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AN ASYMPTOTIC EXPANSION FOR THE NONCENTRAL WISHART DISTRIBUTION

BY GEORGE A. ANDERSON

Trinity College

0. Summary. The noncentral Wishart density depends on a definite integral over the group of orthogonal matrices. This integral defines a function of the latent roots of a matrix involving the parent normal vectors and their means and covariances. An expansion for the integral in increasing powers of the reciprocals of these roots is developed using two distinct methods—an integration procedure and a substitution into a set of differential equations.

1. Introduction. Suppose the columns of an $m \times n$ matrix $\mathbf{X}$ are independently normally distributed with common covariance matrix $\mathbf{\Sigma}$ and suppose $E(\mathbf{X}) = \mathbf{M}$. Then from James [5] the $m \times m$ matrix $\mathbf{XX}'$ has the noncentral Wishart distribution

$$\exp((-\tfrac{1}{2})\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{MM}')[\textstyle\int_{O(n)}\exp(\operatorname{tr}\mathbf{H}'\mathbf{M}'\mathbf{\Sigma}^{-1}\mathbf{X})\,dV(\mathbf{H})]\,dF(\mathbf{XX}', n)$$

where $dF(\mathbf{XX}', n)$ is the (central) Wishart probability element on n degrees of freedom and $dV(\mathbf{H})$ is the normalized invariant measure on the group $O(n)$ of $n \times n$ orthogonal matrices. From James [4] we have

$$\begin{aligned} dV(\mathbf{H}) &= (1/V(n)) \quad (\mathbf{H}'\,d\mathbf{H}) \\ &= (1/V(n)) \quad \textstyle\prod_{i<j}^{n} h_i'\,dh_j \end{aligned}$$

where h_i and dh_j are the ith and jth columns of $\mathbf{H}$ and $d\mathbf{H}$ respectively, and

$$\begin{aligned} V(n) &= \textstyle\int_{O(n)}(\mathbf{H}'\,d\mathbf{H}) \\ &= 2^n\pi^{n(n+1)/4}/\textstyle\prod_{i=1}^{n}\Gamma(i/2) \end{aligned}$$

is the "volume" of the group $O(n)$. To simplify the integration procedure of the next section we define $\psi(\mathbf{A}) = \int_{O(n)}\exp(\operatorname{tr}\mathbf{H}'\mathbf{A})(\mathbf{H}'\,d\mathbf{H})$ so that the noncentral Wishart distribution can be written $(1/V(n))\exp((-\tfrac{1}{2})\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{MM}')\psi(\mathbf{M}'\mathbf{\Sigma}^{-1}\mathbf{X})\,dF(\mathbf{XX}', n)$.

From James [8] $\psi(\mathbf{A})$ can be expressed as $V(n)$ times the hypergeometric function ${}_0F_1(n/2, (\tfrac{1}{4})\mathbf{A}'\mathbf{A})$ which is defined as a series of symmetric polynomials in the latent roots of $(\tfrac{1}{4})\mathbf{A}'\mathbf{A}$. These polynomials are called zonal polynomials and a complete discussion of them is given in James [7]. Since this series converges slowly unless the latent roots are small some other type of expansion appears necessary if the roots are large. The main result of this paper is to develop such an expansion first for $\mathbf{A}$ non-singular and then for the more general case of $\mathbf{A}$ singular.

Since $\mathbf{A}'\mathbf{A}$ is symmetric there is an orthogonal matrix $\mathbf{H}_2$ such that $\mathbf{H}_2'\mathbf{A}'\mathbf{A}\mathbf{H}_2 = \operatorname{diag}\{a_i^2\}$ where $a_1^2 \geqq a_2^2 \geqq \cdots \geqq a_n^2 \geqq 0$. With $\mathbf{H}_1 = \operatorname{diag}\{1/(a_i^2)^{\frac{1}{2}}\}\mathbf{H}_2'\mathbf{A}'$ we have $\mathbf{H}_1\mathbf{A}\mathbf{H}_2 = \operatorname{diag}\{1/(a_i^2)^{\frac{1}{2}}\}$. (The matrix $\mathbf{H}_1$ can be adjusted if the rank of $\mathbf{A}$ is less than n so that $\mathbf{H}_1\mathbf{A}\mathbf{H}_2$ comes to $\operatorname{diag}\{(a_1^2)^{\frac{1}{2}}, \cdots, (a_r^2)^{\frac{1}{2}}, 0 \cdots, 0\}$.) Due to

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the invariance of the measure we have $\psi(\mathbf{A}) = \psi(\mathbf{H}_1\mathbf{A}\mathbf{H}_2)$ and hence the argument matrix $\mathbf{A}$ can be assumed diagonal with nonnegative elements written in decreasing order. Thus we write

$$\psi(a_1, a_2, \cdots, a_n) = \int_{O(n)} \exp(\operatorname{tr} \mathbf{H}'\mathbf{A})(\mathbf{H}' d\mathbf{H})$$

where $\mathbf{A} = \operatorname{diag}\{a_i\}$ and $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$.

With $n = 2$, $\psi(a_1, a_2)$ comes to $2\pi(I_0(a_1+a_2)+I_0(a_1-a_2))$ where $I_0(z) = (1/\pi)\int_0^\pi \exp(z\cos\theta)\,d\theta$ is the imaginary Bessel function of the first kind. If both a_1 and a_2 are large we can assume $\psi(a_1, a_2)$ approximately $2\pi I_0(a_1+a_2)$. From the asymptotic expansion for $I_0(z)$ we finally have

$$\psi(a_1, a_2) \cong (2\pi)^{\frac{1}{2}}(e^{(a_1+a_2)}/(a_1+a_2)^{\frac{1}{2}})(1+\textstyle\sum_{m=1} 1^2\cdot 3^2 \cdots (2m-1)^2/m!8^m(a_1+a_2)^m).$$

The integration procedure of the next section is a generalization of Laplaces' method (Erdélyi [2]) as applied to $I_0(a_1+a_2)$ to generate $e^{a_1+a_2}/(2\pi(a_1+a_2))^{\frac{1}{2}}$. Term by term (approximate) integration of the remaining part of the integrand yields the series in inverse powers of a_1+a_2 as shown. In Section 3 and 4 a set of differential equations for ψ given by James [6] is used. When $n = 2$ this set of equations reduces to the Bessel equation $y''+y'/2-y = 0$ where the independent variable is $z = a_1+a_2$. The procedure followed in Section 3 and 4 is a generalization of the substitution of $(e^z/z^{\frac{1}{2}})(1+\sum_{m=1} a_m/z^m)$ into the Bessel equation above. The resulting recurrence relation yields $a_m = 1^2\cdot 3^2 \cdots (2m-1)^2/m!\,8^m$.

2. The integration procedure. This section follows the methods of Anderson [1] and the results there will be used whenever possible. We assume here that $\mathbf{A} = \operatorname{diag}\{a_i\}$ is non-singular with $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$. The integrand $\exp(\operatorname{tr} \mathbf{H}'\mathbf{A})$ assumes its maximum value $\exp(\operatorname{tr} \mathbf{A})$ only when $\mathbf{H}$ is the identity matrix $\mathbf{I}$. Thus for large a_i the integrand is negligible except on a small neighborhood $N(\mathbf{I})$ of $\mathbf{I}$ on the orthogonal manifold. Since $N(\mathbf{I})$ will consist only of proper orthogonal matrices we can transform $\psi(\mathbf{A})$ under $\mathbf{H} = \exp(\mathbf{S})$ where $\mathbf{S}$ is an $n \times n$ skew-symmetric matrix. From Anderson [1] $(\mathbf{H}'d\mathbf{H})$ comes to $J\prod_{i<j}^n ds_{ij}$ where

$$J = 1+((n-2)/24)\operatorname{tr}\mathbf{S}^2+((8-n)/4(6!))\operatorname{tr}\mathbf{S}^4$$
$$+((5n^2-20n+14)/8(6!))(\operatorname{tr}\mathbf{S}^2)^2+\cdots$$

and $N(\mathbf{I})$ transforms into a neighborhood $N(\mathbf{S} = 0)$ of $\mathbf{S} = 0$. It is not necessary to go further into the nature of $N(\mathbf{I})$ or $N(\mathbf{S} = 0)$ since for large a_i we shall approximate $\psi(\mathbf{A})$ not by integrating over exactly $N(\mathbf{S} = 0)$ but simply over intervals $-\infty < s_{ij} < \infty$ for each s_{ij}. Justification for this is given following equation (2.2) below. Under $\mathbf{H} = \exp(\mathbf{S})$ we have

$$\exp(\operatorname{tr}\mathbf{H}'\mathbf{A}) = \exp(\operatorname{tr}\mathbf{A})\exp((-\tfrac{1}{2})\textstyle\sum_{i<j}^n c_{ij}s_{ij}^2) \cdot\exp\{\operatorname{tr}(\mathbf{S}^4\mathbf{A}/4!)+\operatorname{tr}(\mathbf{S}^6\mathbf{A}/6!)+\cdots\} \tag{2.1}$$

where $c_{ij} = a_i+a_j$. Thus for large a_i

$$\psi(\mathbf{A}) \cong \exp(\operatorname{tr}\mathbf{A})\int_{N(\mathbf{S}=0)} \exp((-\tfrac{1}{2})\textstyle\sum_{i<j}^n c_{ij}s_{ij})(\exp\{\quad\})J\prod_{i<j}^n ds_{ij}.$$

If this integration is performed term by term on the expansion

$$\text{(2.2a)}\quad (\exp\{\quad\})J = 1+\operatorname{tr}(\mathbf{S}^4\mathbf{A}/4!)+((n-2)/24)\operatorname{tr}\mathbf{S}^2$$

$$+\operatorname{tr}(\mathbf{S}^6\mathbf{A}/6!)+(\tfrac{1}{2})(\operatorname{tr}(\mathbf{S}^4\mathbf{A}/4!))^2$$

$$\text{(2.2b)}\quad +((n-2)/24)(\operatorname{tr}\mathbf{S}^2)\operatorname{tr}(\mathbf{S}^4\mathbf{A}/4!)+((8-n)/4(6!))\operatorname{tr}\mathbf{S}^4$$

$$+((5n^2-20n+14)/8(6!))(\operatorname{tr}\mathbf{S}^2)^2+\cdots$$

then for large a_i the limits for each s_{ij} can be set to $\pm\infty$ since each integration is of the form

$$\int_{N(\mathbf{S}=0)}\exp((-\tfrac{1}{2})\textstyle\sum_{i<j}^n c_{ij}s_{ij}^2)\prod_{i<j}^n s_{ij}^{m_{ij}}\prod_{i<j}^n ds_{ij}$$

and most of this integral is given in a small neighborhood of $\mathbf{S}=0$. The m_{ij} are positive even integers or zero since any term containing an odd power of an s_{ij} will integrate to zero. It is also clear that (2.2a) and (2.2b) yield terms in $c_{ij}^{-1}D$ and $(c_{ij}c_{kl})^{-1}D$ respectively where $D=\prod_{i<j}^n(2\pi/c_{ij})^{\frac{1}{2}}$. The precise form for the expansion for $\psi(\mathbf{A})$ is given by the theorem and conjecture below.

THEOREM 1. *For large* a_i

$$\text{(2.3)}\qquad \psi(\mathbf{A})\cong\exp(\operatorname{tr}\mathbf{A})\prod_{i<j}^n(2\pi/c_{ij})^{\frac{1}{2}}(1+(\tfrac{1}{8})\textstyle\sum_{i<j}^n(1/c_{ij})+\cdots)$$

PROOF. We must expand and integrate term by term the two expressions in (2.2a). The even-powered terms of $\operatorname{tr}\mathbf{S}^4\mathbf{A}$ come to $\sum_{i,j,k;k\neq i}^n s_{ij}^2s_{jk}^2c_{ij}+\sum_{i<j}^n s_{ij}^4c_{ij}$ and after integration these terms yield $D(\sum_{i,j,k;i\neq j,k}^n(1/c_{jk})+3\sum_{i<j}^n(1/c_{ij}))$. With S_1 defined as $\sum_{i<j}^n 1/c_{ij}^2$ it is clear that $\operatorname{tr}\mathbf{S}^4\mathbf{A}$ yields $(2(n-2)+3)S_1D$. Finally $\operatorname{tr}\mathbf{S}^2=-2\sum_{i<j}^n s_{ij}$ yields $-2S_1D$ so that the two expressions of (2.2a) combine to give $(\frac{1}{8})S_1D$.

A lemma proved by Hsu [3] and stated in Anderson [1] can be applied here to prove the $\exp(\operatorname{tr}A)\prod_{i<j}^n(2\pi/c_{ij})^{\frac{1}{2}}$ is an asymptotic representation for $\psi(\mathbf{A})$ as a_n (hence all the a_i's) approaches infinity.

Since the proof of Theorem 1 proceeds easily we might try to expand and integrate (2.2b) so as to produce the quadratic (terms in $(c_{ij}c_{kl})^{-1}D$) terms for the expansion (2.3). However the second and third expressions of (2.2b) do not lend themselves to a closed form for arbitrary n. For this reason the author has worked out the (enormous) details involved for both $n=3$ and $n=4$. The results are $(\frac{9}{128})S_2+(\frac{3}{64})S_{11}$ and $(\frac{9}{128})S_2+(\frac{3}{64})S_{11}+(\frac{1}{64})S_{1-1}$ respectively, where

$$S_2=\textstyle\sum_{i<j}^n(1/c_{ij}^2),\qquad S_{11}=\sum_{i<k}^n(1/c_{ij}c_{jk}),$$

and $S_{1-1}=\sum_{i<j,k<l,i<k}^n(1/c_{ij}c_{kl})$. For S_{11} and S_{1-1} the indices are assumed unequal so that the sums include all possible terms of the type shown (assuming $c_{ij}=c_{ji}$) without repetition. For larger n there seems to be no new type of term possible and we suggest the following

CONJECTURE. For arbitrary n the quadratic terms in the expansion (2.3) for $\psi(\mathbf{A})$ are $(\frac{9}{128})S_2+(\frac{3}{64})S_{11}+(\frac{1}{64})S_{1-1}$.

Finally note that with $n=2$ and both a_1 and a_2 large $\psi(\mathbf{A})$ has been shown in the introduction to be a multiple of a Bessel function with known asymptotic expansion

$$\exp(c_{12})(2\pi/c_{12})^{\frac{1}{2}}(1+\textstyle\sum_{m=1}1^2\cdot 3^2\cdots(2m-1)^2/m!(8c_{12})^m).$$

This expansion together with the coefficients of S_1 and S_2 above suggest the following

CONJECTURE. For arbitrary n the coefficient of $\sum_{i<j}^n(1/c_{ij}^m)$ in the expansion (2.3) for $\psi(A)$ is $1^2\cdot 3^2\cdots(2m-1)^2/m!\,8^m$.

3. The differential equation procedure with nonsingular argument matrix. In this section we assume $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$ so that $\mathbf{A}=\operatorname{diag}\{a_i\}$ is nonsingular. James [5] has shown that $\psi(\mathbf{A})$ satisfies

$$\psi_{pp}+\textstyle\sum_{i\neq p}^n(a_p\psi_p-a_i\psi_i)/(a_p{}^2-a_i{}^2)=\psi \tag{3.1}$$

for $p=1,2,\cdots,n$. Here ψ_j and ψ_{jj} denote the first and second partial derivatives of ψ with respect to a_j and $\sum_{i\neq p}^n f(i)=\sum_{i=1}^n f(i)-f(p)$. The form (2.3) for $\psi(A)$ suggests a substitution $\psi=(\exp(\operatorname{tr}\mathbf{A}))(\prod_{i<j}^n(a_i+a_j))^{-\frac{1}{2}}f$. Certainly the equation (3.1) cannot yield the constant $(2\pi)^{\frac{1}{4}n(n-1)}$ of (2.3) so that some sort of approximate integration is necessary both to exhibit the asymptotic form for $\psi(\mathbf{A})$ and to reveal the constant. The substitution is best done in two steps. First let $\psi=(\exp(\operatorname{tr}\mathbf{A}))\phi$. Then (3.1) comes to

$$\phi_{pp}+2\phi_p+\textstyle\sum_{i\neq p}^n(a_p\phi_p-a_i\phi_i)/(a_p{}^2-a_i{}^2)+\phi\sum_{i\neq p}^n 1/(a_p+a_i)=0 \tag{3.2}$$

where again the subscripts for ϕ denote partial differentiation. Now use logarithm ic differentiation with the second substitution

$$\begin{aligned}\phi &= (\textstyle\prod_{i<j}^n(a_i+a_j))^{-\frac{1}{2}}f\\ &= M(a_i)f.\end{aligned}$$

Again with the subscripts for f denoting partial differentiation

$$\phi_i = M(a_i)\qquad (f_i-(f/2)T_i)$$

and

$$\phi_{ii}=M(a_i)(f_{ii}-f_iT_i+f((\tfrac{3}{4})U_i+(\tfrac{1}{2})V_i))$$

where

$$T_i=\textstyle\sum_{j\neq i}^n 1/(a_i+a_j),\qquad U_i=\sum_{j\neq i}^n 1/(a_i+a_j)^2$$

and

$$V_i=\textstyle\sum_{j<k;j,k\neq i}^n 1/(a_i+a_j)(a_i+a_k)).$$

Equation (3.2) comes to

$$\begin{aligned}f_{pp}+2f_p+f_p(-T_p+\textstyle\sum_{i\neq p}^n a_p/(a_p{}^2-a_i{}^2))\\ -\textstyle\sum_{i\neq p}^n a_if_i/(a_p{}^2-a_i{}^2)+f((\tfrac{3}{4})U_p+(\tfrac{1}{2})V_p-(\tfrac{1}{2})(Q))\end{aligned}$$

where

$$\begin{aligned} Q &= \textstyle\sum_{i\neq p}^{n}(a_p T_p - a_i T_i)/(a_p{}^2 - a_i{}^2) \\ &= \textstyle\sum_{i\neq p}^{n}(\sum_{j\neq p,i}^{n} a_j/((a_p+a_i)(a_p+a_j)(a_i+a_j))) + U_p \\ &= (\tfrac{1}{2})\textstyle\sum_{i\neq p}^{n}\sum_{j\neq p,i}^{n} 1/((a_p+a_i)(a_p+a_j)) + U_p \\ &= V_p + U_p. \end{aligned}$$

Here the second line follows directly and the third from writing a_j as $((a_j+a_i)+(a_p+a_j)-(a_i+a_p))/2$. After simplifying the coefficient of f_p to $\sum_{i\neq p}^{n} a_i/(a_p{}^2-a_i{}^2)$ the equation for f comes to

$$f_{pp}+2f_p+\textstyle\sum_{i\neq p}^{n} a_i(f_p-f_i)/(a_p{}^2-a_i{}^2)+(f/4)U_p = 0.$$

Summing over p we finally have

$$\textstyle\sum_{i=1}^{n} f_{ii}+2\sum_{i=1}^{n} f_i+\sum_{i<j}^{n}(f_i-f_j)/(a_i-a_j)+(f/2)\sum_{i<j}^{n} 1/(a_i+a_j)^2 = 0$$

or more briefly

$$D_2 f+2D_1 f+D_3 f+(\tfrac{1}{2})S_2 f = 0 \tag{3.3}$$

where D_1, D_2, and D_3 are the obvious operators and $S_2 = \sum_{i<j}^{n} 1/c_{ij}^2$ with $c_{ij} = a_i + a_j$. By a term of degree m in the formal series for f we mean a multiple of $\prod_{i<j}^{n}(c_{ij})^{-m_{ij}}$ where the m_{ij} are positive integers or zero and $\sum_{i<j}^{n} m_{ij} = m$. Clearly D_1 increases the degree of a term by one while D_2, D_3 and S_2 increase the degree by two. Thus to find the coefficients of the mth degree terms h of f from the previously determined $(m-1)$st degree terms g simply set

$$D_2 g+2D_1 h+D_3 g+(\tfrac{1}{2})S_2 g = 0. \tag{3.4}$$

THEOREM 2. *The formal series solution of* (3.3) *up to the quadratic terms is given by*

$$f = 1+(\tfrac{1}{8})S_1+(\tfrac{9}{128})S_2+(\tfrac{3}{64})S_{11}+(\tfrac{1}{64})S_{1-1}.$$

PROOF. Recall that S_1, S_2, S_{11}, and S_{1-1} have been defined near the end of Section 2. We also define $S_3 = \sum_{i<j}^{n} 1/c_{ij}^3$, $S_{21} = \sum^{n} 1/c_{ij}^2 c_{jk}$ and $S_{2-1} = \sum_{i<j,\,k<l}^{n} 1/c_{ij}^2 c_{kl}$. S_{21} and S_{2-1} include all possible terms of the type shown (assuming the indices unequal and $c_{ij} = c_{ji}$) without repetition.

First let $g = 1$ and $h = d_1 S_1$. Then $D_1 h = -2d_1 S_2$, $D_2 g = D_3 g = 0$ and $(\frac{1}{2})S_2 g = (\frac{1}{2})S_2$. Thus (3.4) forces $(-4d_1+\frac{1}{2})S_2 = 0$ or $d_1 = \frac{1}{8}$.

Now let $g = d_1 S_1$ and $h = d_2 S_2 + d_{11} S_{11} + d_{1-1} S_{1-1}$. Then $D_1 h = -4d_2 S_3 - 2d_{11} S_{21} - 2d_{1-1} S_{2-1}$, $D_2 g = 4d_1 S_3$, $D_3 g = d_1 S_{21}$ and $S_2 g = d_1 S_3 + d_1 S_{21} + d_1 S_{2-1}$. Thus (3.4) forces $d_2 = (\frac{9}{16})d_1$, $d_{11} = (\frac{3}{8})d_1$, and $d_{1-1} = (\frac{1}{8})d_1$.

These methods become a bit clumsy for higher order terms. The author has worked out the coefficients of the cubic terms for $n = 4, 5, 6$ and also for n arbitrary. The results are given in Theorem 3 below but due to the enormous amount of detail required the proof is not included. We define

$$S^1_{111} = \sum^n 1/c_{ij}\, c_{ik}\, c_{jk}$$
$$S^2_{111} = \sum^n 1/c_{ij}\, c_{ik}\, c_{il}$$
$$S^3_{111} = \sum^n 1/c_{ij}\, c_{ik}\, c_{jl}$$
$$S^4_{111} = \sum^n 1/c_{ij}\, c_{ik}\, c_{lm}$$
$$S^5_{111} = \sum^n 1/c_{ij}\, c_{kl}\, c_{mn}$$

where again the indices are assumed unequal and each S^i_{111} includes all terms of the type shown without repetition (with $c_{ij} = c_{ji}$).

THEOREM 3. *The cubic terms of the formal series solution of* (3.3) *are given by*

$$\tfrac{1}{2}(\tfrac{1}{8})^3(75S_3+45S_{21}+9S_{2-1})+(\tfrac{1}{8})^3(15S^1_{111}+15S^2_{111}+9S^3_{111}+3S^4_{111}+S^5_{111}).$$

To close this section we consider the case $n = 3$. f has the form

$$1+\sum_{i_1+i_2+i_3 \geqq 1} c(i_1, i_2, i_3)/c_{12}^{i_1}c_{13}^{i_2}c_{23}^{i_3}$$

and from the conjecture following Theorem 1 it is clear that

$$c(i_1+1,0,0) = ((2i_1+1)^2/8(i_1+1))c(i_1,0,0).$$

With the obvious definitions for S_4, S_{31}, S_{22}, and S_{211} when $n = 3$ the fourth degree terms come to $(\frac{1}{8})^4((3^2\cdot 5^2\cdot 7^2/4!)S_4+(3\cdot 5^2\cdot 7/2)S_{31}+(3^3\cdot 5\cdot 7/4)S_{22}+(3^2\cdot 5\cdot 7/2)S_{211})$. These terms together with Theorem 3 suggest that

$$c(i_1+1, i_2, i_3) = ((2i_1+1)(2(i_1+i_2+i_3)+1)/8(i_1+1))c(i_1, i_2, i_3).$$

Repeated use of this along with the symmetry of the coefficients leads to the following

CONJECTURE.

$$c(i_1, i_2, i_3) = (1\cdot 3\cdots(2i_1-1))(1\cdot 3\cdots(2i_2-1)) \cdot(1\cdot 3\cdots(2i_3-1))(1\cdot 3\cdots(2(i_1+i_2+i_3)-1))/i_1!\, i_2!\, i_3!\, 8^{i_1+i_2+i_3}.$$

4. The differential equation procedure with singular argument matrix. In this section we use the notation and methods of Section 3 but assume $a_1 \geqq a_2 \geqq \cdots \geqq a_r > 0$ so that $\mathbf{A} = \operatorname{diag}\{a_1, a_2, \cdots, a_r, 0, \cdots, 0\}$ is singular. A slight extension of James's [6] work shows that $\psi(\mathbf{A})$ satisfies

$$\psi_{pp}+\sum_{i\neq p}^r(a_p\psi_p-a_i\psi_i)/(a_p^{\,2}-a_i^2)+(n-r)\psi_p/a_p = \psi \tag{4.1}$$

for $p = 1, 2, \cdots, r$. The nonsingular results of Theorem 1 suggest that

$$\psi(a_1,\cdots,a_r) = \exp(\operatorname{tr}\mathbf{A})(\textstyle\prod_{i<j}^r(2\pi/c_{ij})\prod_{i=1}^r(2\pi/a_i)^{n-r})^{\frac{1}{2}}f(a_1,\cdots,a_r)$$

or $\psi = k\exp(\operatorname{tr}\mathbf{A})(\prod_{i<j}^r(a_i+a_j)\prod_{i=1}^r a_i^{\,n-r})^{-\frac{1}{2}}f$ where $f = 1+(\frac{1}{8})\sum_{i<j}^r 1/(a_i+a_j)+d_1'\sum_{i=1}^r 1/a_i+\cdots$ and $k = ((2\pi)^{\frac{1}{2}})^{r(r-1)/2+r(n-r)}$. From (4.1) we can determine the

asymptotic form for ψ to within an unknown constant and the value shown for k is only suggested.

Following Section 3 we first substitute $\psi = (\exp(\mathrm{tr}\,\mathbf{A}))\,\phi$ and (4.1) comes to

(4.2) $$\phi_{pp}+2\phi_p+\sum_{i\neq p}^r(a_p\phi_p-a_i\phi_i)/(a_p^2-a_i^2)+(n-r)\phi_p/a_p + \phi\left(\sum_{i\neq p}^r 1/(a_p+a_i)+(n-r)/a_p\right)=0.$$

Now let

$$\phi = \left(\prod_{i<j}^r(a_i+a_j)\cdot\prod_{i=1}^r a_i^{n-r}\right)^{-\frac{1}{2}}f = M(a_i)f.$$

Then $\phi_i = M(a_i)(f_i-(f/2)T_i)$ and $\phi_{ii} = M(a_i)(f_{ii}-f_iT_i+f((\frac{3}{4})U_i+(\frac{1}{2})V_i))$ where

$$T_i = \sum_{j\neq i}^r(1/(a_i+a_j))+(n-r)/a_i$$
$$U_i = \sum_{j\neq i}^r(1/(a_i+a_j)^2)+(n-r)/a_i^2$$

and

$$V_i = \sum_{j<k;\,j,k\neq i}^r 1/(a_i+a_j)(a_i+a_k)+(n-r)\sum_{j\neq i}^r 1/(a_i+a_j)a_i+(n-r)(n-r-1)/2a_i^2.$$

Equation (4.2) comes to

$$f_{pp}+2f_p+f_p\left(-T_p+\sum_{i\neq p}^r a_p/(a_p^2-a_i^2)+(n-r)/a_p\right) -\sum_{i\neq p}^r a_if_i/(a_p^2-a_i^2)+f((\tfrac{3}{4})U_p+(\tfrac{1}{2})V_p-(\tfrac{1}{2})Q)=0$$

where

$$\begin{aligned} Q &= \sum_{i\neq p}^r \frac{a_pT_p-a_iT_i}{a_p^2-a_i^2}+(n-r)T_p/a_p \\ &= \sum_{i<j,\,i,j\neq p}^r 1/(a_p+a_i)(a_p+a_j)+\sum_{i\neq p}^r 1/(a_p+a_i)^2+(n-r)T_p/a_p \\ &= U_p+V_p+(n-r)(n-r-1)/2a_p^2. \end{aligned}$$

Here the second line follows from the similar arguments in Section 3 and the third from writing $(n-r)^2$ as $(n-r)+(n-r)(n-r-1)$. The coefficient of f_p comes to simply $\sum_{i\neq p}^r a_i/(a_p^2+a_i^2)$ and the equations for f simplify to

$$f_{pp}+2f_p+\sum_{i\neq p}^r a_i(f_p-f_i)/(a_p^2-a_i^2)+(f/4)(U_p-(n-r)(n-r-1)/a_p^2)=0.$$

Summing over p we finally have

$$\sum_{i=1}^r f_{ii}+2\sum_{i=1}^r f_i+\sum_{i<j}^r(f_i-f_j)/(a_i-a_j) +(f/4)\left(2\sum_{i<j}^r 1/(a_i+a_j)^2-(n-r)(n-r-2)\sum_{i=1}^r 1/a_i^2\right)=0$$

or more briefly

(4.3) $$D_2f+2D_1f+D_3f+(f/2)S_2-(f/4)(n-r)(n-r-2)\sum_{i=1}^r 1/a_i^2=0$$

where D_1, D_2, D_3, and S_2 are as defined in Section 3 with n replaced by r. Formal substitution into (4.3) will now yield the coefficients of the formal series for f.

THEOREM 4. *The formal series solution of* (4.3) *up to quadratic terms is given by*

$$\begin{aligned} f = {} & 1+(\tfrac{1}{8})S_1+d_1'\sum_{i=1}^r 1/a_1 \\ & +(\tfrac{9}{128})S_2+(\tfrac{3}{64})S_{11}+(\tfrac{1}{64})S_{1-1} \\ & +(\tfrac{1}{2})d_1'(1+d_1')\sum_{i=1}^r 1/a_i^2+d_1'(\tfrac{5}{8}+d_1')\sum_{i<j}^r 1/a_ia_j \\ & +(\tfrac{1}{8})d_1'\sum_{k=1}^r\left((1/a_k)\sum_{i<j,\,i,j\neq k}^r 1/(a_i+a_j)\right), \end{aligned}$$

where $d_1' = (-\frac{1}{8})(n-r)(n-r-2)$ *and* S_1, S_2, S_{11}, *and* S_{1-1} *are as defined in Section 2 with n replaced by r.*

PROOF. Let $f = 1 + d_1 S_1 + d_1' \sum_{i=1}^r 1/a_i$. Then the terms of degree two in (4.3) come to $(-4d_1 + \frac{1}{2})S_2 + (-2d_1' - (\frac{1}{4})(n-r)(n-r-2))\sum_{i=1}^r 1/a_i^2$ so that $d_1 = \frac{1}{8}$ and $d_1' = (-\frac{1}{8})(n-r)(n-r-2)$. Consider the quadratic terms of f. Note that the first four terms of equation (4.3) make up equation (3.3) with n replaced by r. Thus the S_2, S_{11}, and S_{1-1} terms follow from Theorem 2. To find the additional quadratic terms apply the first, third, and fourth terms of (4.3) to $d_1' \sum_{i=1}^r 1/a_i$ and the fifth term to $d_1 S_1 + d_1' \sum_{i=1}^r 1/a_i$. This comes to

$$d_1'(2 - (\tfrac{1}{4})(n-r)(n-r-2)) \sum_{i=1}^r 1/a_i^3$$
$$+ d_1'(\tfrac{5}{4} - (\tfrac{1}{4})(n-r)(n-r-2)) \sum_{i,j;\, i \neq j}^r 1/a_i^2 a_j$$
$$+ (\tfrac{1}{4}) d_1' \sum_{k=1}^r \sum_{i<j;\, i,j \neq k}^r (2/a_k(a_i + a_j)^2 + 1/a_k^2(a_i + a_j)).$$

Finally apply the second term of equation (4.3) to constant multiples of the additional quadratic terms for f as shown in the theorem to complete the proof.

The results of this section suggest that with $\mathbf{A} = \operatorname{diag}\{a_1, a_2, \cdots, a_r, 0, \cdots, 0\}$, $\psi(\mathbf{A})$ has the asymptotic expansion

$$(\exp(\operatorname{tr}\mathbf{A}))(\prod_{i<j}^r (a_i + a_j) \prod_{i=1}^r (a_i)^{n-r})^{-\frac{1}{2}} f$$

as the a_i increase without bound. Certainly the conjecture given at the end of Section 2 also holds here, namely, that the coefficient of $\sum_{i<j}^r 1/(a_i + a_j)^m$ in the expansion for f is $1^2 \cdot 3^2 \cdots (2m-1)^2/m!\, 8^m$.

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