CORRECTIONS TO "ON THE PROBABILITY OF LARGE DEVIATIONS OF FAMILIES OF SAMPLE MEANS"

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Long ago, Professor H. Chernoff pointed out to me, correctly, that Lemma 3 in the above paper, Sethuraman (1964), required the condition:

There exists a function g such that $|f(x)| \leqq g(x)$ for all $f \in \mathscr{F}$ and $E(\exp(tg)) < \infty$ for all t.

I did not announce this in a correction note since the condition was somewhat implicit and was always satisfied whenever Lemma 3 was used, and was proved later in a more general form as Lemma 6 in Sethuraman (1965).

Recently Professor R. A. Wijsman pointed out that the assumptions of continuity, equicontinuity, etc. may be dropped in many places. Again, Professors R. A. Wijsman and J. K. Ghosh point out a more serious error in my proofs of Theorem 2 and Theorem 3. The error arose from the fact that certain compact sets K_ω were chosen and then functions $f_1, \cdots, f_m$ were chosen dependent on K_ω and hence on ω. This dependence of $f_1, \cdots, f_m$ and m on ω makes the proofs of Theorem 2 and Theorem 3 incorrect. A simple way to correct the proofs is to start with a suitable compact subset K, independent of ω, from the beginning.

In the following we first restate some of the lemmas and theorems in a more general way. In the original versions the Ascolli theorem was used to establish that a certain class of functions $\mathscr{F}$ is pre-compact in the u.c.c. topology. This is unnecessary now since we straightway assume that $\mathscr{F}$ is pre-compact under the u.c.c. topology. The original proofs of Theorem 2 and Theorem 3 namely pages 1312 through 1314 of Section 4 and displays (41) through (52) are to be replaced by the proofs that follow.

LEMMA 3. *Let $\mathscr{F}$ be a class of functions that is compact under the* u.c.c. *topology. Let there exist a function g such that $|f(x)| \leqq g(x)$ for all $f \in \mathscr{F}$ and $E\{\exp(tg)\} < \infty$ for all t. Then $\rho(\mathscr{F}, \varepsilon)$ is continuous from the left at each $\varepsilon > 0$.*

Compare with Lemma 6 of Sethuraman (1965).)

LEMMA 6. *If $\{f_n\}$ is a sequence of functions converging to a function f* a.e. μ, *then $\mu f_n^{-1} \Rightarrow \mu f^{-1}$ and, moreover if μf^{-1} is non-atomic then*

$$\sup_a |\mu(A(f_n, a)) - \mu(A(f, a))| \to 0 \tag{29}$$

where $A(f, a)$ is the set

$$\{x : f(x) \leqq a\}. \tag{30}$$

LEMMA 7. *Let $\mathscr{F}$ be a class of functions that is compact under the* u.c.c. *topology and μf^{-1} be non-atomic for each $f \in \mathscr{F}$. Then as $\delta \to 0$*

$$\sup_{f \in \mathscr{F}} \sup_{a \in R} |\mu(A(f+\delta, a)) - \mu(A(f, a))| \to 0 \tag{31}$$

where $f + \delta$ is the function $f(x) + \delta$ and $A(f, a)$ is as defined in (30).

THEOREM 2. *Let $\mathscr{F}$ be a class of functions from X to the real line which is precompact in the* u.c.c. *topology. Let $g(x)$ be a function such that $|f(x)| \leqq g(x)$ for each $f \in \mathscr{F}$ and such that $E(\exp(tg)) < \infty$ for all t. Then for $\varepsilon > 0$.*

$$(1/n)\log P\{\omega : \sup_{f\in\mathscr{F}} |\int f(x)\,\mu(n,\omega,dx) - \int f(x)\,\mu(dx)| \geqq \varepsilon\} \to \log \rho(\bar{\mathscr{F}}, \varepsilon)$$

where $\bar{\mathscr{F}}$ is the closure of $\mathscr{F}$ under u.c.c. *topology and where $\rho(\bar{\mathscr{F}}, \varepsilon)$ is as defined in* (15).

(In the original paper the log is missing from the r.h.s.)

THEOREM 3. *Let $\mathscr{F}$ be a class of functions from X into R^k that is compact under the* u.c.c. *topology. Let μf^{-1} be non-atomic for each $f \in \mathscr{F}$. For each $\mathbf{a} \in R^k$ and $f \in \mathscr{F}$ let $A(f, \mathbf{a}) = \{x : f_1(x) \leqq a_1, \cdots, f_k(x) \leqq a_k\}$. Then for $\varepsilon > 0$,*

$$(1/n)\log P\{\omega : \sup_{f\in\mathscr{F}} \sup_a |\mu(n,\omega,A(f,\mathbf{a})) - \mu(A(f,\mathbf{a}))| > \varepsilon\} \to \log \rho^*(\varepsilon)$$

where $0 \leqq \rho^*(\varepsilon) < 1$ and $\rho^*(\varepsilon)$ is as defined in (26).

PROOF OF THEOREM 2. Let $\bar{\mathscr{F}}$ be the closure of $\mathscr{F}$ under the u.c.c. topology. We note that

$$\sup_{f\in\mathscr{F}} |\int f(x)\mu(n,\omega,dx) - \int f(x)\mu(dx)| = \sup_{f\in\bar{\mathscr{F}}} |\int f(x)\mu(n,\omega,dx) - \int f(x)\mu(dx)|.$$

It would be thus no loss of generality to assume that $\mathscr{F}$ is closed and therefore compact. Define

$$\Omega(f,\varepsilon) = \{\omega : |\int f(x)\mu(n,\omega,dx) - \int f(x)\mu(dx)| > \varepsilon\}. \tag{41}$$

Since $\bigcup_{f\in\mathscr{F}} \Omega(f,\varepsilon) \supset \Omega(f,\varepsilon)$ for each $f \in \mathscr{F}$,

$$\liminf_n (1/n)\log P\{\bigcup_{f\in\mathscr{F}} \Omega(f,\varepsilon)\} \geqq \log \rho(\mathscr{F},\varepsilon). \tag{42}$$

We now have to prove the opposite inequality for the lim sup.

Let $\delta > 0$ be fixed and let $4\delta < \varepsilon$. There exists a compact set K_δ such that $\mu(K_\delta') < \delta$ and $\int_{K_\delta'} g(x)\mu(dx) < \delta$. Let K be a compact set containing K_δ, to be chosen as in (47). Let $g_K(x) = 0$ if $x \in K$ and $= g(x)$ if $x \in K'$. As K tends to the σ-compact support of μ, $g_K(x)$ tends to 0 almost everywhere μ. There exists a finite collection $\{f_1, \cdots, f_m\}$ in $\mathscr{F}$ such for any $f \in \mathscr{F}$ there is an index i such that $\sup_{x\in K} |f(x) - f_i(x)| < \delta$.

Let $f \in \mathscr{F}$ and i be the index as above. Since $|f(x)| \leqq g(x)$,

$$\begin{aligned} &|\int f(x)\mu(n,\omega,dx) - \int f(x)\mu(dx)| \\ &\leqq \int_{K'} g(x)\mu(n,\omega,dx) + \int_{K'} g(x)\mu(dx) + |\int_K f(x)\mu(n,\omega,dx) - \int_K f(x)\mu(dx)| \\ &\leqq 2\int_{K'} g(x)\mu(n,\omega,dx) + 2\int_{K'} g(x)\mu(dx) \\ &\qquad + |\int f_i(x)\mu(n,\omega,dx) - \int f_i(x)\mu(dx)| + 2\delta \\ &\leqq 2\int g_K(x)\mu(n,\omega,dx) + |\int f_i(x)\mu(n,\omega,dx) - \int f_i(x)\mu(dx)| + 4\delta. \end{aligned} \tag{43}$$

Thus

$$\begin{aligned} &\sup_{f\in\mathscr{F}} |\int f(x)\mu(n,\omega,dx) - \int f(x)\mu(dx)| \\ &\leqq 2\int g_K(x)\mu(n,\omega,dx) + \max_{1\leqq i\leqq m} |\int f_i(x)\mu(n,\omega,dx) - \int f(x)\mu(dx)| + 4\delta. \end{aligned} \tag{44}$$

Let $0 < 2\theta < \varepsilon - 4\delta$. Then for any $t > 0$,

$$P\{\omega : \int g_K(x)\mu(n, \omega, dx) > \theta\} \leqq [e^{-t\theta} \int \exp(tg_K(x))\mu(dx)]^n.$$

Thus

$$\limsup_n (1/n) \log P\{\omega : \int g_K(x)\mu(n, \omega, dx) > \theta\} \leqq \log[e^{-t\theta} \int \exp(tg_K(x))\mu(dx)]. \tag{45}$$

From (44),

$$\bigcup \Omega(f, \varepsilon) \subset \{\omega : \int g_K(x)\mu(n, \omega, dx) > \theta\} \cup \{\bigcup_1^m \Omega(f_k, \varepsilon - 4\delta - 2\theta)\}.$$

Hence

$$\begin{aligned} &\limsup_n (1/n) \log P\{\bigcup_{f \in \mathscr{F}} \Omega(f, \varepsilon)\} \\ &\quad \leqq \max\{\log[e^{-t\theta} \int \exp(tg_K(x))\mu(dx)], \max_{1 \leqq i \leqq m} \log \rho(f_i, \varepsilon - 4\delta - 2\theta)\} \\ &\quad \leqq \max\{\log[e^{-t\theta} \int \exp(tg_K(x))\mu(dx)], \log \rho(\mathscr{F}, \varepsilon - 4\delta - 2\theta)\}. \end{aligned} \tag{46}$$

The second part of the above r.h.s. does not depend on t and K. First choose t so that

$$-t\theta + \log 2 < \log \rho(\mathscr{F}, \varepsilon - 4\delta - 2\theta)$$

and then K so that $K \supset K_\delta$ and

$$\int \exp(tg_K(x))\mu(dx) \leqq 2, \quad \text{which is possible since } g_K \to 0 \quad \text{a.e.} \quad \mu. \tag{47}$$

Then the first term of the r.h.s. of (46) is smaller than the second term. Further $\rho(\mathscr{F}, \varepsilon)$ is left continuous from Lemma 3. Thus, we have

$$\limsup_n (1/n) \log P\{\bigcup_{f \in \mathscr{F}} \Omega(\mathscr{F}, \varepsilon)\} \leqq \log \rho(\mathscr{F}, \varepsilon). \tag{48}$$

(42) and (48) complete the proof of Theorem 2.

Proof of Theorem 3. Let $A(f, \mathbf{a}) = \{x : f_1(x) \leqq a_1, \cdots, f_k(x) \leqq a_k\}$.
Let $0 < \theta < \varepsilon$. From Lemma 7, there is a δ with $0 < \delta < \varepsilon - \theta$ such that

$$\sup_{f \in \mathscr{F}} \sup_a |\mu(A(f + \delta, \mathbf{a})) - \mu(A(f - \delta, \mathbf{a}))| < \theta. \tag{49}$$

Let K_δ be a compact set such that $\mu(K_\delta') < \delta$. Let K be a compact set containing K_δ. Let $I_K(x) = 0$ if $x \in K$, and $= 1$ if $x \in K'$. We will choose this compact set K according to (51). Since $\mathscr{F}$ is compact under the u.c.c. topology, there is a finite collection $\{f_1, \cdots, f_m\}$ in $\mathscr{F}$ such that for each $f \in \mathscr{F}$ there is an index i such that $\sup_{x \in K} |f(x) - f_i(x)| < \delta$. Let $f \in \mathscr{F}$ and i be the index as above.

$$\begin{aligned} &\mu(n, \omega, A(f, \mathbf{a})) - \mu(A(f, \mathbf{a})) \\ &\quad \leqq \mu(n, \omega, K') + \mu(n, \omega, A(f, \mathbf{a}) \cap K) - \mu(A(f, \mathbf{a}) \cap K) \\ &\quad \leqq \mu(n, \omega, K') + \mu(n, \omega, A(f_i - \delta, \mathbf{a}) \cap K) - \mu(A(f_i + \delta, \mathbf{a}) \cap K) \\ &\quad \leqq \mu(n, \omega, K') + \mu(K') + \mu(n, \omega, A(f_i - \delta, \mathbf{a})) - \mu(A(f_i + \delta, \mathbf{a})) \\ &\quad \leqq \mu(n, \omega, K') + \mu(K') + \theta + \mu(n, \omega, A(f_i - \delta, \mathbf{a})) - \mu(A(f_i - \delta, \mathbf{a})). \end{aligned}$$

Similarly,

$$\begin{aligned}\mu(n,\omega,A(f,\mathbf{a}))-\mu(A(f,\mathbf{a}))\\ \geqq \mu(n,\omega,A(f_i+\delta,\mathbf{a}))-\mu(A(f_i-\delta,\mathbf{a}))-\mu(n,\omega,K')-\mu(K')\\ \geqq \mu(n,\omega,A(f_i+\delta,\mathbf{a}))-\mu(A(f_i+\delta,\mathbf{a}))-\theta-\mu(n,\omega,K')-\mu(K').\end{aligned}$$

Combining the above two inequalities we have

$$\text{(50)}\quad \begin{aligned}&\sup_{f\in\mathscr{F}}\sup_a|\mu(n,\omega,A(f,\mathbf{a}))-\mu(A(f,\mathbf{a}))|\\ &\leqq \max_{1\leqq i\leqq m}\max_{1\leqq j\leqq 2}\sup_a|\mu(n,\omega,A(f_i+\phi_j\delta,\mathbf{a}))-\mu(A(f_i+\phi_j\delta,\mathbf{a}))|\\ &\qquad +\int I_K(x)\mu(n,\omega,dx)+\delta+\theta\end{aligned}$$

where $\phi_1=1, \phi_2=-1$. Let $0<\beta<\varepsilon-\delta-\theta$. Let $t>0$.

$$\limsup_n (1/n)\log P\{\omega:\int I_K(x)\mu(n,\omega,dx)\geqq\beta\}\leqq\log[e^{-t\beta}\int\exp(tI_K(x))\mu(dx)].$$

First choose t so that

$$-t\beta+\log 2<\log\rho^*(\varepsilon-\delta-\theta-\beta)$$

and the $K\supset K_\delta$ so that

$$\text{(51)}\qquad \int\exp(tI_K(x))\mu(dx)\leqq 2$$

which is possible since $I_K(x)$ tends to zero a'most everywhere as K tends to the σ-compact support of μ.

Since $f_i+\phi_j\delta$ has a non-atomic induced distribution,

$$(1/n)\log P\{\omega:\sup \mathbf{a}\,|\mu(n,\omega,A(f_i+\phi_j\delta,\mathbf{a}))-\mu(A(f_i+\phi_j\delta,\mathbf{a}))|>\varepsilon\}\to\log\rho^*(\varepsilon)$$

from Theorem 1. Using this in inequality (50)

$$\begin{aligned}&\limsup_n(1/n)\log P\{\omega:\sup_{f\in\mathscr{F}}\sup_a|\mu(n,\omega,A(f,\mathbf{a}))-\mu(A(f,\mathbf{a}))|>\varepsilon\}\\ &\qquad\leqq\max\{\log[e^{-t\beta}\int\exp(tI_K(x))\mu(dx)],\log\rho^*(\varepsilon-\theta-\delta-\beta)\}.\end{aligned}$$

Because of our choice of t and K, the first term of the above r.h.s. is smaller than the second. Further $\rho^*(\varepsilon)$ is left-continuous in ε from Lemma 5. Thus,

$$\text{(52)}\quad \begin{aligned}&\limsup_n(1/n)\log P\{\omega:\sup_{f\in\mathscr{F}}\sup_a|\mu(n,\omega,A(f,\mathbf{a}))-\mu(A(f,\mathbf{a}))|>\varepsilon\}\\ &\qquad\leqq\log\rho^*(\varepsilon).\end{aligned}$$

The opposite inequality for the lim inf follows from the fact that

$$\sup_{f\in\mathscr{F}}\sup_a|\mu(n,\omega,A(f,\mathbf{a}))-\mu(A(f,\mathbf{a}))|\geqq\sup_a|\mu(n,\omega,A(f,\mathbf{a}))-\mu(A(f,\mathbf{a}))|$$

for any $f\in\mathscr{F}$. This completes the proof of Theorem 3.

I would like to thank Professors H. Chernoff, R. A. Wijsman and J. K. Ghosh for pointing out the above errors and for their comments.

Recently Professor N. Glick asked me if it was true that

$$\sqrt{n}\,\sup_{f\in\mathscr{F}}\left|\int f(x)\mu(n,\omega,dx)-\int f(x)\mu(dx)\right| = O_p(1)$$

under the conditions of Theorem 2. A similar question can be asked in relation to Theorems 1, 3, 4, 5, 6, and 7. The answer is yes for the question related to Theorem 1 (since it is related to the Kolmogorov–Smirnov limit distribution) and Theorem 3. The answer is not known in the other cases.

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