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A NOTE ON A SEQUENTIAL PROCEDURE FOR COMPARING TWO KOOPMAN–DARMOIS POPULATIONS[1]

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1. Introduction. Let π_1 and π_2 be two populations with associated independent random variables X_1 and X_2, and suppose the probability distribution of X_i for $i = 1, 2$ belongs to the Koopman–Darmois family with frequency function

$$f(x, \theta_i) = \exp[P(x)Q(\theta_i) + R^*(x) + S(\theta_i)]$$

where $f(x, \theta_i)$ is a probability density in the continuous case and a probability in the discrete case.

We will let $\delta = Q(\theta_1) - Q(\theta_2)$ represent the "difference" between populations π_1 and π_2. In [1], Girshick gave a sequential procedure for testing the hypothesis H_0 that $\delta \leqq -\Delta$ against alternatives $\delta \geqq \Delta$ so that $P[\text{rejecting } H_0 \mid \delta \leqq -\Delta] < \alpha$ and $P[\text{accepting } H_0 \mid \delta \geqq \Delta] < \beta$. This procedure of Girshick apparently cannot be extended to deal with the situation when it is desired to test the hypothesis that $\delta \leqq \delta_1$ against alternatives $\delta \geqq \delta_2$ for arbitrary δ_1 and δ_2 with $\delta_2 > \delta_1$, and it is the purpose of this note to supply such a sequential procedure using a different approach to the problem.

2. Derivation of the sequential procedure. The rth pair of measurements from populations π_1 and π_2 is denoted by (X_{1r}, X_{2r}). It is assumed that all measurements are independent and for each r, $r = 1, 2, 3, \cdots,$ the pair (X_{1r}, X_{2r}) has the same probability distribution as the pair of variables (X_1, X_2) specified in the introduction. Let $Y_{ir} = P(X_{ir})$ for $i = 1, 2$, and $V_r = Y_{1r} + Y_{2r}$. We will assume that $P(x)$, the coefficient of $Q(\theta_i)$ in the specified frequency function is subject to some mild restriction, say $P(x) = x^m$, so that Y_{ir} has a frequency function of the form

$$f(x, \theta_i) = \exp[xQ(\theta_i) + R(x) + S(\theta_i)],$$

where $\exp[R(x)] = 0$ whenever $f(x, \theta) = 0$. This weak condition on $P(x)$ is certainly satisfied in most cases of practical interest, including the normal, binomial, Poisson and exponential distributions.

Now Y_{1r} and V_r have a joint frequency function of the form

$$f(y, v, \theta_1, \theta_2) = \exp[y\{Q(\theta_1) - Q(\theta_2)\} + R(y) + R(v-y) + S(\theta_1) + S(\theta_2) + vQ(\theta_2)].$$

For convenience we will assume that we are dealing with the continuous case. The conditional probability density of Y_{1r} given that $V_r = Y_{1r} + Y_{2r} = v$ is now given by

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$$f(y, \delta \mid v) = \frac{\exp[y\delta + R(y) + R(v-y)]}{\int \exp[y\delta + R(y) + R(v-y)]\, dy} \tag{1}$$

where $\delta = Q(\theta_1) - Q(\theta_2)$, and the set of all values of δ for which the frequency function exists is an interval including $\delta = 0$, δ_1, and δ_2 as interior points.

Since the conditional probability density in (1) as a function of θ_1 and θ_2 depends only on δ, a Wald sequential probability ratio test could be constructed directly to test the simple hypothesis $\delta = \delta_1$ against the simple alternative $\delta = \delta_2$ with specified probabilities of type I and type II errors α and β. However, it seems difficult to prove the resulting procedure has the desired bounds on the two errors for testing the composite hypothesis $\delta \leqq \delta_1$ against the composite alternative $\delta \geqq \delta_2$, and a different approach will be used. Let

$$M(t, \delta \mid v) = \frac{\int \exp[(t+\delta)y + R(y) + R(v-y)]\, dy}{\int \exp[\delta y + R(y) + R(v-y)]\, dy}$$

denote the conditional moment generating function of Y_{1r} when $V_r = v$. Let $\tilde{V} = (V_1, V_2, \cdots)$, let $\lambda_0 = \delta_2 - \delta_1 > 0$, and let $g(x, t, \delta \mid v) = \exp(tx) f(x, \delta \mid v) / M(t, \delta \mid v)$. We now want to show for all $\delta \leqq \delta_1$ that

$$P[\exp(-\lambda_0 \textstyle\sum_{r=1}^n Y_{1r}) \prod_{r=1}^n M(\lambda_0, \delta_1 \mid V_r) < \alpha \text{ for at least one } n,\ n \geqq 1] < \alpha. \tag{2}$$

Now (2) is clearly equivalent to

$$P\left[\frac{\prod_{r=1}^n f(Y_{1r}, \delta \mid V_r)}{\prod_{r=1}^n g(Y_{1r}, \lambda_0, \delta \mid V_r)} < \alpha \frac{\prod_{r=1}^n M(\lambda_0, \delta \mid V_r)}{\prod_{r=1}^n M(\lambda_0, \delta_1 \mid V_r)} \text{ for at least one } n,\ n \geqq 1\right] < \alpha. \tag{3}$$

It follows from a well-known lemma of Wald (page 146 of [2]) that to prove (3) holds for all $\delta \leqq \delta_1$, it suffices to show that $M(\lambda_0, \delta \mid v)$ is a monotonically increasing function of δ for any v. If we regard $M(\lambda_0, \delta \mid v)$ as a quotient of two functions of δ and let N denote the numerator of the derivative of $M(\lambda_0, \delta \mid v)$ with respect to δ, we get

$$N = \int \exp[\delta y + R(y) + R(v-y)\, dy \int y \exp[(\lambda_0 + \delta)y + R(y) + R(v-y)]\, dy$$
$$- \int \exp[(\lambda_0 + \delta)y + R(y) + R(v-y)]\, dy \int y \exp[\delta y + R(y) + R(v-y)]\, dy.$$

This can be written as

$$N = \int\int (y_2 - y_1) \exp[\delta y_1 + (\lambda_0 + \delta) y_2$$
$$+ R(y_1) + R(y_2) + R(v - y_1) + R(v - y_2)]\, dy_1 dy_2.$$

If we let C denote the region in the (y_1, y_2) plane where the integrand is not zero, this is clearly symmetric about $y_1 = y_2$. Let $C_1 = C \cap \{(y_1, y_2) : y_2 > y_1\}$ and let $C_2 = C \cap \{(y_1, y_2) : y_2 < y_1\}$. Then the double integral is equal to the sum of the integrals over C_1 and C_2, and in C_2 we can interchange the variables, which transforms C_2 into C_1, and we get

$$N = \int\int_{C_1} (y_2 - y_1)\{\exp(\lambda_0 y_2) - \exp(\lambda_0 y_1)\} \exp[H]\, dy_1\, dy_2$$

where $H = \delta(y_1+y_2)+R(y_1)+R(y_2)+R(v-y_1)+R(v-y_2)$. Since $\lambda_0 = \delta_2-\delta_1 > 0$ and $y_2 > y_1$ in C_1, we have $N > 0$, which shows that $M(\lambda_0, \delta \mid v)$ is a monotonically increasing function of δ, and the proof that (2) holds for all $\delta \leqq \delta_1$ is complete. In a similar manner it can be shown that for all $\delta \geqq \delta_2$

(4) $P[\exp(\lambda_0 \sum_{r=1}^n Y_{1r})\prod_{r=1}^n M(-\lambda_0, \delta_2 \mid V_r) < \beta$ for at least one n, $n \geqq 1] < \beta$.

Let

$$Z_r = \left\{\lambda_0 Y_{1r}+\log\left[\frac{\int \exp(\delta_1 y+R(y)+R(V_r-y))\,dy}{\int \exp(\delta_2 y+R(y)+R(V_r-y))\,dy}\right]\right\}.$$

Noting that $M(\lambda_0, \delta_1 \mid V_r) = [M(-\lambda_0, \delta_2 \mid V_r)]^{-1}$, we can combine (2) and (4) to get a sequential procedure for testing H_0 that $\delta \leqq \delta_1$ against alternatives $\delta \geqq \delta_2$ so that P[rejecting $H_0 \mid \delta \leqq \delta_1] < \alpha$ and P[accepting $H_0 \mid \delta \geqq \delta_2] < \beta$. This sequential procedure is as follows: we continue taking pairs of measurements (X_{1r}, X_{2r}) as long as $-\log(1/\beta) \leqq \sum_{r=1}^n Z_r \leqq \log(1/\alpha)$, and stop the experiment and reject H_0 as soon as $\sum_{r=1}^n Z_r > \log(1/\alpha)$, and stop and accept H_0 as soon as $\sum_{r=1}^n Z_r < -\log(1/\beta)$. In the discrete case, each integration is replaced by a summation.

3. Two special cases. For the exponential distribution with probability density $f(x, \theta) = \theta\exp(-\theta x) = \exp(-\theta x+\log\theta)$ for $x > 0$, we have $P(x) = x$, $Q(\theta) = -\theta$, and $\delta = \theta_2-\theta_1$. The conditional probability density of X_{1r} given $X_{1r}+X_{2r} = v$ is

$$f(x, \delta \mid v) = \frac{\delta\exp(x\delta)}{\exp(\delta v)-1} \quad \text{for} \quad 0 < x < v.$$

For Z_r we obtain

$$Z_r = (\delta_2-\delta_1)X_{1r}+\log\left[\left(\frac{\delta_2}{\delta_1}\right)\frac{1-\exp\{\delta_1(X_{1r}+X_{2r})\}}{1-\exp\{\delta_2(X_{1r}+X_{2r})\}}\right].$$

To test the hypothesis that $\theta_2-\theta_1 \leqq \delta_1$ against alternatives $\theta_2-\theta_1 \geqq \delta_2$, we continue sampling until either $\sum_{r=1}^n Z_r > \log(1/\alpha)$, in which case we reject the hypothesis, or $\sum_{r=1}^n Z_r < -\log(1/\beta)$, in which case we accept the hypothesis.

For the Poisson distribution, the probability $f(x, \theta) = \exp[x\log\theta-\theta-\log(x!)]$ for $x = 0, 1, 2, \cdots$, so $P(x) = x$, $Q(\theta) = \log\theta$, $\delta = \log(\theta_1/\theta_2)$. The conditional probability that $X_{1r} = x$ given $X_{1r}+X_{2r} = v$ is

$$f(x, \delta \mid v) = \frac{\binom{v}{x}\exp(\delta x)}{(1+\exp\delta)^v} \quad \text{for} \quad x = 0, 1, 2, \cdots, v.$$

Now

$$Z_r = (\delta_2-\delta_1)X_{1r}+(X_{1r}+X_{2r})\log\left[\frac{1+\exp(\delta_1)}{1+\exp(\delta_2)}\right],$$

and to test the hypothesis that $\log(\theta_1/\theta_2) \leqq \delta_1$ against $\log(\theta_1/\theta_2) \geqq \delta_2$ we again take one pair of measurements at each stage until $\sum_{r=1}^{n} Z_r > \log(1/\alpha)$ when we reject the hypothesis, or until $\sum_{r=1}^{n} Z_r < -\log(1/\beta)$, in which case we accept the hypothesis.

REFERENCES

[1] GIRSHICK, M. A. (1946). Contributions to sequential analysis: I. *Ann. Math. Statist.* **17** 123–143.
[2] WALD, A. (1947). *Sequential Analysis.* John Wiley, New York.