y_i) = \sup_{0 \leqq t \leqq j} m[x_i(t), y_i(t)] < 2^{j-1}\varepsilon/[1 - 2^{j-1}\varepsilon]$$

for all $i$ and $j = 1, \cdots, J$. Since

$$\rho_2(x, y) = \textstyle\sum_{j=1}^{\infty} 2^{-j} \rho_j(x, y)/[1 + \rho_j(x, y)],$$

where $\rho_j(x, y) = \sup_{0 \leqq t \leqq j} \{\max_{1 \leqq i \leqq k} m[x_i(t), y_i(t)]\} < 2^{j-1}\varepsilon/[1 - 2^{j-1}\varepsilon]$ for $j = 1, \cdots, J$,

$$\rho_2(x, y) < \sum_{j=1}^{J} 2^{-j} \frac{(2^{j-1}\varepsilon/[1 - 2^{j-1}\varepsilon])}{1 + (2^{j-1}\varepsilon/[1 - 2^{j-1}\varepsilon])} + \sum_{j=J+1}^{\infty} 2^{-j} < J\varepsilon/2 + 2\varepsilon.$$

Recall that $J$ is a function of $\varepsilon$ such that $J\varepsilon \to 0$ as $\varepsilon \to 0$.

Henceforth let $C^k \equiv C^k[0, \infty)$ with the metric $\rho$ represent both $(C[0, \infty), E)^k$ with $\rho_1$ and $(C[0, \infty), E^k)$ with $\rho_2$. Let $\rho_i^j : C \times C \to R$ be defined for any $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k)$ in $C^k$ by $\rho_i^j(x, y) = \sup_{0 \leqq t \leqq j} m[x_i(t), y_i(t)]$.

COROLLARY 6. *The product space* $(C^k, \rho)$ *is a complete separable metric space in which* $\lim_{n \to \infty} \rho(x_n, x) = 0$ *if and only if* $\lim_{n \to \infty} \rho_i^j(x_n, x) = 0$ *for each* $i (1 \leqq i \leqq k)$ *and* $j (j \geqq 1)$.

We now characterize the tightness of sets of probability measures on $C^k[0, \infty)$ in terms of the tightness of associated sets of probability measures on $C[0, j]$, $j \geqq 1$. Let $\pi_i : C^k[0, \infty) \to C[0, \infty)$ and $\pi_i : C^k[0, j] \to C[0, j]$ be defined for any $x = (x_1, \cdots, x_k)$ in $C^k[0, \infty)$ or $C^k[0, j]$ by $\pi_i(x) = x_i$.

THEOREM 8. *The sequence of probability measures* $\{P_n\}$ *on* $C^k[0, \infty)$ *is tight if and only if the sequences of probability measures* $\{P_n \pi_i^{-1} r_j^{-1}\} \equiv \{P_n r_j^{-1} \pi_i^{-1}\}$ *are tight for each* $i (1 \leqq i \leqq k)$ *and* $j$ $(j \geqq 1)$.

PROOF. A set of probability measures on a finite (or countable) product space (complete separable metric space) is tight if and only if each of the families of marginal measures is tight, cf. [1] page 41. It only remains to apply Corollary 5.

COROLLARY 7. *Let* $P_n (n \geqq 1)$ *and* $P$ *be probability measures on* $C^k[0, \infty)$. *Then* $P_n \Rightarrow P$ *if and only if*

(i) *the finite-dimensional distributions of* $P_n$ *converge weakly to the finite-dimensional distributions of* $P$, *and*

(ii) *the families of measures* $\{P_n r_j^{-1} \pi_i^{-1}\}$ *on* $C[0, j]$ *are tight for each* $i (1 \leqq i \leqq k)$ *and* $j (j \geqq 1)$.

PROOF. Apply Theorems 3 and 8.

**3. Conclusion.** It is now clear that many weak convergence theorems in $C[0, 1]$ can be extended to $C[0, \infty)$ or even $C^k[0, \infty)$ with very little extra work. For example, Donsker's theorem, [1] Theorem 10.1, holds in $C^k[0, \infty)$, cf. [2]. The space $C[0, \infty)$ is also more convenient than $C[0, 1]$ for treating first passage times. Let $T_a : C[0, \infty) \to R \cup \{+\infty\}$ be defined for each $x \in C$ by

$$T_a(x) = \inf\{t \geqq 0 : x(t) \geqq a\},$$

where the infimum of an empty set is $+\infty$. The function $T_a$ is not continuous on $C$, but it is measurable and continuous almost everywhere with respect to the Wiener measure, $W$. Therefore, we can apply the continuous mapping theorem, [1] Theorem 5.1, to obtain

THEOREM 9. *Let* $\{X_n\}$ *be any sequence of random functions in* $C[0, \infty)$. *If* $X_n \Rightarrow W$, *then* $T_a(X_n) \Rightarrow T_a(W)$, *where*

$$P\{T_a(W) \leqq t\} = (2/\pi t)^{\frac{1}{2}} \int_a^\infty e^{-y^2/2t} \, dy.$$

Limit theorems for more complicated stopping times can obviously be obtained in the same way. However, it is necessary to check that the stopping time actually constitutes a measurable function on $C[0, \infty)$ which is continuous almost everywhere with respect to the limiting measure.

Stone's (1963) major concern was not $C[0, \infty)$, but $D[0, \infty)$, the space of all right-continuous functions on $[0, \infty)$ with limits from the left and values in a complete separable metric space $(E, m)$. The analysis of $D[0, \infty)$ is more complicated because of the discontinuities in the functions, but a metric can be defined on $D[0, \infty)$ which makes it a complete separable metric space too. The weak convergence theory associated with $D[0, \infty)$ will be studied in a subsequent paper.

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