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ASYMPTOTIC EXPANSIONS FOR DISTRIBUTIONS OF THE ROOTS OF TWO MATRICES FROM CLASSICAL AND COMPLEX GAUSSIAN POPULATIONS[1]

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1. Introduction and summary. The distribution of the characteristic (ch.) roots of a sample covariance matrix **S** (one-sample case) or the matrix $\mathbf{S}_1\mathbf{S}_2^{-1}$ (two-sample case, see Section 3) depends on a definite integral over the group of orthogonal (in the complex case replaced by unitary) matrices. This integral, either in the one-sample case or the two-sample case, involves the ch. roots of both the population and sample matrices. Usually the integral in either case is expressed as a hypergeometric series involving zonal polynomials [4], [6]. Unfortunately, these series converge slowly unless the ch. roots of the argument matrices lie in very limited ranges. Furthermore, the computations of these series are not so easy and not convenient for further development. In the one-sample real case, Anderson [1] has obtained an asymptotic expansion for the distribution of the ch. roots of the sample covariance matrix. In the two-sample case, however, the situation is more complicated. Chang [2] has obtained an asymptotic expansion for the first term. In Section 3 and Section 4 of this paper, we extend Chang's results obtaining the second term and also derive a general formula which includes the formulae of Anderson [1], James [7], Chang [2] and Roy [12] as limiting or special cases. In Section 5 we are dealing with the asymptotic expansions in the two-sample case in the complex Gaussian population, from which the one-sample results are obtained as limiting cases. Finally, Section 6 gives a comparison of the four asymptotic expansions.

2. Notation. Before proceeding further, we list the notations which will be used throughout.

The letters j, k, s, t, p, q, m and n with or without subscripts will denote positive integers, and $i=(-1)^{\frac{1}{2}}$. Matrices will be denoted by bold face capital letters and their dimensions are all $p \times p$ unless otherwise stated. In particular, **S** and **Σ** with or without subscripts denote the sample and population covariance matrices respectively. **A**, **B**, **R**, and **Θ** are diagonal matrices, and **I**, identity matrix. **H**, **Q** and **U** denote Hermitian, orthogonal and unitary matrices respectively. **U**′ is the transpose of **U**, and **U*** is the complex conjugate and transpose of **U**. $O(p)$ and $U(p)$ are the groups of all $p \times p$ orthogonal and unitary matrices respectively. $|\alpha|$ denotes the absolute value of α, and $|\mathbf{X}|$ denotes the determinant of **X**. $\bar{h}_{jk}$ is the conjugate of h_{jk}. h_{jkR} and h_{jkI} are the real and imaginary parts of h_{jk}. h_{jkc} denotes

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either h_{jkR} or h_{jkI}. Summation $\sum_{j=1}$ or $\sum_{j<k}$ means $\sum_{j=1}^{p}$ or $\sum_{j<k}^{p}$. Product $\prod_{j=1}$ or $\prod_{i<k}$ means $\prod_{j=1}^{p}$ or $\prod_{j<k}^{p}$ unless otherwise stated.

3. The asymptotic expansion of $\mathscr{I}$. Let $\mathbf{S}_j$ ($j = 1.2$) be independently distributed as Wishart $(n_j, p, \mathbf{\Sigma}_j)$, and let the ch. roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})^{-1}$ be b_k and a_k $(k = 1, \cdots, p)$ respectively such that $b_1 > b_2 > \cdots > b_p > 0$ and $0 < a_1 < a_2 < \cdots < a_p$. Further, let us denote

$$\mathbf{A} = \operatorname{diag}(a_1, a_2, \cdots, a_p),$$
$$\mathbf{B} = \operatorname{diag}(b_1, b_2, \cdots, b_p)$$

and $n = n_1 + n_2$. Then the joint distribution of $b_1, b_2, \cdots, b_p$ is given by [6], [9]

$$C \prod_{j=1} a_j^{\frac{1}{2}n_1} b_j^{\frac{1}{2}(n_1-p-1)} \prod_{j<k}(b_j - b_k) \prod_{j=1} db_j \cdot \int_{O(p)} |\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(\mathbf{Q}'d\mathbf{Q}), \tag{3.1}$$

where

$$C = \Gamma_p(\tfrac{1}{2}n)\{2^p\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\}^{-1}, \qquad \Gamma_l(x) = \pi^{\frac{1}{4}l(l-1)}\prod_{j=1}^{l}\Gamma(x - \tfrac{1}{2}j + \tfrac{1}{2}), \tag{3.2}$$

and $(\mathbf{Q}'\,d\mathbf{Q})$ is the invariant measure on the group $O(p)$.

From (3.1) we know that the distribution of the ch. roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ depends on the definite integral

$$\mathscr{I} = \int_{O(p)} |\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(\mathbf{Q}'\,d\mathbf{Q}). \tag{3.3}$$

Let us transform first

$$\mathbf{Q} = \exp(\mathbf{S}) \tag{3.4}$$

where $\mathbf{S}$ is a skew symmetric matrix (note that "$\mathbf{S}$" was also used as the sample covariance matrix). The Jacobian of this transformation has been computed by Anderson (cf. (2.3) of [1]), and is given by

$$J = 1 + \frac{p-2}{4!}\operatorname{tr}\mathbf{S}^2 + \frac{8-p}{4(6!)}\operatorname{tr}\mathbf{S}^4 + \frac{5p^2-20p+14}{8(6!)}(\operatorname{tr}\mathbf{S}^2)^2 + \cdots. \tag{3.5}$$

LEMMA 3.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be defined as before, then* $f(\mathbf{Q}) = |\mathbf{I}+\mathbf{AQBQ}'|$, $\mathbf{Q} \in O(p)$, *attains its minimum value* $|\mathbf{I}+\mathbf{AB}|$ *when* $\mathbf{Q}$ *is of the form*

$$\begin{pmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{pmatrix}. \tag{3.6}$$

PROOF. See [1] and [2].

Lemma 3.1 allows us to claim that, for large n, the integrand in (3.3) is negligible except for small neighborhoods about each of these matrices of (3.6) and $\mathbf{I}$ consists of identical contributions from each of these neighborhoods, so that

$$\mathscr{I} = 2^p \int_{N(\mathbf{I})} |\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(\mathbf{Q}'\,d\mathbf{Q}), \tag{3.7}$$

where $N(\mathbf{I})$ is a neighborhood of the identity matrix on the orthogonal manifold.

LEMMA 3.2. *Let $g_j (j = 1, \cdots, p)$ be the ch. roots of* **G** *such that* $\max_{1 \leqq j \leqq p} |g_j| < 1$ *then*

$$|\mathbf{I}+\mathbf{G}|^{-\frac{1}{2}m} = \exp\{-\tfrac{1}{2}m \operatorname{tr}(\mathbf{G}-\tfrac{1}{2}\mathbf{G}^2+\tfrac{1}{3}\mathbf{G}^3-\cdots)\}.$$

PROOF. See [2].

Since we wish to compute up to the second term in the asymptotic expansion of $\mathscr{I}$, we need to investigate the groups of terms up to the fourth order of **S**. Using the transformation (3.4), we have

$$|\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n} = |\mathbf{I}+\mathbf{AB}|^{-\frac{1}{2}n}|\mathbf{I}+\{\mathbf{S}\}+\{\mathbf{S}^2\}+\{\mathbf{S}^3\}+\{\mathbf{S}^4\}+\cdots|^{-\frac{1}{2}n},$$

where

$$\begin{aligned}
\{\mathbf{S}\} &= \mathbf{RSB}-\mathbf{RBS},\\
\{\mathbf{S}^2\} &= \tfrac{1}{2}(\mathbf{RBS}^2+\mathbf{RS}^2\mathbf{B}-2\mathbf{RSBS}),\\
\{\mathbf{S}^3\} &= \tfrac{1}{6}(\mathbf{RS}^3\mathbf{B}-3\mathbf{RS}^2\mathbf{BS}+3\mathbf{RSBS}^2-\mathbf{RBS}^3),\\
\{\mathbf{S}^4\} &= \tfrac{1}{24}(\mathbf{RBS}^4-4\mathbf{RSBS}^3+6\mathbf{RS}^2\mathbf{BS}^2-4\mathbf{RS}^3\mathbf{BS}+\mathbf{RS}^4\mathbf{B})
\end{aligned}$$

and

$$\mathbf{R} = (\mathbf{I}+\mathbf{AB})^{-1}\mathbf{A} = \operatorname{diag}(r_1, r_2, \cdots, r_p), \qquad r_j = \frac{a_j}{1+a_j b_j} \quad (j = 1, \cdots, p).$$

Under transformation (3.4), we have $N(\mathbf{I}) \to N(\mathbf{S} = \mathbf{0})$. If we put $\mathbf{G} = \{\mathbf{S}\}+\{\mathbf{S}^2\}+\{\mathbf{S}^3\}+\{\mathbf{S}^4\}+\cdots$, then in the neighborhood of $\mathbf{S} = \mathbf{0}$, the elements of **S** are very small, and hence the maximum ch. roots of **G** can be assumed to be less than unity. Therefore Lemma 3.2 is applicable. By Lemma 3.2, we obtain

$$\begin{aligned}
|\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n} &= |\mathbf{I}+\mathbf{AB}|^{-\frac{1}{2}n}|\mathbf{I}+\mathbf{G}|^{-\frac{1}{2}n}\\
&= |\mathbf{I}+\mathbf{AB}|^{-\frac{1}{2}n}\exp\{-\tfrac{1}{2}n \operatorname{tr}([\mathbf{S}]+[\mathbf{S}^2]+[\mathbf{S}^3]+[\mathbf{S}^4]+\cdots)\},
\end{aligned}$$

where

$$\begin{aligned}
[\mathbf{S}] &= \{\mathbf{S}\},\\
[\mathbf{S}^2] &= \{\mathbf{S}^2\}-\tfrac{1}{2}\{\mathbf{S}\}^2,\\
[\mathbf{S}^3] &= \{\mathbf{S}^3\}-\tfrac{1}{2}\{\mathbf{S}\}\{\mathbf{S}^2\}-\tfrac{1}{2}\{\mathbf{S}^2\}\{\mathbf{S}\}+\tfrac{1}{3}\{\mathbf{S}\}^3
\end{aligned}$$

and

$$\begin{aligned}
[\mathbf{S}^4] = \{\mathbf{S}^4\}&-\tfrac{1}{2}\{\mathbf{S}\}\{\mathbf{S}^3\}-\tfrac{1}{2}\{\mathbf{S}^3\}\{\mathbf{S}\}-\tfrac{1}{2}\{\mathbf{S}^2\}^2+\tfrac{1}{3}\{\mathbf{S}\}^2\{\mathbf{S}^2\}\\
&+\tfrac{1}{3}\{\mathbf{S}\}\{\mathbf{S}^2\}\{\mathbf{S}\}+\tfrac{1}{3}\{\mathbf{S}^2\}\{\mathbf{S}\}^2-\tfrac{1}{4}\{\mathbf{S}\}^4.
\end{aligned}$$

Since $\mathbf{S} = (s_{jk})$, $s_{kj} = -s_{jk}$ for all $j, k = 1, \cdots, p$, now we have

$$\operatorname{tr}[\mathbf{S}] = \operatorname{tr}(\mathbf{RSB}-\mathbf{RBS}) = 0,$$

$$\begin{aligned}\operatorname{tr}[\mathbf{S}^2] &= \operatorname{tr}(\{\mathbf{S}^2\}-\tfrac{1}{2}\{\mathbf{S}\}^2)\\ &= \operatorname{tr}(\tfrac{1}{2}\mathbf{RBS}^2+\tfrac{1}{2}\mathbf{RS}^2\mathbf{B}-\mathbf{RSBS}-\tfrac{1}{2}(\mathbf{RBSRBS}+\mathbf{RSBRSB}-\mathbf{RBSRSB}\\ &\qquad -\mathbf{RSBRBS}))\\ &= \operatorname{tr}(\mathbf{BS}-\mathbf{SB})(\mathbf{I}-\mathbf{RB})\mathbf{SR}\\ &= \textstyle\sum_{j<k} c_{jk}s_{jk}^2\end{aligned}$$

where

$$c_{jk} = (r_{kj}-r_j r_k b_{jk})b_{jk} = c_{kj} \qquad (3.8)$$

$$r_{jk} = r_j - r_k \quad \text{and} \quad b_{jk} = b_j - b_k.$$

Similarly, after simplification, we find

$$\operatorname{tr}[\mathbf{S}^3] = \operatorname{tr}\{\mathbf{S}^3\}-\operatorname{tr}\{\mathbf{S}\}\{\mathbf{S}^2\}+\tfrac{1}{3}\operatorname{tr}\{\mathbf{S}\}^3 + \textstyle\sum_{j<k<t} f_{jkt}\cdot s_{jk}s_{kt}s_{tj}$$

where

$$\begin{aligned}f_{jkt} &= r_{jk}b_{kt}-r_{kt}b_{jk}+r_j r_{kt}b_{jk}b_{jt}+r_k r_{jt}b_{jk}b_{kt}\\ &\quad + r_t r_{jk}b_{jt}b_{kt}-2r_j r_k r_t b_{jk}b_{kt}b_{tj},\end{aligned}$$

$$\begin{aligned}\operatorname{tr}[\mathbf{S}^4] &= \operatorname{tr}\{\mathbf{S}^4\}-\operatorname{tr}\{\mathbf{S}\}\{\mathbf{S}^3\}-\tfrac{1}{2}\operatorname{tr}\{\mathbf{S}^2\}^2+\operatorname{tr}\{\mathbf{S}\}^2\{\mathbf{S}^2\}-\tfrac{1}{4}\operatorname{tr}\{\mathbf{S}\}^4\\ &= \textstyle\sum_{j<k}\varphi_{jk}\cdot s_{jk}^4 = \sum_{j<k<t}\psi_{jkt}s_{jk}^2 s_{jt}^2+\sum_{j<k<t}\psi_{kjt}\cdot s_{jk}^2 s_{kt}^2\\ &\quad +\textstyle\sum_{j<k<t}\psi_{tjk}\cdot s_{jt}^2 s_{kt}^2+\sum_{j<k\neq t\neq u} g\cdot s_{jk}s_{kt}s_{tu}s_{uj},\end{aligned} \qquad (3.9)$$

where

$$\begin{aligned}\varphi_{jk} &= (r_j r_k b_{jk}^2-\tfrac{1}{3})r_{kj}b_{jk}+(\tfrac{1}{3}r_j r_k-\tfrac{1}{2}r_{kj}^2)b_{jk}^2-\tfrac{1}{2}r_j{}^2 r_k{}^2 b_{jk}^4\\ &= -\tfrac{1}{3}c_{jk}-\tfrac{1}{2}c_{jk}^2,\end{aligned} \qquad (3.10)$$

$$\begin{aligned}\psi_{jkt} &= -\tfrac{1}{3}r_{kj}b_{jk}-\tfrac{1}{3}r_{tj}b_{jt}+\tfrac{1}{4}r_{tk}b_{kt}\\ &\quad +\tfrac{1}{3}r_j(r_k b_{jk}^2+r_t b_{jt}^2)+r_j(r_k+r_t)b_{jk}b_{jt}\\ &\quad -r_j{}^2 b_{jk}b_{jt}-\tfrac{1}{4}r_k r_t(b_{jk}+b_{jt})^2\\ &\quad -r_j(r_k b_{jk}r_{jt}+r_t r_{jk}b_{jt}+r_j r_k r_t b_{jk}b_{jt})b_{jk}b_{jt}\\ &= -\tfrac{1}{3}(c_{jk}+c_{jt})+\tfrac{1}{4}c_{kt}-c_{jk}c_{jt}.\end{aligned} \qquad (3.11)$$

Note that ψ_{kjt} and ψ_{tjk} can be obtained from ψ_{jkt} cyclically, i.e., by changing j to k and k to t, ψ_{jkt} becomes ψ_{kjt} and ψ_{kjt} becomes ψ_{tjk} respectively. Moreover, we need not know the value of g, because any term containing an odd power of a factor s_{jk} when integrated with respect to s_{jk} reduces to zero. From (3.9) it is not difficult to show that $f_{jkt}^2 = c_{jk}^2+c_{jt}^2+c_{kt}^2-2(c_{jk}c_{jt}+c_{jk}c_{kt}+c_{jt}c_{kt})-4c_{jk}c_{kt}c_{jt}$.

Finally, we can write (3.7) to be

$$\mathscr{I} = 2^p \prod_{j=1} (1+a_j b_j)^{-\frac{1}{2}n} \int_{N(S=0)} \exp(-\tfrac{1}{2}n \textstyle\sum_{j<k} c_{jk} s_{jk}^2) \cdot \exp(-\tfrac{1}{2}n \operatorname{tr}[\mathbf{S}^3] - \tfrac{1}{4}n \operatorname{tr}[\mathbf{S}^4] - \cdots) J \prod_{j<k} ds_{jk}. \tag{3.12}$$

If this integration is to be performed term by term on the expansion of $\exp(-\frac{1}{2}n \operatorname{tr}[\mathbf{S}^3] - \cdots)J$ then for large n, the limits for each s_{jk} can be put to $\pm\infty$, since each integration is of the form

$$\int_{N(S=0)} \exp(-\tfrac{1}{2}n \textstyle\sum_{j<k} c_{jk} s_{jk}^2) \prod_{j<k} s_{jk}^{m_{jk}} ds_{jk}$$

and most of this integral is given in a small neighborhood of $\mathbf{S} = \mathbf{0}$. The m_{jk}'s are positive even integers or zero since any term containing an odd power of an s_{jk} as a factor will integrate to zero. We expand $\exp(\frac{1}{2}n \operatorname{tr}[\mathbf{S}^3] - \cdots)J$, writing the terms in groups, each group corresponding to a certain value of m. We have

$$\begin{aligned} \exp\left(-\frac{n}{2}\operatorname{tr}[\mathbf{S}^3] - \frac{n}{2}\operatorname{tr}[\mathbf{S}^4] - \cdots\right)J \\ = 1 - \frac{n}{2}\operatorname{tr}[\mathbf{S}^4] + \frac{n^2}{8}(\operatorname{tr}[\mathbf{S}^3])^2 + \frac{p-2}{4!}\operatorname{tr}\mathbf{S}^2 \\ - \frac{n}{2}\operatorname{tr}[\mathbf{S}^6] + \frac{n^2}{8}(\operatorname{tr}[\mathbf{S}^4])^2 + \cdots. \end{aligned} \tag{3.13}$$

Using (2.6a) and (2.6b) of [1], we obtain the following theorem.

THEOREM 3.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be diagonal matrices with* $0 < a_1 < a_2 < \cdots < a_p$ *and* $b_1 > b_2 > \cdots > b_p > 0$. *Then for large* n, *the first two terms in the expansion for* $\mathscr{I}$ *are given by*

$$\mathscr{I} = 2^p \prod_{j=1} (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j<k} \left(\frac{2\pi}{nc_{jk}}\right)^{\frac{1}{2}} \left\{1 + \frac{1}{2n}\left[\sum_{j<k} c_{jk}^{-1} + \alpha(p)\right] + \cdots\right\}, \tag{3.14}$$

where

$$\alpha(p) = p(p-1)(4p+1)/12. \tag{3.15}$$

PROOF. In the proof, we include only terms without an odd power of an s_{jk}. First note that only the second, third and fourth terms on the right-hand side of (3.13) contribute terms of order n^{-1}. After integration, the first term unity has been shown [1] to give

$$K = \prod_{j<k} \left(\frac{2\pi}{nc_{jk}}\right)^{\frac{1}{2}}. \tag{3.16}$$

The second term $-n \operatorname{tr}[S^4]/2$ contributes

$$K\left\{\frac{1}{2n}\sum_{j<k} c_{jk}^{-1} + \frac{3}{4n}\binom{p}{2} + \frac{p-2}{3n}\sum_{j<k} c_{jk}^{-1} - \frac{1}{8n}\sum_{j<k<t}\left(\frac{c_{kt}}{c_{jk}c_{jt}} + \frac{c_{jt}}{c_{jk}c_{kt}} + \frac{c_{jk}}{c_{jt}c_{kt}}\right) + \frac{3}{2n}\binom{p}{3}\right\}, \tag{3.17}$$

and the third term $n^2(\operatorname{tr}[\mathbf{S}^3])^2/8$ gives

$$K\left\{\frac{1}{8n}\sum_{j<k<t}\left(\frac{c_{kt}}{c_{jk}c_{jt}}+\frac{c_{jt}}{c_{jk}c_{kt}}+\frac{c_{jk}}{c_{jt}c_{kt}}\right)-\frac{p-2}{4n}\sum_{j<k}c_{jk}^{-1}-\frac{1}{2n}\binom{p}{3}\right\}. \tag{3.18}$$

Finally, since $\operatorname{tr}\mathbf{S}^2 = -2\sum_{j<k}s_{jk}^2$, it is easy to see that $(p-2)\operatorname{tr}\mathbf{S}^2/4!$ contributes

$$-\frac{p-2}{12n}K\sum_{j<k}c_{jk}^{-1}. \tag{3.19}$$

Adding (3.16)–(3.19) and factoring K out, we obtain (3.14).

THEOREM 3.2. *The asymptotic distribution of the ch. roots,* $b_1 > b_2 > \cdots > b_p > 0$, *of* $\mathbf{S}_1\mathbf{S}_2^{-1}$ *for large degrees of freedom* $n = n_1+n_2$ *when the roots of* $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ *are* $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ *where* $\lambda_j = a_j^{-1}$ $(j = 1, \cdots, p)$, *is given by*

$$C2^p\prod_{j=1}a_j^{\frac{1}{2}n_1}b_j^{\frac{1}{2}(n_1-p-1)}(1+a_jb_j)^{-\frac{1}{2}n}\prod_{j<k}(b_j-b_k) \cdot\prod_{j=1}db_j\prod_{j<k}(2\pi/nc_{jk})^{\frac{1}{2}}\{1+(2n)^{-1}[\textstyle\sum_{j<k}c_{jk}^{-1}+\alpha(p)]+\cdots\}, \tag{3.20}$$

where C, c_{jk} *and* $\alpha(p)$ *are defined by* (3.2), (3.8) *and* (3.15) *respectively.*

4. The asymptotic expansion of $\mathscr{I}$ when roots are not all distinct. In the previous section we restricted the roots of population matrix $(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})^{-1}$ to be all distinct. However, the roots need not be all so. And when we are interested in the likelihood of equality of population roots, the asymptotic formula of Section 3 breaks down. Overcoming this situation a general formula is derived which includes the case of distinct roots as a special case. The one-sample case has been studied by James [7]; his result would follow from ours as a limiting case.

Now let $0 < a_1 < \cdots < a_k < a_{k+1} = \cdots = a_p = a$, $(1 \leqq k \leqq p-1)$. Then

$$\mathbf{A} = \operatorname{diag}(a_1, \cdots, a_k, a, \cdots, a)$$

and the joint distribution of $b_1, b_2, \cdots, b_p$ of (3.1) becomes

$$Ca^{\frac{1}{2}qn_1}\prod_{j=1}^k a_j^{\frac{1}{2}n_1}\int_{O(p)}|\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(\mathbf{Q}'d\mathbf{Q}) \cdot\prod_{j=1}b_j^{\frac{1}{2}(n_1-p-1)}\prod_{j<t}(b_j-b_t)\prod_{j=1}db_j, \tag{4.1}$$

where $q = p-k$.

As in Section 3, we consider the integral

$$\mathscr{I} = \int_{O(p)}|\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(\mathbf{Q}'d\mathbf{Q}). \tag{4.2}$$

Now we partition the matrix $\mathbf{Q}$ into the submatrices $\mathbf{Q}_1$ consisting of its first k, and $\mathbf{Q}_2$, the remaining q rows. If the integrand of (4.2) does not depend on $\mathbf{Q}_2$, then we can integrate over $\mathbf{Q}_2$ for fixed $\mathbf{Q}_1$ by the formula

$$\int_{Q_2}C_1(d\mathbf{Q}) = C_2(d\mathbf{Q}_1) \tag{4.3}$$

where

$$C_1 = \pi^{\frac{1}{2}p^2}\{\Gamma_p(\tfrac{1}{2}p)\}^{-1}, \qquad C_2 = \pi^{\frac{1}{2}kp}\{\Gamma_k(\tfrac{1}{2}p)\}^{-1},$$

and the symbol $(d\mathbf{Q}_1)$ denotes the invariant volume element on the Stiefel manifold of orthonormal k-frames in p-space normalized to make its integral unity. Make transformation (3.4) whose Jacobian is given by (3.5).

A parameterization of $\mathbf{Q}_1$ may be obtained by writing

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{pmatrix} = \exp\left\{\begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ -\mathbf{S}_{12} & \mathbf{0} \end{pmatrix}\right\} \tag{4.4}$$

where $\mathbf{S}_{11}$ is a $k \times k$ skew symmetric matrix and $\mathbf{S}_{12}$ is a $k \times q$ rectangular matrix. From (3.5), it is not difficult to show that

$$C_2(d\mathbf{Q}_1) = (d\mathbf{S}_{11})(d\mathbf{S}_{12})\{1+O(\text{squares of } s_{jk}\text{'s})\}$$

where the symbols (dS_{11}) and (dS_{12}) stand for $\prod_{j<t}^k ds_{jt}$ and $\prod_{j=1}^k \prod_{t=k+1} ds_{jt}$ respectively.

Since we are only interested in the first term, all we need to investigate is the groups of terms up to the second order of $\mathbf{S}$, which is denoted by $[\mathbf{S}^2]$. As we did in Section 3, but remembering that the last q ch. roots of $\mathbf{A}$ are equal, it is easy to show that

$$\operatorname{tr}[\mathbf{S}^2] = \textstyle\sum_{j<t}^k c_{jt} s_{jt}^2 + \sum_{j=1}^k \sum_{t=k+1} c_{jt}^0 s_{jt}^2,$$

where

$$\begin{aligned} c_{jt} &= r_{tj} b_{jt} - r_j r_t b_{jt}^2 = c_{tj}, & j,t &= 1,\cdots,k, j<t; \\ c_{jt}^0 &= r_{tj} b_{jt} - r_j r_t b_{jt}^2 = c_{tj}^0, & j &= 1,\cdots,k, t = k+1,\cdots,p \end{aligned} \tag{4.5}$$

$$r_j = \frac{a_j}{1+a_j b_j} \quad \text{if } j = 1,\cdots,k,$$

$$= \frac{a}{1+ab_j} \quad \text{if } j = k+1,\cdots,p,$$

$$r_{jt} = r_j - r_t \quad \text{and} \quad b_{jt} = b_j - b_t.$$

Therefore,

$$\begin{aligned} |\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n} &= \textstyle\prod_{j=1}^k (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j=k+1} (1+ab_j)^{-\frac{1}{2}n} \\ &\cdot \textstyle\prod_{j<t}^k \exp(-\frac{1}{2}nc_{jt} s_{jt}^2) \prod_{j=1}^k \prod_{t=k+1} \exp(-\frac{1}{2}nc_{jt}^0 s_{jt}^2) \\ &\cdot \{1+O(\text{squares of } s_{jt}\text{'s})\}. \end{aligned} \tag{4.6}$$

Substituting (4.6) into (3.7) and using

$$\textstyle\int_{O(p)} |\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(\mathbf{Q}'\,d\mathbf{Q}) = 2^p C_1 \int_{O(p)} |\mathbf{I}+\mathbf{AQBQ}'|^{-\frac{1}{2}n}(d\mathbf{Q})$$

yields

$$\begin{aligned} \mathscr{I} &= \frac{\pi^{\frac{1}{2}q^2}}{\Gamma_q(\frac{1}{2}q)} \prod_{j=1}^k (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j=k+1} (1+ab_j)^{-\frac{1}{2}n} \\ &\cdot \textstyle\int_{S_{11}} \int_{S_{12}} \prod_{j<t}^k \exp(-\frac{1}{2}nc_{jt} s_{jt}^2)\, ds_{jt} \\ &\cdot \textstyle\prod_{j=1}^k \prod_{t=k+1} \exp(-\frac{1}{2}nc_{jt}^0 s_{jt}^2)\, ds_{jt} \{1+O(1/n)\}. \end{aligned} \tag{4.7}$$

For large n and a_j's and b_j's ($j = 1, \cdots, k$) well-spaced, most of the integral in (4.7) will be obtained from small values of the elements of $\mathbf{S}_{11}$ and $\mathbf{S}_{12}$. Hence, to obtain an asymptotic series, we can replace the finite range of s_{jt} by the range of all real values of s_{jt}. Thus

$$\mathscr{I} = \frac{\pi^{\frac{1}{2}q^2}}{\Gamma_q(\frac{1}{2}q)} \prod_{j=1}^{k} (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j=k+1} (1+ab_j)^{-\frac{1}{2}n}$$
$$\cdot \prod_{j<t}^{k} \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}nc_{jt}s_{jt}^2)\, ds_{jt}$$
$$\cdot \prod_{j=1}^{k} \prod_{t=k+1} \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}nc_{jt}^0 s_{jt}^2)\, ds_{jt} \{1+O(1/n)\}.$$

Hence we have the following theorem:

THEOREM 4.1. *The asymptotic distribution of the ch. roots, $b_1 > b_2 > \cdots > b_p > 0$ of $\mathbf{S}_1\mathbf{S}_2^{-1}$, for large degrees of freedom $n = n_1 + n_2$, when ch. roots of $(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})^{-1}$ are $0 < a_1 < \cdots < a_k < a_{k+1} = \cdots = a_p = a$, $(1 \leqq k \leqq p-1)$ is given by*

$$(4.8) \qquad C_3\, a^{\frac{1}{2}qn_1} \prod_{j=1}^{k} a_j^{\frac{1}{2}n_1} \prod_{j=1} b_j^{\frac{1}{2}(n_1-p-1)} \prod_{j=1}^{k} (1+a_j b_j)^{-\frac{1}{2}n}$$
$$\cdot \prod_{j=k+1} (1+ab_j)^{-\frac{1}{2}n} \prod_{j<t} (b_j - b_t) \prod_{j<t}^{k} (2\pi/nc_{jt})^{\frac{1}{2}}$$
$$\cdot \prod_{j=1}^{k} \prod_{t=k+1} (2\pi/nc_{jt}^0)^{\frac{1}{2}} \prod_{j=1} db_j,$$

where $C_3 = \pi^{\frac{1}{2}q^2}\, \Gamma_p(\frac{1}{2}n)\{\Gamma_q(\frac{1}{2}q)\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\}^{-1}$ and c_{jt} and c_{jt}^0 defined by (4.5).

The result (4.8) was given by Chang [3], but he had an error in the constant; he had

$$\frac{\pi^{\frac{1}{2}p(p-1)-\frac{1}{2}kp}}{[\Gamma_k(\frac{1}{2}p)]^{-1}} \prod_{j=1} \Gamma(\tfrac{1}{2}j)\Gamma_p(\tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\}^{-1} \prod_{j=1} a_j^{\frac{1}{2}n_1}$$

instead of $C_3\, a^{\frac{1}{2}qn_1} \prod_{j=1}^{k} a_j^{\frac{1}{2}n_1}$. He had also error in the factors; he had

$$\prod_{j=k+1} (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j=1}^{k} (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j=1}^{k} \prod_{t=k+1} \left(\frac{2\pi}{nc_{jt}}\right)^{\frac{1}{2}}$$

instead of

$$\prod_{j=1}^{k} (1+a_j b_j)^{-\frac{1}{2}n} \prod_{j=k+1} (1+ab_j)^{-\frac{1}{2}n} \prod_{j=1}^{k} \prod_{t=k+1} \left(\frac{2\pi}{nc_{jt}^0}\right)^{\frac{1}{2}}.$$

Note that for $k = 0$, $\prod_{j=1}^{k} (1+a_jb_j)^{-\frac{1}{2}n}$, $\prod_{j=1}^{k} \prod_{t=k+1} (2\pi/(nc_{jt}^0))^{\frac{1}{2}}$ products should be assumed to be unity. Similarly for $k = p$, $\prod_{j=k+1} (1+ab_j)^{-\frac{1}{2}n}$ etc. are unity, and define $\Gamma_0(x) = 1$, then $1 \leqq k \leqq p-1$ can be written $0 \leqq k \leqq p$.

(i) If $k = 0$, i.e., $q=p$, then $a_1 = \cdots = a_p = a$ and (4.8) reduces to

$$(4.9) \qquad \pi^{\frac{1}{2}p^2}\Gamma_p(\tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\}^{-1} a^{\frac{1}{2}pn_1}$$
$$\cdot \prod_{j=1} b_j^{\frac{1}{2}(n_1-p-1)} \prod_{j=1} (1+ab_j)^{-\frac{1}{2}n} \prod_{j<t} (b_j - b_t) \prod_{j=1} db_j,$$

(4.9) is the joint distribution of $b_1, b_2, \cdots, b_p$ under null hypothesis $\mathbf{\Sigma}_1 = a\mathbf{\Sigma}_2$ [12], and is an exact form where we assume no asymptotic condition. Moreover, in this case, the integrand of (4.2) is independent of $\mathbf{Q}$.

(ii) If $k = p$, i.e., $q = 0$, then $0 < a_1 < a_2 < \cdots < a_p$, and reduces to

$$(4.10) \qquad \Gamma_p(\tfrac{1}{2}n)\{\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)\}^{-1} \prod_{j=1} a_j^{\frac{1}{2}n_1} b_j^{\frac{1}{2}(n_1-p-1)}(1+a_j b_j)^{-\frac{1}{2}n} \prod_{j<t}(b_j-b_t)\prod_{j<t}\left(\frac{2\pi}{nc_{jt}}\right)^{\frac{1}{2}} \prod_{j=1} db_j.$$

This is Chang's result under condition $0 < a_1 < a_2 < \cdots < a_p$ (cf. [2]).

Now let $b_j = n_1 v_j/n_2$ $(j = 1, \cdots, p)$ and let n_2 tend to infinity, then (4.8) reduces to (3.12) of James [7]; (4.9) becomes the joint distribution of $b_1, b_2, \cdots, b_p$ under the null hypothesis $\mathbf{\Sigma} = a\mathbf{I}$ [12]; and (4.10) is the first approximation of (1.8) in [1]. This is when F is taken as one.

5. Two-sample complex case. Let $\mathbf{S}_j$ $(j = 1, 2)$ be independently distributed as complex Wishart $(n_j, p, \mathbf{\Sigma}_j)$, and let $b_1 \geqq b_2 \geqq \cdots \geqq b_p > 0$ and $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p > 0$ be the ch. roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ respectively. Let $\mathbf{B} = \operatorname{diag}(b_1, b_2, \cdots, b_p)$, $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$, $\mathbf{A} = \mathbf{\Lambda}^{-1}$ so that $a_j = \lambda_j^{-1}$ $(j = 1, \cdots, p)$ $0 < a_1 \leqq a_2 \leqq \cdots \leqq a_p$. Furthermore, let $n = n_1 + n_2$. Then the distribution of $b_1, b_2, \cdots, b_p$ can be expressed in the form [6],

$$(5.1) \qquad C_1 |\mathbf{A}|^{n_1} |\mathbf{B}|^{n_1-p} \prod_{j<k}(b_j-b_k)^2 \int_{U(p)} |\mathbf{I}+\mathbf{AUBU}^*|^{-n}(\mathbf{U}^* d\mathbf{U})$$

where

$$(5.2) \qquad C_1 = \frac{\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)},$$

where $(\mathbf{U}^* d\mathbf{U})$ is the invariant measure on the group $U(p)$. The group $U(p)$ has volume

$$\tilde{v}(p) = \int_{U(p)} (\mathbf{U}^* d\mathbf{U}) = \pi^{p(p-1)}/\tilde{\Gamma}_p(p)$$

where $\tilde{\Gamma}_p(p)$ is defined in [6].

However, this form is not convenient for further development. We have

$$(5.3) \qquad \mathscr{I}_1 = \int_{U(p)} |\mathbf{I}+\mathbf{AUBU}^*|^{-n}(\mathbf{U}^* d\mathbf{U}) = C_2 \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[n]_\kappa \tilde{C}_\kappa(-\mathbf{A})\tilde{C}_\kappa(\mathbf{B})}{k!\,\tilde{C}_\kappa(\mathbf{I})},$$

where $C_2 = \pi^{p(p-1)}\{\tilde{\Gamma}_p(p)\}^{-1}$, and $[b]_\kappa$ and the zonal polynomial of a Hermitian matrix $\mathbf{L}$, $\tilde{C}_\kappa(\mathbf{L})$, are defined in James [6]. The use of (5.3) in (5.1) gives a power series expansion, but the convergence of this series is very slow, unless the ch. roots of the argument matrices are in limited ranges. In this section, we obtain a beta type asymptotic expansion of the roots distribution of $\mathbf{S}_1\mathbf{S}_2^{-1}$ involving linkage factors between sample roots and corresponding population roots. If the roots are distinct, the limiting distribution as n_2 tends to infinity has the form (5.29). If, moreover, n_1 is assumed also large, then it corresponds to Girshick's result [5] in the real case.

We here require that $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$ and $b_1 > b_2 > \cdots > b_p > 0$. It is easy to see that $|\mathbf{I}+\mathbf{AUBU}^*|$ is positive real for all $\mathbf{B}$ and every $\mathbf{U} \in U(p)$.

LEMMA 5.1. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be defined as before, then* $f(\mathbf{U}) = |\mathbf{I}+\mathbf{AUBU}^*|$, $\mathbf{U} \in U(p)$, *attains its minimum value* $|\mathbf{I}+\mathbf{AB}|$ *when* $\mathbf{U}$ *is of the form*

$$\begin{pmatrix} e^{i\varphi_1} & & & 0 \\ & e^{i\varphi_2} & & \\ & & \ddots & \\ 0 & & & e^{i\varphi_p} \end{pmatrix} \tag{5.4}$$

where $0 \leqq \varphi_j < 2\pi, j = 1, \cdots, p$.

PROOF. Since $\mathbf{A}$ is positive definite

$$|\mathbf{I}+\mathbf{AUBU}^*| = |\mathbf{I}+\mathbf{A}^{\frac{1}{2}}\mathbf{UBU}^*\mathbf{A}^{\frac{1}{2}}|$$

$$\begin{aligned} df(\mathbf{U}) &= d\,|\mathbf{I}+\mathbf{A}^{\frac{1}{2}}\mathbf{UBU}^*\mathbf{A}^{\frac{1}{2}}| \\ &= |\mathbf{I}+\mathbf{A}^{\frac{1}{2}}\mathbf{UBU}^*\mathbf{A}^{\frac{1}{2}}|\operatorname{tr}(\mathbf{I}+\mathbf{A}^{\frac{1}{2}}\mathbf{UBU}^*\mathbf{A}^{\frac{1}{2}})^{-1}(\mathbf{A}^{\frac{1}{2}}\,d\mathbf{U}\cdot\mathbf{BU}^*\mathbf{A}^{\frac{1}{2}}+\mathbf{A}^{\frac{1}{2}}\mathbf{UB}\,d\mathbf{U}^*\cdot\mathbf{A}^{\frac{1}{2}}) \\ &= |\mathbf{I}+\mathbf{AUBU}^*|\operatorname{tr}(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}(d\mathbf{U}\cdot\mathbf{BU}^*-\mathbf{UBU}^*\,d\mathbf{U}\cdot\mathbf{U}^*) \\ &= |\mathbf{I}+\mathbf{AUBU}^*|\operatorname{tr}(\mathbf{BU}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}-\mathbf{U}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{UBU}^*)\,d\mathbf{U} \end{aligned}$$

for every $d\mathbf{U}$. Therefore $df(\mathbf{U}) = 0$ implies

$$\operatorname{tr}(\mathbf{BU}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}-\mathbf{U}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{UBU}^*) = 0,$$

for every $\mathbf{B}$ and $\mathbf{U}$, which implies

$$\mathbf{BU}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1} = \mathbf{U}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{UBU}^*,$$

i.e. $\mathbf{BU}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{U} = \mathbf{U}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{UB}$ which means that $\mathbf{B}$ and $\mathbf{U}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{U}$ commute. But $\mathbf{B}$ is a diagonal matrix with positive distinct elements. This implies that $\mathbf{U}^*(\mathbf{A}^{-1}+\mathbf{UBU}^*)^{-1}\mathbf{U}$ is a diagonal matrix, say $\mathbf{\Delta}$. Thus $\mathbf{A}^{-1} = \mathbf{U}(\mathbf{\Delta}^{-1}-\mathbf{B})\mathbf{U}^*$. This can happen only if $\mathbf{U} = \mathbf{DP}$ where $\mathbf{D}$ is of the form (5.4) and $\mathbf{P}$ is a permutation matrix. After substituting those stationary values in $f(\mathbf{U})$, we get

$$\textstyle\prod_{j=1}(1+a_j b_{\tau_j}) \tag{5.5}$$

where b_{τ_j} is any permutation of b_j $(j = 1, \cdots, p)$. It is easy to see that (5.5) attains its minimum value $|\mathbf{I}+\mathbf{AB}|$ when $\mathbf{U}$ is of the form (5.4).

Now we impose conditions on $\mathbf{U}$ (see reasons later), that all $e^{i\varphi_j}$ $(j = 1, \cdots, p)$ are positive real, say. Then $e^{i\varphi_j} = 1$ for all j, and (5.4) reduces to $\mathbf{I}$.

The above lemma allows us to claim that, for large n, the integrand of $\mathscr{I}_1$ is negligible except for small neighborhood of **I**. Therefore

$$\mathscr{I}_1 = \int_{N(I)} |\mathbf{I}+\mathbf{AUBU}^*|^{-n}(\mathbf{U}^* d\mathbf{U}) \tag{5.6}$$

where $N(\mathbf{I})$ is a neighborhood of the identity matrix on the unitary manifold.

LEMMA 5.2. *Let* **U** *be a unitary matrix, and make the transformation*

$$\mathbf{U} = \exp(i\mathbf{H}) \tag{5.7}$$

where **H** *is Hermitian matrix. Then the Jacobian of this transformation is*

$$J = 1 - \frac{p}{12}\operatorname{tr}\mathbf{H}^2 + \frac{1}{12}(\operatorname{tr}\mathbf{H})^2 + \frac{1}{2(6!)}\{5(\operatorname{tr}\mathbf{H})^4 - p\operatorname{tr}\mathbf{H}^4 - 11\operatorname{tr}\mathbf{H}^3\operatorname{tr}\mathbf{H} - 10p\operatorname{tr}\mathbf{H}^2(\operatorname{tr}\mathbf{H})^2 + (5p^2-3)(\operatorname{tr}\mathbf{H}^2)^2\} + \cdots. \tag{5.8}$$

PROOF. Let $\mathbf{\Theta} = \operatorname{diag}(\theta_1, \theta_2, \cdots, \theta_p)$ where θ_j $(j = 1, \cdots, p)$ are distinct numbers. Since **U** is unitary, there exists a unitary matrix $\mathbf{U}_1$ with real diagonal elements, such that

$$\mathbf{U} = \exp(i\mathbf{U}_1{}^*\mathbf{\Theta}\mathbf{U}_1).$$

Put $\mathbf{H} = (h_{jk}) = \mathbf{U}_1{}^*\mathbf{\Theta}\mathbf{U}_1$, then from Murnaghan [11], we have

$$(\mathbf{U}^* d\mathbf{U}) = \prod_{j<k} 4\sin^2 \tfrac{1}{2}(\theta_j - \theta_k) \prod_{j=1} d\theta_j (U_1{}^* d\mathbf{U}_1). \tag{5.9}$$

Since **H** is Hermitian, from Khatri [8], we have

$$\prod_{j=1} dh_{jj} \prod_{j<k} dh_{jkR}\, dh_{jkI} = \prod_{j<k}(\theta_j - \theta_k)^2 \prod_{j=1} d\theta_j (\mathbf{U}_1{}^* d\mathbf{U}_1) \tag{5.10}$$

where h_{jj} $(j = 1, \cdots, p)$ are real diagonal elements of **H**. Note that

$$\operatorname{tr}\mathbf{H}^m = \sum_{j=1} \theta_j{}^m.$$

Then using (5.9) and (5.10) we obtain (5.8).

Since we want to compute up to the second term in the asymptotic expansion of $\mathscr{I}_1$, we need to investigate the groups of terms up to the fourth order of **H**. Under transformation (5.7) we have

$$\begin{aligned}\mathbf{AUBU}^* = {} & \mathbf{AB} + i(\mathbf{AHB} - \mathbf{ABH}) + (\mathbf{AHBH} - \tfrac{1}{2}\mathbf{ABH}^2 - \tfrac{1}{2}\mathbf{AH}^2\mathbf{B}) \\ & + (i/6)(\mathbf{ABH}^3 - 3\mathbf{AHBH}^2 + 3\mathbf{AH}^2\mathbf{BH} - \mathbf{AH}^3\mathbf{B}) \\ & + \tfrac{1}{24}(\mathbf{ABH}^4 - 4\mathbf{AHBH}^3 + 6\mathbf{AH}^2\mathbf{BH}^2 - 4\mathbf{AH}^3\mathbf{BH} + \mathbf{AH}^4\mathbf{B}) + \cdots.\end{aligned}$$

Hence

$$|\mathbf{I}+\mathbf{AUBU}^*|^{-n} = |\mathbf{I}+\mathbf{AB}|^{-n}\,|\mathbf{I} + \{\mathbf{H}\} + \{\mathbf{H}^2\} + \{\mathbf{H}^3\} + \{\mathbf{H}^4\} + \cdots|^{-n},$$

where

$$\mathbf{R} = (\mathbf{I}+\mathbf{AB})^{-1}\mathbf{A} = \operatorname{diag}(r_1, r_2, \cdots, r_p), \qquad r_j = \frac{a_j}{1+a_j l_j} \quad (j = 1, \cdots, p),$$

$$\{\mathbf{H}\} = i(\mathbf{RHB}-\mathbf{RBH}),$$

$$\{\mathbf{H}^2\} = \mathbf{RHBH}-\tfrac{1}{2}\mathbf{RBH}^2-\tfrac{1}{2}\mathbf{RH}^2\mathbf{B},$$

$$\{\mathbf{H}^3\} = (i/6)(\mathbf{RBH}^3-3\mathbf{RHBH}^2+3\mathbf{RH}^2\mathbf{BH}-\mathbf{RH}^3\mathbf{B}) \quad \text{and}$$

$$\{\mathbf{H}^4\} = \tfrac{1}{24}(\mathbf{RBH}^4-4\mathbf{RHBH}^3+6\mathbf{RH}^2\mathbf{BH}^2-4\mathbf{RH}^3\mathbf{BH}+\mathbf{RH}^4\mathbf{B}).$$

After the transformation (5.7), one has $\mathbf{N}(\mathbf{I}) \to \mathbf{N}(\mathbf{H} = \mathbf{0})$. If we put $\mathbf{G} = \{\mathbf{H}\}+\{\mathbf{H}^2\}+\{\mathbf{H}^3\}+\{\mathbf{H}^4\}+\cdots$, then in the neighborhood of $\mathbf{H} = \mathbf{0}$, the absolute values of the elements of $\mathbf{H}$ are very small, and hence the absolute values of maximum ch. roots of $\mathbf{G}$ can be assumed to be less than unity. Since $|\mathbf{I}+\mathbf{G}|^{-n}$ is positive real, Lemma 3.2 is applicable. Thus we have

$$\begin{aligned}|\mathbf{I}+\mathbf{AUBU}^*|^{-n} &= |\mathbf{I}+\mathbf{AB}|^{-n}\cdot|\mathbf{I}+\mathbf{G}|^{-n} \\ &= |\mathbf{I}+\mathbf{AB}|^{-n}\exp\{-n\operatorname{tr}([\mathbf{H}]+[\mathbf{H}^2]+[\mathbf{H}^3]+[\mathbf{H}^4]+\cdots)\},\end{aligned}$$

where $[\mathbf{H}], \cdots, [\mathbf{H}^4]$ are of the same form as $[\mathbf{S}], \cdots, [\mathbf{S}^4]$ in Section 3, only that $\mathbf{S}$ should be replaced by $\mathbf{H}$.

Since $\mathbf{H} = (h_{jk})$, $h_{jk} = \bar{h}_{kj}$ for all $j, k = 1, \cdots, p$, we have

$$\operatorname{tr}[\mathbf{H}] = i\operatorname{tr}(\mathbf{RHB}-\mathbf{RBH}) = 0, \qquad \operatorname{tr}[\mathbf{H}^2] = \textstyle\sum_{j<k} c_{jk} h_{jk} \bar{h}_{jk}, \tag{5.11}$$

where

$$c_{jk} = (r_{kj}-r_j r_k b_{jk}) b_{jk} = c_{kj} \tag{5.12}$$

$$r_{jk} = r_j - r_k \quad \text{and} \quad b_{jk} = b_j - b_k.$$

Since h_{jj} $(j = 1, \cdots, p)$ are real, each one may range in a certain interval, and since they do not occur in the right-hand side of (5.11), may lead to the divergence of the integral [10]. So we need to impose conditions on $\mathbf{H}$. We may put h_{jj} $(j = 1, \cdots, p)$ to be constants, but the result is quite complicated. For simplicity, we set $h_{jj} = 0 (j = 1, \cdots, p)$. In view of (5.7), this is equivalent to imposing p conditions on $\mathbf{U}$. Thus each side of (5.7) contains p^2-p parameters. Under these conditions, (5.8) reduces to

$$J = 1-\frac{p}{12}\operatorname{tr}\mathbf{H}^2+\frac{1}{2(6!)}[(5p^2-3)(\operatorname{tr}\mathbf{H}^2)^2-p\operatorname{tr}\mathbf{H}^4]+\cdots. \tag{5.13}$$

As before, after simplification, we find

$$\operatorname{tr}[\mathbf{H}^3] = \textstyle\sum_{j<k<s} F_{jks}\cdot(h_{jk}h_{ks}h_{sj}-\overline{h_{jk}h_{ks}h_{sj}}),$$

where

$$\begin{aligned}F_{jks} = \tfrac{1}{2}i(r_{kj}b_{ks}-r_{sk}b_{jk}+r_j r_{sk} b_{jk} b_{js} \\ +r_k r_{sj} b_{jk} b_{ks}+r_s r_{sk} b_{js} b_{ks}-2r_j r_k r_s b_{jk} b_{ks} b_{js}),\end{aligned} \tag{5.14}$$

and

$$\operatorname{tr}[\mathbf{H}^4] = \sum_{j<k}\Phi_{jk}\cdot(h_{jk}\bar{h}_{jk})^2 + \sum_{j<k<s}\psi_{jks}\cdot h_{jk}\bar{h}_{jk}h_{js}\bar{h}_{js}$$
$$+\sum_{j<k<s}\psi_{kjs}\cdot h_{jk}\bar{h}_{jk}h_{ks}\bar{h}_{ks} + \sum_{j<k<s}\psi_{sjk}\cdot h_{js}\bar{h}_{js}h_{ks}\bar{h}_{ks}$$
$$+\sum_{j<k\neq s\neq t} G\cdot(h_{jk}h_{ks}h_{st}h_{tj} + \overline{h_{jk}h_{ks}h_{st}k_{tj}}),$$

where (as in 3.10)

$$\Phi_{jk} = -\tfrac{1}{3}c_{jk} - \tfrac{1}{2}c_{jk}^2, \tag{5.15}$$

and (as in 3.11)

$$\psi_{jks} = -\tfrac{1}{3}(c_{jk}+c_{js}) + \tfrac{1}{4}c_{ks} - c_{jk}c_{js}. \tag{5.16}$$

From (5.14), it is not difficult to show that $F_{jks}^2 = -\frac{1}{4}\{c_{jk}^2 + c_{js}^2 + c_{ks}^2 - 2(c_{jk}c_{js} + c_{jk}c_{ks} + c_{js}c_{ks}) - 4c_{jk}c_{ks}c_{js}\}$. Also note that ψ_{kjs} and ψ_{sjk} can be obtained from ψ_{jks} cyclically as in Section 3. Moreover, we need not know the value of G, because any term containing an odd power of a factor h_{jkc} will integrate to zero.

Finally, we can write (5.6) as

$$\mathscr{I}_1 = \prod_{j=1}(1+a_j b_j)^{-n}\int_{N(H=0)}\exp(-n\sum_{j<k}c_{jk}h_{jk}\bar{h}_{jk}) \cdot\exp(-n\operatorname{tr}[\mathbf{H}^3] - n\operatorname{tr}[\mathbf{H}^4] - \cdots)J\prod_{j<k}dh_{jkR}\,dh_{jkI} \tag{5.17}$$

where J is found in (5.13).

If this integration is to be performed term by term on the expansion of $\exp(-n\operatorname{tr}[\mathbf{H}^3] - \cdots)J$, then for large n the limits for each h_{jkc} can be put to $\pm\infty$, since each integration is of the form

$$\int_{N(H=0)}\exp(-n\sum_{j<k}c_{jk}h_{jk}\bar{h}_{jk})\prod_{j<k}h_{jkc}^{m_{jk}}\prod_{j<k}dh_{jkR}\,dh_{jkI},$$

and most of this integral is concentrated in a small neighborhood of $\mathbf{H} = \mathbf{0}$. The m_{jk}'s are positive even integers or zero, since any term containing an odd power of an h_{jkc} will integrate to zero. Now we expand $\exp(-n\operatorname{tr}[\mathbf{H}^3] - \cdots)J$, writing the terms in groups, each group corresponding to a certain value of m. We have

$$\begin{aligned}\exp(-n\operatorname{tr}[\mathbf{H}^3] &- n\operatorname{tr}[\mathbf{H}^4] - \cdots)J\\ &= 1 - n\operatorname{tr}[\mathbf{H}^4] + \tfrac{1}{2}n^2(\operatorname{tr}[\mathbf{H}^3])^2 - (p/12)\operatorname{tr}\mathbf{H}^2\\ &\quad + \frac{1}{2(6!)}\{(5p^2-3)(\operatorname{tr}\mathbf{H}^2)^2 - p\operatorname{tr}\mathbf{H}^4\} + \cdots.\end{aligned} \tag{5.18}$$

Now

$$\begin{aligned}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}&\exp(-n\sum_{j<k}\gamma_{jk}h_{jk}\bar{h}_{jk})\prod_{j<k}dh_{jkR}\,dh_{jkI}\\ &= \prod_{j<k}\frac{\pi}{n\gamma_{jk}} = \left(\frac{\pi}{n}\right)^{\frac{1}{2}p(p-1)}\prod_{j<k}\gamma_{jk}^{-1} = C,\end{aligned} \tag{5.19}$$

$$\begin{aligned}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}&\exp(-n\sum_{j<k}\gamma_{jk}h_{jk}\bar{h}_{jk})h_{stc}^{2m}\prod_{j<k}dh_{jkR}\,dh_{jkI}\\ &= C\cdot 1\cdot 3\cdot 5\cdots(2m-1)(2n\gamma_{st})^{-m}\end{aligned} \tag{5.20}$$

and

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp(-n\textstyle\sum_{j<k}\gamma_{jk}h_{jk}\bar{h}_{jk})(h_{st}\bar{h}_{st})^m\prod_{j<k}dh_{jkR}\,dh_{jkI} = C(m!)/(n\gamma_{st})^m. \tag{5.21}$$

Using formulae (5.19), (5.20) and (5.21), we obtain the following theorem:

THEOREM 5.1. *Let* **A** *and* **B** *be diagonal matrices with* $0 < a_1 < a_2 < \cdots < a_p$ *and* $b_1 > b_2 > \cdots > b_p > 0$. *Then for large n, the first two terms in the expansion for* $\mathscr{I}_1$ *are given by*

$$\mathscr{I}_1 = \prod_{j=1}(1+a_jb_j)^{-n}\prod_{j<k}\frac{\pi}{nc_{jk}}\left\{1+\frac{1}{3n}\left[\sum_{j<k}c_{jk}^{-1}+\beta(p)\right]+\cdots\right\}, \tag{5.22}$$

where

$$\beta(p) = p(p-1)(2p-1)/2. \tag{5.23}$$

PROOF. In the proof, we include only terms without an odd power of an h_{jkc}, and do not write C (where C is defined in (5.19)) which appears with each term after integration, and denote

$$S' = \textstyle\sum_{j<k}c_{jk}^{-1} \quad\text{and}$$

$$S'' = \textstyle\sum_{j<k<s}(c_{ks}/c_{jk}c_{js}+c_{js}/c_{jk}c_{ks}+c_{jk}/c_{js}c_{ks}).$$

Note that only the second, third and fourth terms on the right-hand side of (5.18) contribute the factor n^{-1}, using formulae (5.19)–(5.21). After integration, the second term $-n\operatorname{tr}[\mathbf{H}^4]$ contributes

$$\frac{2}{3n}S'+\frac{1}{n}\binom{p}{2}+\frac{2(p-2)}{3n}S'-\frac{1}{4n}S''+\frac{3}{n}\binom{p}{3}, \tag{5.24}$$

and the third term $n^2(\operatorname{tr}[\mathbf{H}^3])^2/2$ gives

$$\frac{1}{4n}S''-\frac{p-2}{2n}S'-\frac{1}{n}\binom{p}{3}. \tag{5.25}$$

Since $\operatorname{tr}\mathbf{H}^2 = 2\sum_{j<k}h_{jk}\bar{h}_{jk}$, it is not difficult to see that $-p\operatorname{tr}\mathbf{H}^2/12$ gives

$$-\frac{p}{6n}S'. \tag{5.26}$$

Adding (5.24)–(5.26) we obtain (5.22).

THEOREM 5.2. *The asymptotic distribution of the ch. roots,* $b_1 > b_2 > \cdots > b_p > 0$, *of* $\mathbf{S}_1\mathbf{S}_2^{-1}$ *for large degrees of freedom* $n = n_1+n_2$ *when the roots of* $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ *are* $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$, *where* $\lambda_j = a_j^{-1}$ $(j = 1, \cdots, p)$ *is given by*

$$C_1\textstyle\prod_{j=1}a_j^{n_1}b_j^{n_1-p}(1+a_jb_j)^{-n}\prod_{j<k}(b_j-b_k)^2\prod_{j=1}db_j \cdot\prod_{j<k}\frac{\pi}{nc_{jk}}\left\{1+\frac{1}{3n}\left[\sum_{j<k}c_{jk}^{-1}+\beta(p)\right]+\cdots\right\}, \tag{5.27}$$

where C_1, c_{jk} *and* $\beta(p)$ *are defined in* (5.2), (5.12) *and* (5.23) *respectively.*

Now replace b_j by $n_1 b_j/n_2$ ($j = 1, \cdots, p$) and let n_2 tend to infinity, then (5.22) and (5.27) reduce respectively to

$$\mathscr{I}_2 = \exp\left(-n_1 \sum_{j=1} a_j b_j\right) \prod_{j<k} \frac{\pi}{n_1 \gamma_{jk}} \left\{1 + \frac{1}{3n_1} \sum_{j<k} \gamma_{jk}^{-1} + \cdots\right\}, \tag{5.28}$$

and

$$C_3 \exp(-n_1 \textstyle\sum_{j=1} a_j b_j) \prod_{j<k} \gamma_{jk}^{-1} (b_j - b_k)^2 \prod_{j=1} a_j^{n_1} b_j^{n_1 - p} \cdot db_j \left\{1 + \frac{1}{3n_1} \sum_{j<k} \gamma_{jk}^{-1} + \cdots\right\}, \tag{5.29}$$

where

$$C_3 = n_1^{\frac{1}{2}p(2n_1 - p + 1)} \pi^{\frac{1}{2}p(p-1)} \{\tilde{\Gamma}_p(n_1)\}^{-1}$$

and

$$\gamma_{jk} = (a_k - a_j)(b_j - b_k) \qquad \text{for} \quad j, k = 1, \cdots, p, j < k.$$

(5.28) and (5.29) give the formulae in the one-sample case corresponding to (5.22) and (5.27) respectively.

6. Comparison. It is interesting to compare the formulae in the one-sample case with the corresponding ones in the two-sample case, and the real situation with the complex situation. In the real case, there is a factor $1/2n$ but a factor $1/3n$ arises in the complex case. Unlike the one-sample case, in the two-sample formulae, we find that there is an extra term $\alpha(p)/2n$ in the real case and $\beta(p)/3n$ in the complex case (in the second term of the asymptotic expansion for $\mathscr{I}$ in (3.14) and $\mathscr{I}_1$ in (5.22)), which is a function of n and p only. In (3.14), if we write

$$\omega = \omega(a, b) = 2^p \prod_{j=1} (1 + a_j b_j)^{-\frac{1}{2}n} \prod_{j<k} \left(\frac{2\pi}{nc_{jk}}\right)^{\frac{1}{2}},$$

then the expansion for $\mathscr{I}$ with the first term alone, and with both the first and second terms included, are respectively ω and $\omega\{1 + [\sum c_{jk}^{-1} + \alpha(p)]/2n\}$. A similar comparison can be made from (5.22) for the complex case.

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