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A SELECTION PROBLEM

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1. Introduction and formulation of the problem. In many fields of research, one is faced with the problem of selecting the better ones from a given collection. We consider such a selection problem. We assume that there are k populations ($k \geqq 2$) populations $\Pi_1, \Pi_2, \cdots, \Pi_k$ at our disposal from which we want to select a subset. These may be varieties of a grain or some treatments or some production methods. The quality of the ith population is characterized by a real-valued parameter θ_i. The population with the largest θ-value is called the best population. A population is considered as a superior one if its quality measure does not fall too much below that of the best population. If $d(\theta_i, \theta_j)$ is a suitable distance measure between θ_i and θ_j and if $\theta_{\max} = \max(\theta_1, \theta_2, \cdots, \theta_k)$, population Π_i is

$$\begin{aligned} &\text{superior (or good)} \quad &&\text{if} \quad d(\theta_{\max}, \theta_i) \leqq \Delta, \\ &\text{inferior (or bad)} \quad &&\text{if} \quad d(\theta_{\max}, \theta_i) > \Delta, \end{aligned}$$

where Δ is a given positive constant. It must be emphasized that this definition is different from the usual one considered in the literature [6], where θ_i is compared with θ_0, the quality measure of the standard or control population. Our definition is appropriate to situations where comparisons with a standard or control population are not possible. As pointed out by Lehmann [6], such a situation arises when a new product is being developed and one is interested in selecting the most promising of a number of production methods. In such cases each method must be compared with the totality of the remaining methods. A population is then considered superior if it does not fall too much below the best. In such cases our definition is a natural (or appropriate) one.

In some cases, it is reasonable to assume that whenever $d(\theta_{\max}, \theta_i) = \Delta$ one is indifferent towards branding Π_i as superior or inferior. In view of such cases, we may assume that there exist two positive constants Δ_1, Δ_2 (both, presumably, small compared with Δ) such that considering Π_i as inferior when $d(\theta_{\max}, \theta_i) \leqq \Delta - \Delta_1$ and considering Π_i as superior when $d(\theta_{\max}, \theta_i) \geqq \Delta + \Delta_2$.

Further it is of no serious consequence in whatever way one classifies Π_i when $\Delta - \Delta_1 < d(\theta_{\max}, \theta_i) < \Delta + \Delta_2$. In view of these remarks, we modify our previous definition as follows: A population Π_i is said to be

$$\begin{aligned} &\text{superior (or good)} \quad &&\text{if} \quad d(\theta_{\max}, \theta_i) \leqq \delta_1^*, \\ &\text{inferior (or bad)} \quad &&\text{if} \quad d(\theta_{\max}, \theta_i) \geqq \delta_2^*, \end{aligned} \tag{1}$$

where δ_1^*, δ_2^* are specified constants such that $0 < \delta_1^* < \delta_2^*$.

With this modified definition of superior and inferior populations, we are

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interested in devising a procedure which selects a random size subset, that does not contain all the inferior populations with a probability not less than P^*, a specified constant. In the selection problems considered in the literature even though the subset is random (not fixed in advance), like our problem, the interest is on including the best one or all those better than a standard or control population [4], [5]. Once again, it should be emphasized that the problem considered here is different from those considered earlier in the literature, but the solution has some similarities with the problems considered by Gupta [4].

We propose a procedure, which determines the selection or nonselection of the ith population on the basis of a real-valued statistic Y_i based on a random sample size n. After stating the procedure, we determine a constant whose specification completes the definition of the procedure. This constant will be chosen to meet the above mentioned probability requirement. Following this determination two operating characteristics of the procedure will be studied. The entire discussion has been carried out in relation to two cases—(i) Y_i has the density of the form $f(y-\theta_i)$ and (ii) Y_i has the density of the form $f(y/\theta_i)\theta_i^{-1}$ for $0 < y < \infty$ and 0 for $y \leqq 0$.

In definition (1) we take d as d_L or d_s whenever θ_i is a location or θ_i is a scale parameter for the density of Y_i. d_L and d_s are defined as

$$d_L(a, b) = a-b, \qquad d_s = (a, b) = a/b. \tag{2}$$

Proposed Procedure R. Select Π_i whenever $d(Y_{\max}, Y_i) \leqq d(\delta_2^*, c)$ where c is some specified constant, such that $0 < c < \delta_2^*$ and $Y_{\max} = \max(Y_1, Y_2, \cdots, Y_k)$.

Here the distance measure d is to be taken as d_L or d_s according as θ_i is a location or a scale parameter for the density of Y_i.

Once we specify the constant c, the above procedure is completely defined. This constant will be chosen to meet the probability requirement mentioned before. This type of procedure has been considered by Paulson [8] and Gupta [5] for different objectives.

2. The results for location parameter case. In this section, on the assumption that θ_i is a location parameter for Y_i, we first determine the constant c. Later subsections give results on the expected number of inferior populations that are included in the selected subset and the expected number of superior populations that enter the selected subset.

2.1. *Determination of c.* Let us denote the ordered θ-values by $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$ and $Y_{(i)}$ be the unknown statistic associated with $\theta_{[i]}$ $(1 \leqq i \leqq k)$. Let Ω be the parameter space which is the collection of all possible parameter vectors $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)$. Let t_1 and t_2 denote, respectively, the unknown number of inferior and superior populations in the given collection of k populations. Clearly we have $t_1 \geqq 0$, $t_2 \geqq 1$ and $t_1 + t_2 \leqq k$. For specified δ_1^* and δ_2^*, let

$$\begin{aligned} \Omega(t_1, t_2) = \{\boldsymbol{\theta}: \theta_{[1]} \leqq \cdots \leqq \theta_{[t_1]} \leqq \theta_{[k]} - \delta_2^* < \theta_{[t_1+1]} \\ \leqq \theta_{[t_1+2]} \leqq \cdots \leqq \theta_{[k-t_2]} < \theta_{[k]} - \delta_1^* \leqq \theta_{[k-t_2+1]} \\ \leqq \theta_{[k-t_2]} \leqq \cdots \leqq \theta_{[k]}\}. \end{aligned} \tag{3}$$

Then

$$\Omega = \bigcup_{t_1,t_2}\{\Omega(t_1, t_2)\}. \tag{4}$$

Let CD stand for correct decision which is defined to be the selection of the subset which excludes all the inferior populations. Now we have to determine the constant c such that

$$P(CD \mid R) \geqq P^* \quad \text{for all} \quad \boldsymbol{\theta} \in \Omega.$$

The required constant c is chosen such that the infimum of $P(CD \mid R)$ is not less than P^*. Now we determine the infimum of $P(CD \mid R)$. If $\boldsymbol{\theta} \in \Omega(t_1, t_2)$

$$\begin{aligned} P(CD \mid R) &= P\{\max_{1 \leqq \alpha \leqq t_1} Y_{(\alpha)} < \max_{1 \leqq \beta \leqq k} Y_{(\beta)} - \delta_2{}^* + c\} \\ &= \textstyle\sum_{i=t_1+1}^{k} \int_{-\infty}^{\infty} \prod_{\alpha=1}^{t_1} F(x+\theta_{[i]}-\theta_{[\alpha]}-\delta_2{}^*+c) \\ &\quad \times \textstyle\prod_{\beta=t_1+1,\beta\neq i}^{k} F(x+\theta_{[i]}-\theta_{[\beta]}) f(x)\,dx \end{aligned} \tag{5}$$

where $F(\cdot)$ is cdf of the distribution defined by $f(\cdot)$.

LEMMA 1. *$P\{CD \mid R\}$ is a non-increasing function of $\theta_{[\alpha]}(\alpha = 1, \cdots, t_1)$ and a non-decreasing function of $\theta_{[\beta]}(\beta = t_1+1, \cdots, k)$.*

PROOF. Let $\psi(x_1, \cdots, x_k)$ be the function

$$\begin{aligned} \psi &= 1 \quad \text{if} \quad \max(x_1, \cdots, x_{t_1}) < \max(x_{t_1+1}, \cdots, x_k) - \delta_2{}^* + c; \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{6}$$

It is easily seen that, for each α ($\alpha = 1, \cdots, t_1$), ψ is a non-increasing function of x_α when all x_j ($j = 1, \cdots, k; j \neq \alpha$) are held fixed and for each β ($\beta = t_1+1, \cdots, k$), it is a non-decreasing function of x_β when all x_j ($j = 1, \cdots, k; j \neq \beta$) are held fixed. Now, by Lemma 4.2 of [7], it follows that $P\{CD \mid R\} = E\psi(Y_{(1)}, \cdots, Y_{(k)})$ has the required monotone properties.

In view of the above lemma, it follows that $P(CD \mid R)$ can be made smaller by increasing $\theta_{[t_1-1]}, \cdots, \theta_{[1]}$ in turn to $\theta_{[t_1]}$ and by decreasing $\theta_{[k-t_2]}, \cdots, \theta_{[t_1+2]}$ in turn to $\theta_{[t_1+1]}$ and by decreasing $\theta_{[k-1]}, \cdots, \theta_{[k-t_2+2]}$ in turn to $\theta_{[k-t_2+1]}$.

Thus it is sufficient to restrict our attention to those points $\boldsymbol{\theta}$ in $\Omega(t_1, t_2)$ for which

$$\begin{aligned} \theta_{[1]} &= \cdots = \theta_{[t_1]} = m \quad \text{say} \\ \theta_{[t_1+1]} &= \cdots = \theta_{[k-t_2]} = m' \quad \text{say} \\ \theta_{[k-t_2+1]} &= \cdots = \theta_{[k-1]} = m'' \quad \text{say, and} \quad \theta_{[k]} = \theta \quad \text{say,} \end{aligned} \tag{7}$$

and minimize $P(CD \mid R)$ at such points, which will be denoted by P'. For fixed θ it is easy to see that (in view of Lemma 1) P' can be minimized by letting m, m', and m'' approach $\theta - \delta_2{}^*$, $\theta - \delta_2{}^*$ and $\theta - \delta_1{}^*$ respectively. Thus the infimum of P' over possible values of m, m' and m'' is the same as the infimum of $P(CD \mid R)$ over $\Omega(t_1, t_2)$. Hence

$$\begin{aligned}
&\inf_{\boldsymbol{\theta} \in \Omega(t_1,t_2)} P(CD \mid R) \\
(8) \qquad &= (k-t_1-t_2)\textstyle\int_{-\infty}^{\infty} [F(x-\delta_2^*+c)]^{t_1}[F(x)]^{k-t_1-t_2-1} \\
&\qquad\qquad \cdot [F(x+\delta_1^*-\delta_2^*)]^{t_2-1}F(x-\delta_2^*)f(x)\,dx \\
&= (t_2-1)\textstyle\int_{-\infty}^{\infty} [F(x+\delta_1^*+c)]^{t_1}[F(x+\delta_2^*-\delta_1^*)]^{k-t_1-t_2} \\
&\qquad\qquad \cdot [F(x)]^{t_2-2}F(x-\delta_1^*)f(x)\,dx \\
&\quad + \textstyle\int_{-\infty}^{\infty} [F(x+c)]^{t_1}[F(x+\delta_2^*)]^{k-t_1-t_2}[F(x+\delta_1^*)]^{t_2-1}f(x)\,dx.
\end{aligned}$$

Replacing the middle integral by an equivalent expression obtained through integration by parts, we obtain

$$\inf_{\boldsymbol{\theta} \in \Omega(t_1,t_2)} P(CD \mid R) = 1 - G(t_1, t_2) \tag{9}$$

where

$$G(t_1, t_2) = t_1 \int_{-\infty}^{\infty} \left\{\frac{F(x_1+\delta_1^*-c)}{F(x+\delta_2^*-c)}\right\}^{t_2-1} [F(x+\delta_2^*-c)]^{k-t_1-1} \cdot F(x-c)[F(x)]^{t_1-1}f(x)\,dx. \tag{10}$$

Hence

$$\inf_{\boldsymbol{\theta} \in \Omega} P(CD \mid R) = 1 - \sup_{t_1,t_2} G(t_1, t_2). \tag{11}$$

The supremum of $G(t_1, t_2)$ can be obtained in two steps. For fixed t_1, the supremum of $G(t_1, t_2)$ over possible values of t_2 (all integral values between 1 and $k-t_1$) is seen to be $G(t_1, 1)$, since $\delta_1^*-\delta_2^* < 0$. Hence, if $1-H(t_1) = G(t_1, 1)$, we have

$$\begin{aligned}
(12) \qquad \sup_{t_1,t_2} G(t_1, t_2) &= \sup_{t_1} \{G(t_1, 1)\} = \sup_{t_1} \{1-H(t_1)\} \\
&= 1-H(k-1)(\equiv G(k-1, 1));
\end{aligned}$$

and this implies that

$$\inf_{\boldsymbol{\theta} \in \Omega} P(CD \mid R) = \textstyle\int_{-\infty}^{\infty} F^{k-1}(x+c)f(x)\,dx. \tag{13}$$

Thus the required c value is the largest real number c (less than δ_2^*) such that

$$\textstyle\int_{-\infty}^{\infty} F^{k-1}(x+c)f(x)\,dx = P^*. \tag{14}$$

2.2. *Expected number of inferior populations included in the selected subset and its supremum.* For the procedure proposed the number η of inferior populations that enter into the selected subset is a random variable. For fixed values of k and P^*, the expected value of η is a function of $\boldsymbol{\theta}$. The supremum of this expected value, in analogy with the power function associated with the usual tests of hypotheses, can be regarded as a measure of the efficiency of the procedure.

For $\theta \in \Omega(t_1, t_2)$ it is easy to see that

$$(15) \qquad E(\eta) = \sum_{i=1}^{t_1} P[Y_{(i)} + \delta_2^* - c > Y_{\max}]$$
$$= \sum_{i=1}^{t_1} \int_{-\infty}^{\infty} \{\prod_{j=1, j\neq i}^{k} F(x+\delta_2^* - c - \theta_{[j]} + \theta_{[i]})\} f(x)\, dx.$$

If Q stands for the value of $E(\eta \mid \theta_{[1]} = \cdots = \theta_{[r]} = \theta)$, where $1 \leqq r \leqq t_1$, then

$$(16) \qquad Q = r\int_{-\infty}^{\infty} F^{r-1}(x+\delta_2^*-c)\{\prod_{j=r+1}^{k} F(x+\delta_2^*-c-\theta_{[j]}+\theta)\} f(x)\, dx$$
$$+\sum_{i=r+1}^{t_1} \int_{-\infty}^{\infty} F^{r}(x+\delta_2^*-c+\theta_{[i]}-\theta)$$
$$\cdot\{\prod_{j=r+1, j\neq i}^{k} F(x+\delta_2^*-c-\theta_{[j]}+\theta_{[i]})\} f(x)\, dx.$$

As a step towards finding the supremum of $E(\eta)$ we observe the following result.

THEOREM 1. *For any integer r ($1 \leqq r \leqq t_1$), Q is a non-decreasing function of θ provided $f(x-\theta)$ has monotone likelihood ratio.*

The proof is similar to that of Theorem 1 on page 231 of [4] and hence it is omitted.

By Theorem 1, $E(\eta)$ can be increased by setting $\theta_{[1]} = \cdots = \theta_{[t_1]} = \theta$ and letting θ approach $\theta_{[k]} - \delta_2^*$. Thus, from (15) we have

$$(17) \qquad \sup_{\theta\in\Omega} E(\eta) = \max_{t_1,t_2}\{\sup_{\theta_{[t_1+1]},\cdots,\theta_{[k]}} S(t_1,t_2)\} = \max_{t_1,t_2} S_1(t_1,t_2), \quad \text{say;}$$

where

$$(18) \qquad S(t_1, t_2) = t_1\int_{-\infty}^{\infty} F^{t_1-1}(x+\delta_2^*-c)\{\prod_{j=t_1+1}^{k} F(x-c+\theta_{[k]}-\theta_{[j]})\} f(x)\, dx.$$

It is easy to see that for fixed $\theta_{[k]}$, when $\theta_{[t_1+1]}, \cdots, \theta_{[k-t_2]}$ approach $\theta_{[k]} - \delta_2^*$ and $\theta_{[k-t_2+1]}, \cdots, \theta_{[k-1]}$ approach $\theta_{[k]} - \delta_1^*$, $S(t_1, t_2)$ is maximum. Now for fixed t_1, $S_1(t_1, t_2)$ is maximum when t_2 is minimum since $F(x+\delta_1^*-c) < F(x+\delta_2^*-c)$ and this maximum is independent of t_1. Hence

$$(19) \qquad \sup_{\theta\in\Omega} E(\eta) = (k-1)\int_{-\infty}^{\infty} [F(x+\delta_2^*-c)]^{k-2} F(x-c) f(x)\, dx.$$

If c were chosen to satisfy the basic probability requirement, from (14) we obtain

$$(k-1)\int_{-\infty}^{\infty} F^{k-2}(x) F(x-c) f(x)\, dx = 1 - P^*,$$

so that for such a value of c, we have

$$(20) \qquad \sup_{\theta\in\Omega} E(\eta) > 1 - P^*.$$

Another measure of the efficiency of the procedure R is the supremum of the expected proportion of inferior populations that enter the selected subset. Assuming that $f(x-\theta)$ has monotone likelihood ratio, calculations similar to the above will show that

$$(21) \qquad \sup_{t_1,t_2}\{\sup_{\Omega(t_1,t_2)} E(\eta/t_1)\} = \int_{-\infty}^{\infty} F^{k-2}(x+\delta_2^*-c) F(x-c) f(x)\, dx.$$

It is also of some interest to examine how well R identifies the superior populations. As a measure of the efficiency of R with regards to this aspect we can consider either the infimum of the expected number of superior ones that enter the selected subset or the infimum of the expected proportion of the superior populations entering the selected subset.

2.3. *Expected number of superior populations that enter the selected subset and its infimum.* Let ζ denote the random number of superior populations that enter the selected subset. If $\boldsymbol{\theta} \in \Omega(t_1, t_2)$, then

$$(22)\quad E(\zeta \mid \boldsymbol{\theta}) = \sum_{i=k-t_2+1}^{k} P[Y_{(i)} > Y_{\max} - \delta_2^* + c]$$
$$= \sum_{i=k-t_2+1}^{k} \int_{-\infty}^{\infty} [\prod_{j=1, j\neq i}^{k} F(x+\delta_2^* - c - \theta_{[j]} + \theta_{[i]})] f(x)\, dx.$$

Let S stand for the value $E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-\alpha]} = \theta_{[k]} - \delta)$ where $1 \leqq \alpha \leqq t_2 - 1$ and $\delta_1 \geqq \delta \geqq 0$. Note that

$$(23)\quad S = \sum_{i=k-t_2+1}^{k-\alpha-1} \int_{-\infty}^{\infty} \{\prod_{j=1, j\neq i}^{k-\alpha-1} F(x+\delta_2^* - c - \theta_{[j]} + \theta_{[i]})\} \cdot F^{\alpha}(x+\delta_2^* - c - \theta_{[k]} + \delta + \theta_{[i]}) F(x+\delta_2^* - c - \theta_{[k]} + \theta_{[i]}) f(x)\, dx.$$

A monotone property of S as a function of δ can be obtained by finding the sign of $dS/d\theta$.

THEOREM 2. *For any integer α $(1 \leqq \alpha \leqq t_2 - 1)$, S is a non-increasing function of δ, provided $f(x-\theta)$ has monotone likelihood ratio in x.*

Proof of this Theorem is similar to that of Theorem 1 and hence omitted. In view of Theorem 2, $E(\zeta)$ can be decreased by setting $\theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta$ and letting δ approach δ_1^*. We note that

$$(24)\quad E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*)$$
$$= (t_2 - 1) \int_{-\infty}^{\infty} \{\prod_{j=1}^{k-t_2} F(x+\delta_2^* - c - \theta_{[j]} + \theta_{[k]} - \delta_1^*)\} \cdot [F(x+\delta_2^* - c]^{t_2-2} F(x+\delta_2^* - c - \delta_1^*) f(x)\, dx$$
$$+ \int_{-\infty}^{\infty} \{\prod_{j=1}^{k-t_2} F(x+\delta_2^* - c - \theta_{[j]} + \theta_{[k]})\} \cdot [F(x+\delta_2^* + \delta_1^* - c]^{t_2-1} f(x)\, dx.$$

So the infimum of $E(\zeta \mid \boldsymbol{\theta})$ over $\Omega(t_1, t_2)$ is the same as the infimum of $E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*)$ over the same set. Thus

$$(25)\quad \inf_{\boldsymbol{\theta} \in \Omega} E(\zeta \mid \boldsymbol{\theta})$$
$$= \min_{t_1, t_2} \{\inf_{\boldsymbol{\theta} \in \Omega(t_1, t_2)} E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*)\}.$$

Examining the expression for $E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*)$, it is seen that this expression can be decreased by increasing the values of $\theta_{[1]}, \cdots, \theta_{[t_1]}$. So in our search for the infimum we need to confine our attention to points in that part of Ω for which t_1 is minimum for fixed t_2. Clearly the minimum of t_1 is zero; so we have

$$(26)\quad \inf_{\boldsymbol{\theta} \in \Omega} E(\zeta \mid \boldsymbol{\theta}) = \inf_{\boldsymbol{\theta} \in \Omega^*} E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*)$$

where

$$(27)\quad \Omega^* = \bigcup_{1 \leqq t_2 \leqq k} \{\Omega(0, t_2)\}.$$

It may be recalled that

(28) $$\Omega(0, t_2) = \{\boldsymbol{\theta}: \theta_{[k]} - \delta_2^* < \theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k-t_2]} < \theta_{[k]} - \delta_1^* \leqq \theta_{[k-t_2+1]} \leqq \cdots \leqq \theta_{[k]}\}.$$

The expression for $E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*)$ when $\boldsymbol{\theta} \in \Omega^*$ is the same as (24). So to obtain the infimum we need to push each $\theta_{[i]}$ $(i = 1, \cdots, k-t_2)$ to approach $\theta_{[k]} - \delta_1^*$. Thus for fixed t_2,

(29) $$\begin{aligned} \inf_{\boldsymbol{\theta} \in \Omega(0,t_2)} E(\zeta \mid \theta_{[k-1]} = \cdots = \theta_{[k-t_2+1]} = \theta_{[k]} - \delta_1^*) \\ = (t_2 - 1) \int_{-\infty}^{\infty} F^{k-2}(x + \delta_2^* - c) F(x + \delta_2^* - \delta_1^* - c) f(x)\, dx \\ + \int_{-\infty}^{\infty} F^{k-1}(x + \delta_2^* + \delta_1^* - c) f(x)\, dx. \end{aligned}$$

Clearly, the infimum of the above expression is attained when $t_2 = 1$. That is

(30) $$\inf_{\boldsymbol{\theta} \in \Omega} E(\zeta \mid \boldsymbol{\theta}) = \int_{-\infty}^{\infty} F^{k-1}(x + \delta_2^* + \delta_1^* - c) f(x)\, dx.$$

3. Results for the scale parameter case. Using the same arguments as in Section 2, we obtain

(31) $$\inf_{\boldsymbol{\theta} \in \Omega} P(CD \mid \boldsymbol{\theta}) = \int_0^{\infty} F^{k-1}(cx) f(x)\, dx,$$

so that we need to select a c-value such that $0 < c < \delta_2^*$ and

(32) $$\int_0^{\infty} F^{k-1}(cx) f(x)\, dx = P^*.$$

As before, on the assumption that $\theta^{-1} f(x/\theta)$ has monotone likelihood ratio property in x, we obtain

(33) $$\sup_{\boldsymbol{\theta} \in \Omega} E(\eta) = (k-1) \int_0^{\infty} [F(x\delta_2^*/c)]^{k-2} F(x/c) f(x)\, dx$$

and

(34) $$\inf_{\boldsymbol{\theta} \in \Omega} E(\zeta) = \int_0^{\infty} [F(x\delta_1^* \cdot \delta_2^*/c)]^{k-1} f(x)\, dx.$$

4. Some examples. We shall discuss the above problem for the means of normal populations and for the scale parameters of the gamma populations.

Let $\Pi_1, \cdots, \Pi_k$ be k normal populations with means $\mu_1, \cdots, \mu_k$ and a common known variance σ^2; without loss of generality we can assume that σ^2 is 1. Suppose that μ is the characterizing parameter. Let $\bar{X}_i$ be the sample mean from Π_i based on a sample of size n. We know that $\bar{X}_i$ is distributed as a normal variable with mean μ_i and variance n^{-1}, so that μ_i is a location parameter for the density of $\bar{X}_i$. In fact, the pdf of $\bar{X}_i$ is $n^{\frac{1}{2}} \phi\{(x - \mu_i) n^{\frac{1}{2}}\}$ where $\phi(\cdot)$ is the density function of standard normal distribution. From (14) we need to find a c-value such that $0 < c < \delta_2^*$ and

(35) $$\int_{-\infty}^{\infty} \Phi^{k-1}(x + cn^{\frac{1}{2}}) \phi(x)\, dx = P^*.$$

It is clear that the integral (35) tends to one as n tends to infinity. Thus, the required c-value exists provided n is sufficiently large. The c-values can be obtained from Table I of [1]. It is of some interest to get the expressions for $\sup E(\eta \mid \boldsymbol{\theta})$ and

$\inf E(\zeta \mid \boldsymbol{\theta})$. Using (19) and (30) we obtain

(36) $$\sup_{\boldsymbol{\theta} \in \Omega} E(\eta \mid \boldsymbol{\theta}) = (k-1)\int_{-\infty}^{\infty} \Phi^{k-2}\{x + n^{\frac{1}{2}}(\delta_2^* - c)\}\Phi(x - n^{\frac{1}{2}}c)\phi(x)\,dx,$$

and

(37) $$\inf_{\boldsymbol{\theta} \in \Omega} E(\zeta \mid \boldsymbol{\theta}) = \int_{-\infty}^{\infty} \Phi^{k-1}\{x + n^{\frac{1}{2}}(\delta_1^* + \delta_2^* - c)\}\phi(x)\,dx.$$

It may be noted that as $n \to \infty$

(38) $$\sup_{\boldsymbol{\theta} \in \Omega} E(\eta \mid \boldsymbol{\theta}) \to 0 \quad \text{and} \quad \inf_{\boldsymbol{\theta} \in \Omega} E(\zeta \mid \boldsymbol{\theta}) \to 1.$$

The integral (except for the factor $k-1$) in (36) has been tabulated by Deely and Gupta [2].

Now let us consider the problem in relation to the scale parameters of the populations. $\Pi_1, \cdots, \Pi_k$ with gamma distributions. The distribution associated with Π_i has the density function $g(x \mid \alpha, \theta_i)$ where

(39) $$g(x \mid \alpha, \theta_i) = [\theta_i^\alpha \Gamma(\alpha)]^{-1} x^{\alpha-1} e^{-x/\theta_i} \quad \text{for} \quad x > 0; \qquad = 0 \quad \text{otherwise.}$$

We assume that the shape parameter α is the same for all populations and the common value of this parameter is known. Let X_i be the sample sum from Π_i based on n observations. It has gamma distribution with scale parameter θ_i and shape parameter $n\alpha$. We base our procedure on X_i. Now using (31), we obtain that the required c-value is the value satisfying the equation

(40) $$\int_{-\infty}^{\infty} G^{k-1}(cx \mid n\alpha, 1)\,dG(x \mid n\alpha, 1) = P^*,$$

where G is the cdf corresponding to the density function g. Tables of Gupta [3] can be used to find the required c-values. Here

(41) $$\sup_{\boldsymbol{\theta} \in \Omega} E(\eta \mid \boldsymbol{\theta}) = (k-1)\int_0^\infty G^{k-2}(x\delta_2^*/c \mid n\alpha, 1)G(x/c \mid n\alpha, 1)\,dG(x \mid n\alpha, 1),$$

and

(42) $$\inf_{\boldsymbol{\theta} \in \Omega} E(\zeta \mid \boldsymbol{\theta}) = \int_0^\infty G^{k-1}(x\delta_1^*\delta_2^*/c \mid n\alpha, 1)\,dG(x \mid n\alpha, 1).$$

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