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NOTES

MULTIPLICATION OF POLYKAYS USING ORDERED PARTITIONS

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1. Introduction and summary. The multiplication of k-statistics is treated by Fisher [4], Tukey [8], [9], Wishart [10], Kendall [5], Abdel-Aty [1], Dwyer and Tracy [3], and Tracy [7]. Various rules, procedures, functions and tables have been devised to aid in this task. The rules and procedures derive from the combinatorial properties of the k-statistics and other symmetric functions linearly related to them.

When dealing with generalized polykays it was found convenient to ignore commutativity of multiplication in denoting the various symmetric functions and use ordered partitions to represent them rather than the usual partitions [2]. The simplicity of the relationships among the symmetric functions, which results from the use of ordered partitions, may be used to obtain simple multiplication procedures.

2. Ordered Partitions. Following [2] let (α_t), $\langle\alpha_t\rangle$, $|\alpha_t|$, $[\alpha_t]$, denote, respectively, the polykay, symmetric mean, augmented symmetric function, and unrestricted (power) sum of degree d corresponding to some tth ordered partition of weight d, say $\alpha_t = \alpha_{t1}\,\alpha_{t2}\cdots\alpha_{td}$. The notation can be interpreted as follows:

$$|\alpha_t| = \textstyle\sum_{\neq} X_i\alpha_{t1}\, X_i\alpha_{t2}\cdots X_i\alpha_{td}$$

$$[\alpha_t] = \textstyle\sum_1^n \sum_1^n \cdots \sum_1^n X_i\alpha_{t1}\, X_i\alpha_{t2}\cdots X_i\alpha_{td}$$

$$\langle\alpha_t\rangle = |\alpha_t|/(n)_{\phi(\alpha_t)}$$

where $\phi(\alpha_t)$ is the number of parts of α_t (i.e. the number of distinct symbols among $\alpha_{t1}, \alpha_{t2}, \cdots \alpha_{td}$), and $(n)_r$ is the falling factorial $n(n-1)\cdots(n-r+1)$. The polykays, (α_t), are certain linear functions of the symmetric means.

The set of ordered partitions is partially ordered by subpartitioning, α_t being considered a subpartition of α_u if $\alpha_{ti} = \alpha_{tj}$ implies $\alpha_{ui} = \alpha_{uj}$ for every pair i, j. The set of ordered partitions of a given weight forms a lattice with the greatest lower bound, $\alpha_t \wedge \alpha_u$, the ordered partition formed by considering the equality-inequality of pairs α_{ti}, α_{ui}; $i = 1, 2, \cdots, d$. The least upper bound, $\alpha_t \vee \alpha_u$, of two ordered partitions is the ordered partition which is formed by making any positions equal which are equal in either α_t or α_u or which are linked by positions which are equal in either [2].

Let the ordered partitions of a given degree be put in a fixed order $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that no ordered partition precedes one of which it is a subpartition. Let $(\boldsymbol{\alpha})$, $\langle\boldsymbol{\alpha}\rangle$, $|\boldsymbol{\alpha}|$, and $[\boldsymbol{\alpha}]$ denote the corresponding vectors of polykays, symmetric

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means, augmented symmetric functions and unrestricted sums. Define a matrix Λ with elements

$$\lambda_{ij} = 1 \quad \text{if} \quad \alpha_i \geqq \alpha_j;$$
$$\lambda_{ij} = 0 \quad \text{otherwise.}$$

Define the diagonal matrix N with elements $n_{ii} = (n)_{\phi(\alpha_i)}$. It is shown in [2] that $\langle\boldsymbol{\alpha}\rangle = \Lambda(\boldsymbol{\alpha})$ and $[\boldsymbol{\alpha}] = \Lambda'|\boldsymbol{\alpha}|$, from which $(\boldsymbol{\alpha}) = (\Lambda' N \Lambda)^{-1}[\boldsymbol{\alpha}]$. The last relationship is used in computing (generalized) polykays of degree four.

3. Multiplication of Polykays. The multiplication procedures proposed here are based upon the particularly simple form of the matrix: $A = \Lambda' N \Lambda$. Two lemmas are helpful in determining this form. Let $C(P)$ denote the combinatorial coefficient [3] associated with the partition P. $C(P)$ is the number of ordered partitions which form the partition P and if $P = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ then $C(P) = (\sum p_i t_i)!/\prod_i (p_i!)^{t_i} t_i!$.

LEMMA 1. *Let $d < n$ be positive integers. Then $n^d = \sum C(P_i)(n)_{\phi(P_i)}$ where the sum is over all partitions P_i of d.*

The lemma is similar to several identities in the combinatorial theory of distributions and occupancy [6] and probably a combinatorial proof can be made. However, the lemma is a special case of the last equation in the relationship $[\boldsymbol{\alpha}] = \Lambda'|\boldsymbol{\alpha}|$, the left-hand side of which is $(\sum X)^d$ while the right-hand side is the sum of the augmented symmetric functions corresponding to all ordered partitions of weight d. The desired result is obtained by letting each $X_i = 1$, so that $\sum X = n$. Since the number of terms in the symmetric function $|\alpha_t|$ is $(n)_{\phi(\alpha_t)}$ and since there are $C(P_i)$ ordered partitions α_t corresponding to each partition of d, all having the same number of parts, the result follows.

The second lemma could be obtained by considering the remaining equations in the same relationship.

LEMMA 2. *Let α_t be a given ordered partition and let S be the set of all ordered partitions α_i such that α_t is a subpartition of α_i. $n^{\phi(\alpha_t)} = \sum_S (n)_{\phi(\alpha_i)}$ where the summation extends over all ordered partitions α_i which are elements of S.*

The result can be obtained from Lemma 1 by noting that the set S forms a lattice which is isomorphic to the lattice of ordered partitions of weight $\phi(\alpha_t)$.

THEOREM 1. *Let Λ and N be as defined above. Then with $A = \Lambda' N \Lambda$, the elements of A are $a_{ij} = n^{\phi(\alpha_i \vee \alpha_j)}$.*

PROOF. $a_{ij} = \sum_r \sum_s \lambda_{ri} \lambda_{sj} n_{rs} = \sum_r \lambda_{ri} \lambda_{rj} n_{rr}$. Since $\lambda_{ri} \lambda_{rj} = 1$ if α_i is a subpartition of α_r, and α_j is a subpartition of α_r, and $\lambda_{ri} \lambda_{rj} = 0$ otherwise, it follows that a_{ij} is exactly the sum over the set S of Lemma 2 with $\alpha_t = \alpha_i \vee \alpha_j$.

Let α_i and β_j be ordered partitions of weight r and s, respectively, and define $\sigma_k = \alpha_i \oplus \beta_j$ as follows. The ordered partitions α_i and β_j can be represented by the digits $0, 1, 2, \cdots, \phi(\alpha_i) - 1$ and $0, 1, 2, \cdots, \phi(\beta_j) - 1$. With this representation let $\sigma_{ku} = \alpha_{iu}$, $u = 1, 2, \cdots, r$ and $\sigma_{ku} = \phi(\alpha_i) + \beta_{j,u-r}$, $u = r+1, r+2, \cdots, r+s$. For unrestricted sums, we have $[\sigma_k] = [\alpha_i \oplus \beta_j] = [\alpha_i][\beta_j]$. Let $[\boldsymbol{\sigma}]^*$ denote the vector of unrestricted sums of weight $r+s$ of products $[\alpha_i][\beta_j]$. Using $\otimes$ to denote the direct product of matrices we have $[\boldsymbol{\sigma}]^* = [\boldsymbol{\alpha}] \otimes [\boldsymbol{\beta}]$ where $[\boldsymbol{\alpha}]$ and $[\boldsymbol{\beta}]$ are vectors of unrestricted sums of weight r and s. Let A_r, A_s, and A_{r+s} denote the matrix A of Theorem 1 for ordered partitions of weight r, s, and $r+s$, respectively. Let A_{r+s}^* denote those rows of A_{r+s} corresponding to ordered partitions of weight $r+s$ which are of the form $\sigma_k = \alpha_i \oplus \beta_j$, so that $[\boldsymbol{\sigma}]^* = A_{r+s}^*(\boldsymbol{\sigma})$.

THEOREM 2.

$$(\boldsymbol{\alpha}) \otimes (\boldsymbol{\beta}) = (A_r^{-1} \otimes A_s^{-1}) A_{r+s}^*(\boldsymbol{\sigma}).$$

PROOF.

$$\begin{aligned}(\boldsymbol{\alpha}) \otimes (\boldsymbol{\beta}) &= (A_r^{-1}[\boldsymbol{\alpha}]) \otimes (A_s^{-1}[\boldsymbol{\beta}]) \\ &= (A_r^{-1} \otimes A_s^{-1})([\boldsymbol{\alpha}] \otimes [\boldsymbol{\beta}]) \\ &= (A_r^{-1} \otimes A_s^{-1})[\boldsymbol{\sigma}]^* = (A_r^{-1} \otimes A_s^{-1}) A_{r+s}^*(\boldsymbol{\sigma}).\end{aligned}$$

4. Example. For a simple example, let $r=s=2$ giving

$$(\boldsymbol{\alpha}) = (\boldsymbol{\beta}) = \begin{pmatrix}(00)\\(01)\end{pmatrix}; \qquad A_r = A_s = A_2 = \begin{pmatrix} n & n \\ n & n^2 \end{pmatrix},$$

since $00 \vee 00 = 00$, $00 \vee 01 = 00$, $01 \vee 00 = 00$ and $01 \vee 01 = 01$; while $\phi(00) = 1$, $\phi(01) = 2$.

$$A_2^{-1} = \frac{1}{n^2(n-1)} \begin{bmatrix} n^2 & -n \\ -n & n \end{bmatrix}$$

$$A_2^{-1} \otimes A_2^{-1} = \frac{1}{n^4(n-1)^2} \begin{bmatrix} n^4 & -n^3 & -n^3 & n^2 \\ -n^3 & n^3 & n^2 & -n^2 \\ -n^3 & n^2 & n^3 & -n^2 \\ n^2 & -n^2 & -n^2 & n^2 \end{bmatrix}$$

$$A_4^* = \left[\begin{array}{c:cccc:ccc:cccccc:c} n & n & n & n & n & n^2 & n & n & n^2 & n & n & n & n & n^2 & n^2 \\ n & n^2 & n^2 & n & n & n^2 & n & n & n^3 & n^2 & n^2 & n^2 & n^2 & n^2 & n^3 \\ n & n & n & n^2 & n^2 & n^2 & n & n & n^2 & n^2 & n^2 & n^2 & n^2 & n^3 & n^3 \\ n & n^2 & n^2 & n^2 & n^2 & n^2 & n^2 & n^2 & n^3 & n^3 & n^3 & n^3 & n^3 & n^3 & n^4 \end{array}\right].$$

The redundancy resulting from use of the ordered partitions instead of partitions may be removed by deleting rows of $A_r^{-1} \otimes A_s^{-1}$ corresponding to redundant products, and adding together those columns of A_{r+s}^* which correspond to the same partition. In the present example, we delete row 3 of $A_2^{-1} \otimes A_2^{-1}$, since $k_2 k_{11} = k_{11} k_2$, and add together the columns of A_4^* indicated by the dotted lines.

The result is

$$\begin{pmatrix} k_2{}^2 \\ k_2 k_{11} \\ k_{11}^2 \end{pmatrix} = \frac{1}{n^4(n-1)^2} \begin{pmatrix} n^4 & -n^3 & -n^3 & n^2 \\ -n^3 & n^3 & n^2 & -n^2 \\ n^2 & -n^2 & -n^2 & n^2 \end{pmatrix}$$

$$\times \begin{bmatrix} n & 4n & n^2+2n & 2n^2+4n & n^2 \\ n & 2n^2+2n & n^2+2n & n^3+5n^2 & n^3 \\ n & 2n^2+2n & n^2+2n & n^3+5n^2 & n^3 \\ n^2 & 4n^2 & 3n^2 & 6n^2 & n^4 \end{bmatrix} \begin{bmatrix} k_4 \\ k_{31} \\ k_{22} \\ k_{211} \\ k_{1111} \end{bmatrix}$$

which, when multiplied out, gives the usual result for products of 2*d* degree polykays.

5. Computations. The greatest appeal of the technique outlined here is that the elements of the A matrices have the simple form of Theorem 1 and hence they can be represented functionally rather than as storage arrays in the computer.

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