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DISTRIBUTIONS OF Z^{γ} AND Z^* FOR COMPLEX Z WITH RESULTS APPLIED TO THE COMPLEX NORMAL DISTRIBUTION[1]

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1. Introduction. Let $Z = Re^{i\Theta}$ be a complex random variable such that the density of (R, Θ) is given by

$$f(r, \theta) = \frac{|bc_4| \, c_2^{c_1}(1-a^2)^{\frac{1}{2}}}{2\pi m \Gamma(c_1)} r^{c_1 c_4 - 1} \lambda^{c_1 c_3 - 1} \exp(-c_2 \lambda^{c_3} r^{c_4}) \tag{1.1}$$

where all parameters are real, $r > 0$, $m\pi/|b| < \theta < m\pi/|b|$, $\lambda = 1 - a \sin(b\theta + \alpha)$, $|a| < 1$, $b \neq 0$, $c_1 > 0$, $c_2 > 0$, $c_4 \neq 0$, m a natural number, and $0 \leqq \alpha < 2\pi$ and where $f(r, \theta)$ is zero otherwise. The generalized Mellin transform (GMT) (see [2]) will be used in order to obtain the distribution of Z^{γ} and Z^* when γ is a nonzero real number and Z^* is the complex conjugate of Z. The density function (1.1) is of special interest due to the importance of certain special cases of the family. In particular: (i) Weibull-uniform, (ii) chi-uniform, (iii) gamma-uniform, (iv) complex normal.

2. The distributions of Z^{γ} and Z^*.

THEOREM 2.1. *The* GMT *of the complex random variable* $Z = Re^{i\Theta}$, *with density of* (R, Θ) *given by* (1.1), *is*

$$h(s, t) = \frac{-(1-a^2)^{\frac{1}{2}} \Gamma(s/c_4 + c_1) \sin(mt\pi/b)}{4\pi m c_2^{s/c_4} \Gamma(c_1) \Gamma(c_3 s/c_4 + 1)} \cdot \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} H_{jl}[(t/b) \sin^j \alpha + ij \sin^{j-1} \alpha \cos \alpha] \tag{2.1}$$

where

$$H_{jl} = \frac{(-1)^{mj} \Gamma(2l + j + c_3 s/c_4 + 1) \Gamma(j/2 + t/2b) \Gamma(j/2 - t/2b) a^{2l+j}}{2^{2l} \Gamma(j+1) \Gamma(j/2 + t/2 + l + 1) \Gamma(j/2 - t/2 + l + 1)} \tag{2.2}$$

and $t \neq 0, \pm b, \pm 2b, \cdots$, and $s/c_4 + c > 0$. [for any integer k, $h(s, kb)$ is evaluated by taking the limit of $h(s, t)$ as $t \to kb$.]

PROOF. By definition, the GMT of Z is

$$h(s, t) = \frac{|bc_4| \, c_2^{c_1}(1-a^2)^{\frac{1}{2}}}{2\pi m \Gamma(c_1)} \int_{-m\pi/|b|}^{m\pi/|b|} \int_0^{\infty} r^{s + c_1 c_4 - 1} \lambda^{c_1 c_3 - 1} \cdot [\exp(it\theta - c_2 \lambda^{c_3} r^{c_4})] \, dr \, d\theta. \tag{2.3}$$

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The transformation $w = c_2\lambda^{c_3}r^{c_4}$, $\phi = b\theta+\alpha$ changes (2.3) into

$$(2.4)\qquad h(s,t) = \frac{(1-a^2)^{\frac{1}{2}}\Gamma(s/c_4+c_1)\,e^{-it\alpha/b}}{2m\pi c_2{}^{s/c_4}\Gamma(c_1)}\int_{\alpha-m\pi}^{\alpha+m\pi}\frac{e^{it\phi/b}\,d\phi}{(1-a\sin\phi)^{1+c_3s/c_4}},$$

provided $s/c_4+c_1 > 0$. An application of the binomial theorem yields

$$(2.5)\qquad h(s,t) = \frac{(1-a^2)^{\frac{1}{2}}\Gamma(s/c_4+c_1)\,e^{-it\alpha/b}}{2m\pi c_2{}^{s/c_4}\Gamma(c_1)}\int_{\alpha-m\pi}^{\alpha+m\pi} e^{it\phi/b} \cdot\left[\sum_{j=0}^{\infty}\frac{\Gamma(c_3\,s/c_4+j+1)a^j\sin^j\phi}{\Gamma(c_3\,s/c_4+1)\Gamma(j+1)}\right]d\phi.$$

Since the series in (2.5) converges uniformly for all ϕ (see [1]) we obtain

$$(2.6)\qquad h(s,t) = \frac{(1-a^2)\Gamma(s/c_4+c_1)\,e^{-it\alpha/b}}{2\pi mc_2{}^{s/c_4}\Gamma(c_1)}\sum_{j=0}^{\infty}\frac{\Gamma(c_3\,s/c_4+j+1)a^j}{\Gamma(c_3\,s/c_4+1)\Gamma(j+1)} \cdot\int_{\alpha-m\pi}^{\alpha+m\pi} e^{it\phi/b}\sin^j\phi\,d\phi.$$

Repeated application of the reduction formula

$$(2.7)\qquad \int e^{ax}\sin^n bx\,dx = \frac{(a\sin bx-nb\cos bx)\,e^{ax}\sin bx}{n^2b^2+a^2} + \frac{n(n-1)}{n^2b^2+a^2}\int e^{ax}\sin^{n-2}bx\,dx, \qquad n = 0,1,2,\cdots$$

yields

$$(2.8)\quad h(s,t) = \frac{(1-a^2)^{\frac{1}{2}}\Gamma(s/c_4+c_1)\sin(mt\pi/b)}{4\pi mc_2{}^{s/c_4}\Gamma(c_1)\Gamma(c_3\,s/c_4+1)} \cdot\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{mj}\Gamma(c_3\,s/c_4+j+1)\Gamma(j/2+t/2b-l)\Gamma(j/2-t/2b-l)}{2^{2l}\Gamma(j-2l+1)\Gamma(j/2+t/2b+1)\Gamma(j/2-t/2b+1)} \cdot a^j\sin^{j-2l-1}\alpha[(t/b)\sin\alpha+i(j-2l)\cos\alpha]$$

where

$$\textstyle\sum_{l=0}^{\infty}a_l = \sum_{l=0}^{j/2}a_l \qquad \text{if } j \text{ is even;}$$
$$\textstyle = \sum_{l=0}^{(j-1)/2}a_l \qquad \text{if } j \text{ is odd.}$$

Since the series in (2.8) is uniformly convergent for all α, we may rearrange terms. Observe

$$(2.9)\qquad \textstyle\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}a_{jl}\,x^{j-2l} = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}a_{2l+j,l}\,x^j.$$

Consequently (2.8) may also be represented as

$$(2.10)\quad h(s,t) = \frac{-(1-a^2)^{\frac{1}{2}}\Gamma(s/c_4+c_1)\sin(mt\pi/b)}{4\pi mc_2{}^{s/c_4}\Gamma(c_1)\Gamma(c_3\,s/c_4+1)} \cdot\textstyle\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}h_{jl}[(t/b)\sin^j\alpha+ij\sin^{j-1}\alpha\cos\alpha]$$

where $s/c_4+c_1>0$ and where

$$(2.11)\qquad H_{jl}=\frac{(-1)^{mj}\Gamma(c_3 s/c_4+j+2l+1)\Gamma(j/2+t/2b)\Gamma(j/2-t/2b)a^{2l+j}}{2^{2l}\Gamma(j+1)\Gamma(j/2+t/2b+l+1)\Gamma(j/2-t/2b+l+1)}.$$

THEOREM 2.2. *If $Z=Re^{i\Theta}$ is a complex random variable such that the density of (R,Θ) is given by* (1.1) *then the density of (Ω,Φ), where $W=Z^*=\Omega e^{i\Phi}$, is*

$$(2.12)\qquad f(\omega,\phi)=\frac{|bc_4|\,c_2^{\,c_1}(1-a^2)^{\frac{1}{2}}}{2m\pi\Gamma(c_1)}\,\omega^{c_1c_2-1}\eta^{c_1c_3-1}\exp(-c_2\eta^{c_3}\omega^{c_4})$$

where $\eta=1+a\sin(b\phi-\alpha)$, $\omega>0$, $-m|b|^{-1}\pi<\phi<m|b|^{-1}\pi$, and the remaining parameters satisfy the restrictions below (1.1). *The function $f(\omega,\phi)$ is zero elsewhere.*

PROOF. The GMT of $W=Z^*$ is

$$(2.13)\qquad h_W(s,t)=h_Z(s,-t)$$
$$=\frac{-(1-a^2)^{\frac{1}{2}}\Gamma(s/c_4+c_1)\sin(mt\pi/b)}{4m\pi c_2^{\,s/c_4}\Gamma(c_1)\Gamma(c_3 s/c_4+1)}$$
$$\cdot\textstyle\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}H_{jl}[(t/b)\sin^j\alpha-ij\sin^{j-1}\alpha\cos\alpha]$$

where H_{jl} is defined by (2.11). Observe that (2.13) is the GMT of the complex random variable $W=\Omega e^{i\Phi}$ where the density of (Ω,Φ) is (1.1) with α replaced by $\pi-\alpha$.

THEOREM 2.3. *If $Z=Re^{i\Theta}$ is a complex random variable such that the density of (R,Θ) is* (1.1) *then the density of (Ω,Φ), where $W=Z^{\gamma}=\Omega e^{i\Phi}$, is*

$$(2.14)\qquad f(\omega,\phi)=\frac{|bc_4|\,c_2^{\,c_1}(1-a^2)^{\frac{1}{2}}}{2m\pi\gamma^2\Gamma(c_1)}\,\omega^{c_1c_4/\gamma-1}\lambda^{c_1c_3-1}\exp(-c_2\lambda^{c_3}\omega^{c_4/\gamma})$$

where $\gamma\neq 0$, $\omega>0$, $-m\pi|\gamma|/|b|<\phi<m\pi|\gamma|/|b|$, $\lambda=1-a\sin(b\phi/\gamma+\alpha)$ and the remaining parameters satisfy the restrictions given after (1.1). *The function $f(\omega,\phi)$ is zero elsewhere.*

PROOF. The GMT of W is

$$(2.15)\qquad h_W(s,t)=h_Z(\gamma s,\gamma t)$$
$$=\frac{-(1-a^2)^{\frac{1}{2}}\Gamma(s\gamma/c_4+c_1)\sin(mt\gamma\pi/b)}{4m\pi c_2^{\,\gamma s/c_4}\Gamma(c_1)\Gamma(c_3\gamma s/c_4+1)}$$
$$\cdot\textstyle\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}G_{jl}[(t\gamma/b)\sin^j\alpha+ij\sin^{j-1}\alpha\cos\alpha]$$

where $s\gamma/c_4+c_1>0$ and

$$(2.16)\qquad G_{jl}=\frac{(-1)^{mj}\Gamma(j+2l+c_3\gamma s/c_4+1)\Gamma(j/2+t\gamma/2b)\Gamma(j/2-t\gamma/2b)a^{2l+j}}{2^{2l}\Gamma(j+1)\Gamma(j/2+t\gamma/2b+l+1)\Gamma(j/2-t\gamma/2b+l+1)}.$$

If we replace c_4 by c_4/γ and b by b/γ in (2.1) and (2.2) then we obtain (2.15) and (2.16). Consequently the density of (Ω,Φ) is given by (2.14).

Observe that the family is closed under power and conjugation. That is, if $Z = Re^{i\Theta}$ is a complex random variable such that the density of (R, Θ) is given by (1.1) then the density of (Ω, Φ), where $W = Z^{\gamma} = \Omega e^{i\Phi}$ or $W = Z^* = \Omega e^{i\Phi}$, is also given by (1.1).

3. Special cases. There are four density functions which are interesting special cases of (1.1). For the first three of these we set $a = \alpha = 0$ in (1.1). The density function is then

$$f(r, \theta) = \frac{|bc_4|\, c_2^{c_1}}{2\pi m \Gamma(c_1)} r^{c_1 c_4 - 1} \exp(-c_2 r^{c_4}) \tag{3.1}$$

where $r > 0$, $-m\pi/|b| < \theta < m\pi/|b|$, $c_1 > 0$, $c_2 > 0$, $c_4 \neq 0$, $b \neq 0$ and m is a natural number. Let $f(r, \theta)$ be zero elsewhere. Observe that r and θ are independent random variables and the density of θ is uniform over $-m\pi/|b| < \theta < m\pi/|b|$.

By taking $c_1 = 1$, $c_2 = \alpha$ and $c_4 = \beta$ in (3.1) we have the product of a Weibull-density function and a uniform density function. We refer to this as a Weibull-uniform density function.

By taking $c_1 = n/2$, $c_2 = n/2\sigma^2$ and $c_4 = 2$ in (3.1) where $\sigma > 0$ we obtain a chi-uniform density function.

If we set $c_1 = \alpha + 1$, $c_2 = 1/\beta$ and $c_4 = 1$ in (3.1) we obtain a gamma-uniform density function.

Observe that each of the above density functions is closed under conjugation while only the Weibull-uniform is closed under powers.

The remaining density function is obtained by setting $c_2 = (\sigma_1^2 + \sigma_2^2)/4\sigma_1^2\sigma_2^2(1-\delta^2)$, $a \sin\alpha = (\sigma_1^2 - \sigma_2^2)/(\sigma_1^2 + \sigma_2^2)$, $a \cos\alpha = 2\delta\sigma_1\sigma_2/(\sigma_1^2 + \sigma_2^2)$, $b = c_4 = m = 2$ and $c_1 = c_3 = 1$ in (1.1). The new parameters satisfy the restrictions $\sigma_1 > 0$, $\sigma^2 > 0$ and $|\delta| < 1$. The reparameterized density function is now transformed to the (x, y) plane by means of the inverse of the transformation $X = R\cos\Theta$, $Y = R\sin\Theta$. We then obtain the bivariate normal density function with zero means, correlation δ and variances σ_1^2 and σ_2^2. Consequently $Z = X = iY$ is a complex normal random variable with zero means, correlation δ and variances σ_1^2 and σ_2^2. Observe that the complex normal density function is closed under conjugation.

4. Some applications. Let Z be a complex normal random variable. The density of (Ω, Φ), where $W = Z^{\gamma} = \Omega e^{i\Phi}$, is

$$f(\omega, \phi) = \frac{\omega^{2/\gamma - 1}}{2\pi\sigma_1\sigma_2(1-\delta^2)^{\frac{1}{2}}\gamma^2} \exp\left\{\frac{-\omega^{2/\gamma}}{2(1-\delta^2)}\left[\frac{\cos^2(\phi/\gamma)}{\sigma_1^2} - \frac{\delta \sin(2\phi/\gamma)}{\sigma_1\sigma_2} + \frac{\sin^2(\phi/\gamma)}{\sigma_2^2}\right]\right\} \tag{4.1}$$

where $\omega > 0$, $\gamma \neq 0$ and $-|\gamma|\pi < \phi < |\gamma|\pi$. This is obtained from Theorem 2.3.

For $\gamma = -1$ we obtain

$$f(\omega, \phi) = \frac{\omega^{-3}}{2\pi\sigma_1\sigma_2(1-\delta^2)^{\frac{1}{2}}} \exp\left[\frac{-\omega^{-2}}{2(1-\delta^2)}\left(\frac{\cos^2\phi}{\sigma_1^2} + \frac{\delta \sin 2\phi}{\sigma_1\sigma_2} + \frac{\sin^2\phi}{\sigma_2^2}\right)\right] \tag{4.2}$$

where $\omega > 0$ and $-\pi < \phi < \pi$. In the (u, v) plane we have

$$(4.3)\qquad g(u, v) = \frac{\exp[-(u^2/\sigma_1^2 + 2\delta uv/\sigma_1\sigma_2 + v^2/\sigma_2^2)/2(1-\delta^2)(u^2+v^2)^2]}{2\pi\sigma_1\sigma_2(1-\delta^2)^{\frac{1}{2}}(u^2+v^2)^2}$$

where $-\infty < u < \infty$ and $-\infty < v < \infty$. Since $W = Z^{-1} = (X-iY)/(X^2+Y^2)$ we have $U = X/(X^2+Y^2)$ and $V = -Y/(X^2+Y^2)$. Consequently (4.3) is the joint density of $\{X/(X^2+Y^2), -Y/(X^2+Y^2)\}$ where the density of (X, Y) is the bivariate normal density with zero means. If in (4.1) we set $\delta = 0$, $\sigma_1 = \sigma_2 = \sigma$ then we obtain

$$(4.4)\qquad f(\omega, \phi) = \frac{\omega^{2/\gamma-1}\exp(-\frac{1}{2}\omega^{2/\gamma}/\sigma^2)}{2\pi\sigma^2\gamma^2}$$

where $\omega > 0$, $-|\gamma|\pi < \phi < |\gamma|\pi$ and $\gamma \neq 0$. Observe that this is a Weibull-uniform density function. Consequently, if $Z = X+iY$ is a complex normal random variable with zero means, zero correlation and equal variances then the density of (Ω, Φ), where $W = Z^\gamma = \Omega e^{i\Phi}$, is a Weibull-uniform density function.

The GMT of (4.1) with $\gamma = 1$, $\delta = 0$ and $\sigma_1 = \sigma_2 = \sigma$ is

$$(4.5)\qquad h(s, t) = 2^{s/2}\sigma^s\Gamma(s/2+1)(\sin t\pi)/t$$

where $s > -2$. (It can be seen by putting $s = t = k$ in (4.5) that a complex normal random variable with zero means, zero correlation and equal variances is a complex random variable such that all its moments are zero.)

REFERENCES

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