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A MULTI-PARAMETER GAUSSIAN PROCESS[1]

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1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let A be the p-dimensional unit rectangle ($p \geqq 2$). We denote by $(A, \mathscr{A}, \mu_p)$ the ordinary Lebesgue measure space. Let $\{X(u, \omega): u \in A\}$ be a Gaussian process defined on $(\Omega, \mathscr{F}, P)$ with the properties:

$$X(u, \omega) = 0 \quad \text{a.s.} \quad \text{for every} \quad u \quad \text{in} \quad A_0 \quad \text{where} \tag{1.1}$$

$$A_0 = \{(u_1, \cdots, u_p) \in A : u_j = 0 \ \text{for some } j \text{ with } 1 \leqq j \leqq p\}.$$

$$E[X(u, \omega)] = \int_\Omega X(u, \omega)\, dP(\omega) = 0 \quad \text{for every} \quad u \quad \text{in} \quad A. \tag{1.2}$$

$$E[X(u, \omega)X(v, \omega)] = \min(u_1, v_1) \cdots \min(u_p, v_p) = R(u, v) \tag{1.3}$$

$$\text{for every} \quad u = (u_1, \cdots, u_p) \quad \text{and} \quad v = (v_1, \cdots, v_p) \quad \text{in} \quad A.$$

By considering an expansion in terms of Haar functions on A, it is shown that $X(u, \omega)$ can be realized in the space $C(A)$ of real continuous functions on A which vanish at A_0, i.e.

(1.4) Almost all sample functions of $X(u, \omega)$ are continuous.

For $p = 2$, the existence of the above Gaussian process $X(u, \omega)$ is shown by Yeh [15] and Kuelbs [10]. We will call a Gaussian process $X(u, \omega)$ with the properties (1.1)–(1.4) the p-parameter Gaussian process. We then examine the interrelationship between the p-parameter Gaussian process and its reproducing kernel Hilbert space $H(R)$. Let $L^2(A)$ denote the space of Lebesgue square-integrable functions on A with an inner product $(f, g) = \int_A f(u)g(u)\, d\mu_p(u)$ and norm $\| \ \|$. We also define a stochastic integral $I(f) = \int_A f(u)\, dX(u, \omega)$ for $f \in L^2(A)$ with respect to the p-parameter Gaussian process in two different ways and show that they are identical. From this we show that the p-parameter Gaussian process has an a.s. uniformly convergent orthonormal expansion.

Defining a Gaussian random set function by

$$X(F, \omega) = \int_A 1_F(t)\, dX(t, \omega) \tag{1.5}$$

where $F \in \mathscr{A}$ and 1_F is the indicator function of F, we define the multiple Wiener integral (see Itô [6]) and show that any L^2-functional of the process has an orthogonal representation.

By appealing to the results obtained by Parzen [13], Kallianpur [9] and Oodaira [11], we can simply deduce the results: a translation theorem, equivalence of

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Gaussian measures and a zero-one law for the r-module of the p-parameter Gaussian process, which are generalizations of some known results due to Park [12], Shepp [14] and Yeh [16].

2. Haar functions on $L^2(A)$ and construction of the process. Let $\{\bar{g}_{n,j}\}$ denote the Haar functions on $L^2(I)$, where $I = [0, 1]$, i.e.

$$\bar{g}_{0,0} \equiv 1$$

$$\begin{aligned} \bar{g}_{n,j}(s) &= 2^{(n-1)/2} && \text{if } s \in [j/2^{n-1}, j+\tfrac{1}{2}/2^{n-1}), \\ &= -2^{(n-1)/2} && \text{if } s \in [j+\tfrac{1}{2}/2^{n-1}, j+1/2^{n-1}), \\ &= 0 && \text{otherwise}; \end{aligned} \tag{2.1}$$

for $n = 1, 2, \cdots$ and $j = 0, 1, \cdots, 2^{n-1}-1$.

It is shown by Haar [5] that $\{\bar{g}_{n,j}\}$ is a complete orthonormal system (C.O.N.S.) in $L^2(I)$. We shall use the following notations:

$u = (u_1, \cdots, u_p)$ for $u \in A$.

$1_u =$ the indicator function of $[0, u_1] \times \cdots \times [0, u_p]$.

$D =$ set of all p-tuples $n = (n_1, \cdots, n_p)$ with nonnegative integers $n_i (i = 1, 2, \cdots, p)$.

$|n| = n_1 + \cdots + n_p$ for $n = (n_1, \cdots, n_p)$ in D.

$S_n = \{j = (j_1, \cdots, j_p): 0 \leqq j_i \leqq 2^{n_i - 1} - 1, i = 1, 2, \cdots, p\}$ for a fixed $n = (n_1, \cdots, n_p)$ in D.

For $u \in A$, $n \in D$ and $j \in S_n$, define

$$g_{n,j}(u) = \bar{g}_{n_1,j_1}(u_1) \cdots \bar{g}_{n_p,j_p}(u_p) \tag{2.2}$$

where $\{\bar{g}_{n,j}\}$ are Haar functions on $L^2(I)$. Then it is easy to see that $\{g_{n,j}\}$ $(n \in D, j \in S_n)$ is a C.O.N.S. in $L^2(A)$.

For each $n \in D$ and $j \in S_n$, let us define

$$G_{n,j}(u) = (1_u, g_{n,j}) \qquad (u \in A) \tag{2.3}$$

where $g_{n,j}$ is given by (2.2). It follows clearly from the definition that $G_{n,j}$ is a continuous function on A.

Now we shall define a multi-parameter Gaussian process. Let $\{y_{n,j}\}$ $(n \in D, j \in S_n)$ be mutually independent $N(0, 1)$ random variables defined on $(\Omega, \mathscr{F}, P)$, where $N(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2.

Consider the series

$$\textstyle\sum_{N=0}^{\infty} \sum_{|n|=N} \sum_{j \in S_n} y_{n,j}(\omega) G_{n,j}(u) \qquad (\omega \in \Omega, u \in A) \tag{2.4}$$

where $\sum_{|n|=N}$ means summing over all possible $n \in D$ with $|n| = N$.

THEOREM 1. *The series* (2.4) *converges uniformly in* $u \in A$ *with probability one.*

PROOF. Let

$$f_N(u, \omega) = \sum_{|n|=N} \sum_{j \in S_n} y_{n,j}(\omega) G_{n,j}(u) \tag{2.5}$$

and

$$Y_N(\omega) = \max_{|n|=N, j \in S_n} |y_{n,j}(\omega)| \tag{2.6}$$

for $N = 0, 1, 2, \cdots$; then

$$\max_{u \in A} |f_N(u, \omega)| \leqq Y_N(\omega) \binom{N+p-1}{p-1} 2^{-(N+p)/2}, \tag{2.7}$$

since for a fixed $n \in D$, $G_{n,j}$ are non-overlapping for different $j \in S_n$, $\max_{u \in A} |G_{n,j}(u)| \leqq 2^{-(|n|+p)/2}$ and the number of terms in the summation $\sum_{|n|=N}$ is $\binom{N+p-1}{p-1}$.

Let $a_N > 0$. Then since $y_{n,j}$ are mutually independent $N(0, 1)$ random variables,

$$\begin{aligned} P\{\omega: Y_N(\omega) > a_N\} &\leqq \binom{N+p-1}{p-1} 2^{N-p} (2\pi)^{-\frac{1}{2}} 2 \int_{a_N}^{\infty} \exp(-s^2/2)\, ds \\ &\leqq \binom{N+p-1}{p-1} 2^{N-p} (2\pi)^{-\frac{1}{2}} 2 a_N^{-(2p+2)} \\ &\quad \cdot \exp(-a_N^2/4) \int_{a_N}^{\infty} s^{2p+2} \exp(-s^2/4)\, ds. \end{aligned} \tag{2.8}$$

Choosing $a_N = 2[(N+p-1)\ln 2]^{\frac{1}{2}}$, it is easy to check that

$$P\{\omega: Y_N(\omega) > 2[(N+p-1)\ln 2]^{\frac{1}{2}}\} \geqq C_p (N+p-1)^{-2}. \tag{2.9}$$

where C_p is a finite constant which does not depend on N. Therefore

$$\sum_{N=0}^{\infty} P\{\omega: Y_N(\omega) > 2[(N+p-1)\ln 2]^{\frac{1}{2}}\} < \infty:$$

hence by the Borel–Cantelli lemma,

$$P\{\omega: Y_N(\omega) > 2[(N+p-1)\ln 2]^{\frac{1}{2}}, \text{ infinitely often}\} = 0 \tag{2.10}$$

i.e.,

$$P\{\omega: Y_N(\omega) \leqq 2[(N+p-1)\ln 2]^{\frac{1}{2}}, \text{ for all } N \geqq N_0(\omega)\} = 1. \tag{2.11}$$

Now $\sum_{N=N_0(\omega)} \max_{u \in A} |f_N(u, \omega)| < \infty$ follows from (2.7) and (2.11). Consequently $\sum_{N=0}^{\infty} \max_{u \in A} |f_N(u, \omega)|$ converges a.s., i.e. the series (2.4) converges uniformly in $u \in A$ with probability one. This completes the proof of the theorem.

The above result is a generalization of a similar theorem proved by Ciesielski [3] for the standard Wiener process.

Let Ω^* denote the set of ω in Ω such that the series (2.4) converges uniformly in $u \in A$. We note that $P(\Omega^*) = 1$. We now define a stochastic process

$$\begin{aligned} X(u, \omega) &= \sum_{N=0}^{\infty} \sum_{|n|=N} \sum_{j \in S_n} y_{n,j}(\omega) G_{n,j}(u) \quad (u \in A) \quad \text{if } \omega \in \Omega^* \\ X(u, \omega) &= 0 \quad \text{for every } u \in A \quad \text{if } \omega \notin \Omega^*. \end{aligned} \tag{2.12}$$

The following corollary is obvious.

COROLLARY 1. *All sample functions of the stochastic process* $\{X(u, \omega): u \in A\}$ *defined by* (2.12) *are continuous.*

THEOREM 2. *The stochastic process* $\{X(u, \omega): u \in A\}$ *defined by* (2.12) *is Gaussian with mean function zero and covariance function*

$$R(u, v) = \min(u_1, v_1) \cdots \min(u_p, v_p) \qquad (u, v \in A) \tag{2.13}$$

and satisfies

$$X(u, \omega) = 0 \quad \text{a.s.} \quad \textit{for every} \quad u \in A_0. \tag{2.14}$$

PROOF. It follows immediately from the definition of the process that $\{X(u, \omega): u \in A\}$ is a Gaussian process with mean function zero, and (2.14) holds. Denote the process $X(u, \omega)$ simply as

$$X(u, \omega) = \sum_{j=0}^{\infty} y_j(\omega) G_j(u) \quad \text{a.s.} \tag{2.15}$$

Since $y_j(\omega)G_j(u)$ are mutually independent $N(0, G_j^2(u))$ random variables and $\sum_{j=0}^{\infty} G_j^2(u) < \infty$, we have

$$E[X(u, \omega) - \sum_{j=0}^{n} y_j(\omega) G_j(u)]^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Now it is easy to see that

$$\begin{aligned} E[X(u, \omega)X(v, \omega)] \\ = \lim_{n\to\infty} E[\{\textstyle\sum_{j=0}^{n} y_j(\omega)G_j(u)\}\{\sum_{j=0}^{n} y_j(\omega)G_j(v)\}]. \end{aligned} \tag{2.16}$$

By identifying the right-hand side (r.h.s) of (2.16) to

$$\begin{aligned} \lim_{n\to\infty} \textstyle\sum_{j=0}^{n} G_j(u)G_j(v) &= \textstyle\sum_{j=0}^{\infty} (1_u, g_j)(1_v, g_j) \\ &= (1_u, 1_v) = \min(u_1, v_1) \cdots \min(u_p, v_p), \end{aligned}$$

we obtain (2.13) and thus the proof is complete.

It should be pointed out that Theorem 2 and Corollary 1 together imply the existence of the p-parameter Gaussian process defined in the introduction.

3. The realization of the process on $C(A)$. Here we shall relate the results obtained in the previous section to those obtained by Yeh [15].

Let $(C(I^2), \underline{B}(C), m_W)$ be the Wiener space given in [15], where $C(I^2)$ is the space of real-valued continuous functions on I^2 which vanish on $\{(s, t): s = 0 \text{ or } t = 0\}$, $\underline{B}(C)$ is a σ-field generated by the cylinder sets in $C(I^2)$ and m_W is the Wiener measure on $C(I^2)$. This measure space is also constructed by Kuelbs [10] by using different techniques—mainly Prokhorov's weak convergence of measures. The existence of the probability space $(C(I^2), \underline{B}(C), m_W)$ was also shown by N. N. Cencov (*Akademica Nauk SSSR Doklady* **106**, 1956).

The triplet $(C(I^2), \underline{B}(C), m_W)$ can be considered as a stochastic process where the random variables of the process are given by the coordinate variables $x(s, t)$, $x \in C(I^2)$.

THEOREM 3. *The Gaussian process* $(C(I^2), \underline{B}(C), m_W)$ *has mean function zero and covariance function*

$$R((s_1, t_1), (s_2, t_2)) = E_{m_W}[x(s_1, t_1)x(s_2, t_2)] = \min(s_1, t_1)\min(s_2, t_2) \tag{3.1}$$

for (s_1, t_1) *and* (s_2, t_2) *in* I^2.

PROOF. It is easy to check that the mean function is zero. (3.1) follows from the evaluation of the following integral: For $0 < s_1 < s_2$ and $0 < t_1 < t_2$,

$$\begin{aligned}&\int_{C(I^2)} x(s_1, t_1)x(s_2, t_2)\, dm_W(x) \\ &\quad = \{(2\pi)^4[s_1(s_2-s_1)]^2[t_1(t_2-t_1)]^2\}^{-\frac{1}{2}}\int_{-\infty}^{\infty}(4)\int_{-\infty}^{\infty} u_{11}u_{22} \\ &\qquad \cdot \exp\{-\tfrac{1}{2}[u_{11}^2/s_1 t_1 + (u_{12}-u_{11})^2/s_1(t_2-t_1) \\ &\qquad + (u_{21}-u_{11})^2/(s_2-s_1)t_1 + (u_{22}-u_{21}-u_{12}+u_{11})^2/(s_2-s_1)(t_2-t_1)]\} \\ &\qquad \cdot du_{11}\, du_{12}\, du_{21}\, du_{22} = s_1 t_1.\end{aligned}$$

Thus the theorem is proved.

Let $\{X(u, \omega): u \in A\}$ be the p-parameter Gaussian process and let $B(X(u): u \in A)$ be the σ-field generated by the process $X(u, \omega)$ and $\underline{B}(X(u): u \in A)$ or simply $\underline{B}(X)$ be the completion of $B(X(u): u \in A)$ under P. We usually replace the σ-field $\mathscr{F}$ by $\underline{B}(X)$ and consider $(\Omega, \underline{B}(X), P, \{X(u): u \in A\})$ to be the p-parameter Gaussian process. We shall write $L^2(\Omega)$ for $L^2(\Omega, \underline{B}(X), P)$, the Hilbert space of $\underline{B}(X)$-measurable, real-valued functions square integrable with respect to P with an inner product

$$(x, y)_{L^2(\Omega)} = \int_\Omega x(\omega)y(\omega)\, dP(\omega)$$

and norm $\| \|_{L^2(\Omega)}$. From now if it is obvious that $X(u, \omega)$ is a random variable then we may write $X(u)$ instead of $X(u, \omega)$.

Let $B(C(A))$ or simply $B(C)$ denote the σ-field generated by all cylinder sets in $C(A)$. Let us define a map $S: \Omega \to C(A)$ by

$$S(\omega) = X(\cdot, \omega) \quad \text{for} \quad \omega \in \Omega \tag{3.2}$$

and also define a probability measure m_w on $(C(A), B(C))$ by

$$m_w(E) = P\{S^{-1}(E)\} \quad \text{for} \quad E \in B(C). \tag{3.3}$$

Then m_w has the following property for a cylinder set in $C(A)$:

$$m_w\{x \in C(A): [x(u_1), \cdots, x(u_n)] \in F\} = P\{\omega \in \Omega: [X(u_1, \omega), \cdots, X(u_n, \omega)] \in F\} \tag{3.4}$$

where F is a Borel set in n-dimensional Euclidean space R^n and $u_1, \cdots, u_n$ are in A. The probability space $(C(A), \underline{B}(C), m_w)$, where $\underline{B}(C)$ is the completion of $B(C)$ under m_w, is a generalization of $(C(I^2), \underline{B}(C), m_W)$ given by Yeh, since the mean function being zero and the covariance function determines a Gaussian measure uniquely and when $p = 2$ $(C(A), \underline{B}(C), m_w)$ has the same covariance function as Yeh's.

4. A stochastic integral with respect to $X(u, \omega)$. Let $X(u, \omega)$ be the p-parameter Gaussian process. We shall use the following notations:

(4.1) For $u = (u_1, \cdots, u_p)$ and $v = (v_1, \cdots, v_p)$ in A,

$u < v$ mean $u_i < v_i$ for $i = 1, 2, \cdots, p$.

(4.2) $\triangle_{u,v} = \prod_{i=1}^{p} [u_i, v_i)$ for $u, v \in A$ with $u < v$.

(4.3) $V(u, v, k)$ denotes the set of p-tuples $s = (s_1, \cdots, s_p)$ such that each s_i is either u_i or v_i and exactly k of s_i are u_i for $u = (u_1, \cdots, u_p)$ and $v = (v_1, \cdots, v_p)$.

(4.4) $$\triangle_{u,v} X(\omega) = \sum_{k=0}^{p} (-1)^k \sum_{s \in V(u,v,k)} X(s, \omega) \qquad (u < v).$$

(4.5) $\bar{V}(n, j, k) = V(j+\frac{1}{2}/2^{n-1}, j+1/2^{n-1}, k)$ for $n \in D$ and $j \in S_n$,

where V in the r.h.s. is given by (4.3) and,

for $n = (n_1, \cdots, n_p)$ and $j = (j_1, \cdots, j_p)$,

$j+\frac{1}{2}/2^{n-1} = (j_1+\frac{1}{2}/2^{n_1-1}, \cdots, j_p+\frac{1}{2}/2^{n_p-1})$ and

$j+1/2^{n-1} = (j_1+1/2^{n_1-1}, \cdots, j_p+1/2^{n_p-1})$.

(4.6) $\bar{V}(n, j) = \bigcup_{k=0}^{p} \bar{V}(n, j, k)$ for $n \in D$ and $j \in S_n$.

(4.7) For $s \in \bar{V}(n, j)$ with $s = (r_1/2^{n_1-1}, \cdots, r_p/2^{n_p-1})$

$s-\frac{1}{2}$ means $(r_1-\frac{1}{2}/2^{n_1-1}, \cdots, r_p-\frac{1}{2}/2^{n_p-1})$.

For any $n \in D$ let $0 = u_i^0 < u_i^1 < \cdots < u_i^{n_i} \leqq 1$ $(i = 1, \cdots, p)$ be a partition of A.

(4.8) $S_n^* = \{j = (j_1, \cdots, j_p): 1 \leqq j_i \leqq n_i \text{ for } \iota = 1, 2, \cdots, p\}$.

(4.9) $A_n = \{u^j = (u_1^{j_1}, \cdots, u_p^{j_p}) \in A : j = (j_1, \cdots, j_p) \in S_n^*\}$.

(4.10) If $j = (j_1, \cdots, j_p)$ in S_n^*, then $j-1 = (j_1-1, \cdots, j_p-1)$.

(4.11) $$\bar{y}_{n,j}(\omega) = 2^{(|n|-p)/2} \sum_{k=0}^{p} (-1)^k \sum_{s \in \bar{V}(n,j,k)} \triangle_{s-\frac{1}{2},s} X(\omega)$$

for $n \in D$ and $j \in S_n$.

LEMMA 1. *For u, v, s and t in A with $u < v$ and $s < t$*

$$E[\triangle_{u,v} X(\omega) \cdot \triangle_{s,t} X(\omega)] = \mu_p(\triangle_{u,v} \cap \triangle_{s,t}).$$

LEMMA 2. $\{\bar{y}_{n,j}(\omega)\}$ $(n \in D, j \in S_n)$ *are* $\underline{B}(X)$*-measurable and mutually independent* $N(0, 1)$ *random variables.*

The proofs of Lemma 1 and Lemma 2 are purely computational and they are omitted here. Now we shall define a stochastic integral with respect to the p-parameter Gaussian process $X(u, \omega)$, denoted by $I(f) = \int_A f(u)\, dX(u)$ for $f \in L^2(A)$.

We first define $I(f)$ when f is a step function. If

$$f(u) = c_j \quad \text{for} \quad u \in \triangle_{u^{j-1},u^j} \qquad (u^j \in A_n) \tag{4.12}$$

where $\triangle_{u^{j-1},u^j}$ and a partition A_n are given by (4.2) and (4.9), then we define

$$I(f) = \int_A f(u)\, dX(u) = \sum_{j \in S_n^*} c_j \triangle_{u^{j-1},u^j} X(\omega) \tag{4.13}$$

where S_n^* is given in (4.8).

In fact, we shall accept as $I(f)$ any random variable equal almost surely to the sum on the right. As defined in (4.13) the integral is determined uniquely by f, neglecting $I(f)$ values on a set of zero probability. For each step function f, $I(f)$ is clearly $\underline{B}(X)$-measurable. Let g be a step function of the same type as (4.12).

LEMMA 3. *The stochastic integral satisfies*:

$$I(af+bg) = aI(f)+bI(g) \tag{4.14}$$

for real numbers a and b;

$$E[I(f)\cdot I(g)] = (f, g), \tag{4.15}$$

$$I(1_u) = X(u) \quad \text{a.s.}, \tag{4.16}$$

$$I(g_{n,j}) = \bar{y}_{n,j} \quad \text{a.s.}, \qquad (n \in D, j \in S_n) \tag{4.17}$$

where $g_{n,j}$ are Haar functions as defined in (2.2) *and $\bar{y}_{n,j}$ as defined in* (4.11).

PROOF. (4.14), (4.15) and (4.16) are trivial. To show (4.17) let $n \in D$ and $j \in S_n$ be fixed. Since

$$g_{n,j}(u) = (-1)^k 2^{(|n|-p)/2} \quad \text{if} \quad u \in \triangle_{s-\frac{1}{2},s} \quad \text{and} \quad s \in \bar{V}(n,j,k) \qquad \text{for } k = 0, 1, \cdots, p$$
$$= 0 \qquad \text{otherwise};$$

and since $\{\triangle_{s-\frac{1}{2},s}\}$ $(s \in \bar{V}(n,j))$ are mutually disjoint, we obtain by the definition of the stochastic integral (4.13)

$$I(g_{n,j}) = 2^{(|n|-p)/2} \sum_{k=0}^{p} (-1)^k \sum_{s \in \bar{V}(n,j,k)} \triangle_{s-\frac{1}{2},s} X(\omega)$$
$$= \bar{y}_{n,j} \quad \text{a.s.}$$

This completes the proof of the lemma.

From the property (4.15), it follows that

$$E[\{I(f)\}^2] = ||f||^2 \tag{4.18}$$

and this implies that the correspondence between f and $I(f)$ is an isometry between $L^2(A)$ and $L^2(\Omega)$. Now suppose that f is a limit (in $||\ ||$ norm) of a sequence $\{f_n\}$ of step functions of the above type. Then l.i.m. $I(f_n)$ exists defining a random variable Y. This random variable, as a limit in the mean (in $||\ ||_{L^2|\Omega|}$ norm), is defined uniquely, neglecting values on a set of zero probability. Also Y is independent of the particular sequence $\{f_n\}$ chosen, since two sequences converging to f in $||\ ||$

norm can be combined to form a single sequence converging to f in $\| \ \|$ norm. We define $\int_A f(u)\,dX(u)$ as the limit obtained in this way. For $f \in L^2(A)$ $I(f)$ can be taken to be $\underline{B}(X)$-measurable. Now since the family of step functions on A is dense in $L^2(A)$, we have defined a stochastic integral

$$I(f) = \int_A f(u)\,dX(u) \quad \text{for} \quad f \in L^2(A), \tag{4.19}$$

which satisfies all properties listed in Lemma 3.

We shall show now that the stochastic integral (4.19) can also be defined by a different method. Let $\{g_{n,j}\}$ $(n \in D, j \in S_n)$ be the Haar functions on A and $\{\bar{y}_{n,j}\}$ be a sequence of random variables given in (4.11). Consider the series

$$\textstyle\sum_{N=0}^{\infty} \sum_{|n|=N} \sum_{j \in S_n} (f, g_{n,j})\bar{y}_{n,j} \quad \text{for} \quad f \in L^2(A). \tag{4.20}$$

The above series converges a.s. by the Three Series Theorem since

$$\textstyle\sum_{N=0}^{\infty} \sum_{|n|=N} \sum_{j \in S_n} E[\{(f, g_{n,j})\bar{y}_{n,j}\}^2] = \|f\|^2 < \infty. \tag{4.21}$$

Let us write

$$I(f) = \textstyle\sum_{N=0}^{\infty} \sum_{|n|=N} \sum_{j \in S_n} (f, g_{n,j})\bar{y}_{n,j} \quad \text{for} \quad f \in L^2(A). \tag{4.22}$$

We shall accept as $I(f)$ any random variable equal almost surely to the series on the right.

THEOREM 4. *The stochastic integral defined by* (4.22) *satisfies*:

$$I(af + bg) = aI(f) + bI(g) \tag{4.23}$$

for real numbers a and b and $f, g \in L^2(A)$.

$$E[I(f) \cdot I(g)] = (f, g). \tag{4.24}$$

$$I(f) \quad \textit{is an} \quad N(0, \|f\|^2) \quad \textit{random variable.} \tag{4.25}$$

$$I(g_{n,j}) = \bar{y}_{n,j} \tag{4.26}$$

where $g_{n,j}$ *are Haar functions on* A.

PROOF. (4.23) and (4.26) are obvious from the definition of the stochastic integral. (4.25) follows easily from (4.21) and (4.24) is obtained immediately by using the Parseval's Theorem.

LEMMA 4. *Let* $I_1(f)$ *and* $I_2(f)$ *denote the stochastic integral defined by* (4.19) *and* (4.22) *respectively for* $f \in L^2(A)$. *Then*

$$I_1(f) = I_2(f) \quad \text{a.s.} \tag{4.27}$$

PROOF. From (4.17) and (4.26), $I_1(g_{n,j}) = \bar{y}_{n,j} = I_2(g_{n,j})$ a.s. Since $\{g_{n,j}\}$ is a C.O.N.S. in $L^2(A)$, the lemma follows from the isometry given by (4.18) and (4.24).

COROLLARY 2. *Let*

$$\bar{X}(u, \omega) = \sum_{N=0}^{\infty} \sum_{|n|=N} \sum_{j \in S_n} \bar{y}_{n,j} G_{n,j}(u) \qquad (u \in A) \tag{4.28}$$

where $G_{n,j}(u) = (1_u, g_{n,j})$.

Then $\bar{X}(u, \omega) = X(u, \omega)$ a.s. *for every* $u \in A$.

PROOF. Since $\bar{X}(u, \omega) = I_2(1_u)$ a.s. from (4.22) and $X(u, \omega) = I_1(1_u)$ a.s. from (4.16), the corollary follows immediately from Lemma 4.

The above corollary gives an orthonormal expansion of the p-parameter Gaussian process. Furthermore, by Theorem 1 this expansion converges uniformly a.s.

We define a random set function

$$X(F, \omega) = \int_A 1_F(u)\, dX(u, \omega) \quad \text{a.s.} \tag{4.29}$$

for $F \in \mathscr{A}$, where 1_F is the indicator function of F. Then clearly from (4.24)

$$E[X(F, \omega)X(F^*, \omega)] = (1_F, 1_{F^*}) = \mu_p(F \cap F^*) \tag{4.30}$$

for $F, F^* \in \mathscr{A}$. Thus $X(F, \omega)$ is a normal random measure on Lebesgue measure space $(A, \mathscr{A}, \mu_p)$. According to Itô [6] it is now possible to define the multiple Wiener integral with respect to the normal random measure $X(F, \omega)$

$$I_q(f_q) = \int_{A^{(q)}} \int_A f_q(u_1, \cdots, u_q)\, dX(u_1) \cdots dX(u_q) \tag{4.31}$$

for $f_q \in L^2(A^q)$ and positive integer $q \geqq 1$.

5. The closed linear subspace $L^*(X(u): u \in A)$**.** Let $L^*(X(u): u \in A)$ or simply $L^*(X)$ denote the closed linear subspace in $L^2(\Omega)$ spanned by all finite linear combinations of the form $\sum_{i=1}^{n} c_i X(u_i)$ where c_i's are real numbers, $u_i \in A$, and $X(u, \omega)$ is the p-parameter Gaussian process. Let $H(R)$ denote the reproducing kernel Hilbert space with the reproducing kernel R, where R is the covariance function of the process $X(u, \omega)$, with the inner product $\langle\, ,\rangle_{H(R)}$ and norm $\|\ \|_{H(R)}$ (see [1], [13]).

THEOREM 5.

$$H(R) \cong_J L^*(X) \quad \textit{with} \quad J(R(\cdot, u)) = X(u) \qquad (u \in A). \tag{5.1}$$

Furthermore, if $f \in H(R)$ *and* $J(f) = \xi$, *then*

$$f(u) = E[\xi \cdot X(u)] \quad \textit{for every} \quad u \in A. \tag{5.2}$$

Here $H(R) \cong_J L^*(X)$ means that J is a congruence (inner product preserving isomorphism) from $H(R)$ onto $L^*(X)$.

PROOF. First we note that for each $f \in H(R)$ there corresponds a unique $\xi \in L^*(X)$ such that $f(u) = E[\xi X(u)]$ for $u \in A$, since if ξ and η in $L^*(X)$ both correspond to f in $H(R)$ then $E[(\xi - \eta)X(u)] = 0$ for every $u \in A$, hence $\xi = \eta$ a.s. as $X(u)$, $(u \in A)$

span $L^*(X)$. Let J denote such a map, then clearly

$$J(R(\cdot, u)) = X(u) \quad \text{for every} \quad u \in A, \tag{5.3}$$

and

$$\langle R(\cdot, u), R(\cdot, v)\rangle_{H(R)} = R(u, v) = E[X(u)X(v)] \qquad (u, v \in A). \tag{5.4}$$

Now $R(\cdot, u)$, $(u \in A)$ span $H(R)$ and $f(u) = \langle f, R(\cdot, u)\rangle_{H(R)} = (\xi, X(u)) = E[\xi X(u)]$, hence the Basic Congruence Theorem (see [13]) implies (5.1). This completes the proof.

Let $L^*(1_u\colon u \in A)$ denote the closed linear space spanned by the elements of the form $\sum_{i=1}^n c_i 1_{u_i}(\cdot)$ for real numbers $c_1, \cdots, c_n$ and $u_1, \cdots, u_n$ are in A.

THEOREM 6.

$$H(R) \cong_{J_1} L^*(1_u\colon u \in A) = L^2(A) \tag{5.5}$$

and

$$J_1(R(\cdot, u)) = 1_u(\cdot) \quad \textit{for each} \quad u \in A.$$

Furthermore, if $f \in H(R)$ *and* $J_1(f) = g$, *then*

$$f(u) = (1_u, g) \quad \textit{for each} \quad u \in A. \tag{5.6}$$

The proof is similar to the proof of Theorem 5.

THEOREM 7.

$$L^*(X) = \{I(f)\colon f \in L^2(A)\}. \tag{5.7}$$

PROOF. Clearly $I(f) \in L^*(X)$ for each f in $L^2(A)$ by the definition of the stochastic integral $I(f)$. Let $\xi \in L^*(X)$. Then by (5.2) there is an $f \in H(R)$ corresponding to ξ with $f(u) = E[\xi X(u)]$ and in turn there exists g in $L^2(A)$ corresponding to f such that $f(u) = (g, 1_u)$ by (5.6). Let $\eta = I(g)$, then for each $u \in A$ $f_\eta(u) = E[\eta X(u)] = E[I(g)X(u)] = (g, 1_u) = f(u)$, i.e. $E[I(g)X(u)] = E[\xi X(u)]$ for every $u \in A$. Hence $\xi = I(g)$ a.s. and $\xi \in \{I(f)\colon f \in L^2(A)\}$.

COROLLARY 3. *Let* ξ *be in* $L^*(X)$. *Then there exists a function* g *in* $L^2(A)$ *such that*

$$\xi = \int_A g(u)\, dX(u, \omega) \quad \text{a.s.} \tag{5.8}$$

Furthermore g can be found under the congruences J and J_1 given by (5.1) and (5.5) respectively.

6. Applications. In this section we shall simply deduce a few results regarding the p-parameter Gaussian process from the results in [6], [9], [11] and [13].

Let $(X, B(X))$ be the measurable space where X is the space of real-valued continuous functions on A and $B(X)$ is the σ-field generated by the cylinder sets in X. Let $(X, \underline{B}(X), P)$ be the p-parameter Gaussian process with the mean function zero and the covariance R where $\underline{B}(X)$ is the completion of $B(X)$ under P.

(A) *A translation theorem.* For $m \in X$, the transformation $\sigma_m\colon X \to X$ defined by

$$\sigma_m x = x + m \tag{6.1}$$

clearly sends $B(X)$-measurable set into $B(X)$-measurable set. The probability measure P_m given by

$$P_m(E) = P(\sigma_m^{-1}E) \quad \text{for} \quad E \in B(X) \tag{6.2}$$

is Gaussian with the mean function m and the same covariance function R as P. By a direct application of a result from [13] the following theorem is obtained:

THEOREM 8. *$P_m \equiv P$ relative to $B(X)$ if and only if $m \in H(R)$. If $m \in H(R)$, then the Radon–Nikodym derivative is given by*

$$\frac{dP_m}{dP}(x) = \exp\{u_m(x) - \tfrac{1}{2}\|m\|^2_{H(R)}\} \tag{6.3}$$

where $u_m(x)$ is in $L^*(X)$ which corresponds to $m \in H(R)$ under the congruence of (5.1).

The notation $P_m \equiv P$ means that P_m and P are mutually absolutely continuous.

Now from Corollary 3 and (5.6) there exists g in $L^2(A)$ which corresponds to m in $H(R)$ such that $u_m(x) = \int_A g(u)\,dx(u)$ and $\|m\|_{H(R)} = \|g\|$. Hence the Radon–Nikodym derivative in (6.3) becomes

$$\frac{dP_m}{dP}(x) = \exp\left\{\int_A g(u)\,dx(u) - \tfrac{1}{2}\|g\|^2\right\}. \tag{6.4}$$

For $p = 2$ similar results are obtained in Park [12] and Yeh [16] but their approaches differ from ours.

(B) *Equivalence of Gaussian measures.* Let $(X, B(X), Q)$ be a Gaussian measure space with the mean function m and the covariance function Γ_Q. Then we can deduce the following theorem from a result of Oodaira [11] using the same method as Kailath [7].

THEOREM 9. *$Q \equiv P$ relative to $B(X)$ if and only if there is a symmetric kernel $K \in L^2(A \times A)$ such that*

$$\Gamma_Q(u, v) = R(u, v) - \int_{A \times A} 1_u(s) 1_v(t) K(s, t)\,d\mu_p(s)\,d\mu_p(t), \tag{6.5}$$

$$1 \notin \sigma(K) \quad \textit{and} \tag{6.6}$$

$$m \in H(R) \tag{6.7}$$

where $\sigma(K)$ denotes the spectrum of the operator K given by

$$(Kf)(u) = \int_A K(s, u) f(s)\,d\mu_p(s), \qquad u \in A \quad \textit{and} \quad f \in L^2(A). \tag{6.8}$$

We note that if $m \in H(R)$ there is a function k in $L^2(A)$ such that $m(u) = \int_A 1_u(s) k(s)\,d\mu_p(s)$, by Theorem 6. Let λ_j and φ_j $(j = 1, 2, \cdots)$ be eigenvalues and eigenfunctions respectively of the operator K and let $D(\cdot)$ be the Fredholm determinant of K, i.e.

$$D(\lambda) = \prod_{j=1}^{\infty} (1 - \lambda\lambda_j). \tag{6.9}$$

For each value of λ for which $\lambda^{-1} \notin \sigma(K)$ there exists a unique kernel $H_\lambda \in L^2(A \times A)$ called the Fredholm resolvent of K at λ satisfying

(6.10) $$H_\lambda - K = \lambda H_\lambda K = \lambda K H_\lambda.$$

We denote H_1 by H.

THEOREM 10. *If $Q \equiv P$, then*

(6.11) $$\frac{dQ}{dP}(x) = D(1)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}I_2(H) - \tfrac{1}{2}\int_A H(s,s)\,d\mu_p(s) - \tfrac{1}{2}\|k\| - \tfrac{1}{2}(Hk, k) + I(k) + \int_{A\times A} H(s,t)k(s)\,d\mu_p(s)\,dx(t)\right\}$$

where $I_2(\cdot)$ is the 2nd degree multiple Wiener integral.

PROOF. Define

(6.12) $$\xi_j = \textstyle\int_A \varphi_j(s)\,dx(s).$$

Then clearly ξ_j are independent $N(0, 1)$ random variables under P and independent $N(k_j, 1-\lambda_j)$ random variables under Q where $k_j = (k, \varphi_j)$.

Let

(6.13) $$F_j(x) = (1-\lambda_j)^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}(\xi_j - k_j)^2/(1-\lambda_j)\right]/\exp\left[-\tfrac{1}{2}\xi_j^2\right],$$

then it can be shown by using the same method as Shepp [14]:

(6.14) $$F(x) = \textstyle\prod_{j=1}^{\infty} F_j(x) \quad \text{converges} \quad \text{a.s.}\,(P),$$

(6.15) $$F(x) = \frac{dQ}{dP}(x),$$

(6.16) $$\textstyle\sum_{j=1}^{\infty} k_j^2/(1-\lambda_j) = (k, k) + (Hk, k),$$

(6.17) $$\textstyle\sum_{j=1}^{\infty} \lambda_j \xi_j^2/(1-\lambda_j) = I_2(H) + \int_A H(s,s)\,d\mu_p(s),$$ and

(6.18) $$\textstyle\sum_{j=1}^{\infty} k_j \xi_j/(1-\lambda_j) = I(k) + \int_{A\times A} H(s,t)k(s)\,d\mu_p(s)\,dx(t).$$

Hence we can obtain (6.11) and the theorem is proved.

Shepp [14] has recently proved the same results for the standard Wiener process.

(C) *A zero-one law.* We shall obtain a zero-one law for the r-module of the p-parameter Gaussian process by a direct application of the result of Kallianpur [9], which we shall state here first.

Let Q be a Gaussian probability measure on the measurable space $(X, B(X))$, where X is a family of real-valued functions $x(\cdot)$ defined on a set T, $B(X)$ is the σ-field generated by the cylinder sets in X, and $\underline{B}(X)$ is its completion under Q. Let Γ_Q denote the covariance function of Q and assume the mean function to be zero.

A subset M of X is called an r-module if for every x_1 and x_2 in M and rational numbers r_1 and r_2, $r_1x_1+r_2x_2 \in M$. Kallianpur [9] shows that if M is a $\underline{B}(X)$-measurable r-module, then $Q(M) = 0$ or 1, under the following general assumptions:

(6.19) T is a complete separable metric space,

(6.20) X is a linear space of functions under the usual operation of addition of functions and multiplication by real scalars,

(6.21) Γ_Q is continuous on $T \times T$,

(6.22) $H(\Gamma_Q) \subset X$.

THEOREM 11. *Let $(X, \underline{B}(X), P)$ be the p-parameter Gaussian process and let M be a $\underline{B}(X)$-measurable r-module. Then $P(M) = 0$ or 1.*

PROOF. The assumptions (6.19)–(6.21) are clearly satisfied by the process $(X, \underline{B}(X), P)$. Let $h \in H(R)$. Then there exists a function $g \in L^2(A)$ such that $h(u) = (1_u, g)$ by (5.6). Now h is clearly continuous and $h(u) = 0$ for $u \in A_0$, i.e., $h \in X$. Thus (6.22) is satisfied and the conclusion follows from [9].

(D) *Homogeneous chaos.* We shall give an orthogonal expansion of the L^2-functional of the p-parameter Gaussian process $(X, \underline{B}(X), P)$. It is easy to see that ξ is a L^2-functional of the normal random measure $X(F, \omega)$ (in the sense of Itô [6]) if and only if $\xi \in L^2(X, \underline{B}(X), P)$. The following theorem is deduced from Itô [6].

THEOREM 12. *Let $\xi \in L^2(X, \underline{B}(X), P)$. Then ξ can be expressible in the form:*

$$\xi = \sum_{q \geqq 0} I_q(f_q) \tag{6.23}$$

where $I_q(\cdot)$ is the multiple Wiener integral, f_q is given by the following orthogonal development

$$\begin{aligned} f_q(u_1, \cdots, u_q) = 2^{\frac{1}{2}} \sum_{n_1+\cdots+n_r=q} \sum_{\lambda_1, \cdots, \lambda_r} a_{\lambda_1 \cdots \lambda_r}^{n_1 \cdots n_r} \\ g_{\lambda_1}(u_1) \cdots g_{\lambda_1}(u_{n_1}) g_{\lambda_2}(u_{n_1+1}) \cdots g_{\lambda_2}(u_{n_1+n_2}) \cdots \\ g_{\lambda_r}(u_{n_1+\cdots+n_{r-1}+1}) \cdots g_{\lambda_r}(u_{n_1+\cdots+n_r}), \end{aligned} \tag{6.24}$$

$\{g_\lambda\}$ is a C.O.N.S. *in $L^2(A)$, and $\{a_{\lambda_1 \cdots \lambda_r}^{n_1 \cdots n_r}\}$ is the Fourier Hermite coefficient given in* [2].

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