The Annals of Mathematical Statistics
1970, Vol. 41, No. 5, 1466–1471

THE DECISION PROCESS FOR MAXIMIZING THE PROBABILITY OF OBTAINING THE RANDOM VARIABLE NEAREST TO AN ARBITRARY REAL NUMBER[1]

BY E. G. ENNS

University of Calgary

1. Introduction. Let x_i, for $i=1, 2, \cdots, N$ be N independent random variables (rv's) with a common continuous distribution $H(x) = P\{x_i \leqq x\}$, $x \in (-\infty, \infty)$. These N rv's will be sampled sequentially beginning with x_1. Exactly one rv is to be chosen, hence on each presentation the rv sampled is either accepted or forever rejected. Once an rv is accepted, the process ends, as this rv cannot later be rejected. For an arbitrary real number $r \in (-\infty, \infty)$, define the decision function $D(N, r, x)$ as the probability of accepting the first of these N independent rv's if it has the value x. The optimum decision function $D^*(N, r, x)$ will then be defined as that decision function which maximizes the probability of obtaining the rv nearest to r. It will also be assumed that if x_1 is rejected, the problem then becomes that of finding the rv nearest r for the remaining $(N-1)$ rv's, irrespective of the value of x_1. The special cases when $r = +\infty$ or $-\infty$ then correspond to finding the maximum or minimum rv respectively.

Let the order statistics of the N independent rv's be y_i, $i = 1, 2, \cdots, N$, the ordering being defined by $|y_i - r| < |y_{i+1} - r|$, $i = 1, 2, \cdots, N-1$. Also define $P_N(i)$ as the probability of obtaining the ith order statistic when following the optimum decision function on a sample of size N. It will be shown that $P_N(i)$ is independent of the distribution $H(x)$ and the real number r.

A special case of this problem, namely when $r = \infty$, has previously been considered by Enns (1969). A somewhat similar problem has also been considered by Karlin (1962). He considers a decision function which maximizes the expected value of the rv chosen. The "optimized" expected value he obtains is, however, a function of the distribution of the rv's.

2. The optimum decision function. For simplicity, let $P_N(1) = P_N$. If the decision $D(N, r, x)$ is made when $x_1 = x$ and the optimum decision function is followed thereafter, then let $P_N(D)$ be the probability of obtaining y_1 in this case. One can then write:

$$(2.1) \qquad P_{N+1}(D) = \int_{-\infty}^{\infty} D(N+1, r, x)(f(r, r-x))^N \, dH(x) + P_N \int_{-\infty}^{\infty} (1 - D(N+1, r, x))(1 - (f(r, r-x))^N) \, dH(x)$$

for $N = 1, 2, \cdots, P_1 = 1$ and

$$(2.2) \qquad f(r, u) = 1 + H(r - |u|) - H(r + |u|), \qquad r, u \in (-\infty, \infty).$$

Received November 10, 1969.

[1] The author acknowledges the support of the National Research Council of Canada.

The probabilistic interpretation is:

(2.3) $(f(r, r-x))^N = P\{x_1 = y_1 \mid x_1 = x$ and the sample size is $N+1\}$.

Obviously then:

(2.4) $$\int_{-\infty}^{\infty} (f(r, r-x))^N \, dH(x) = (N+1)^{-1}.$$

(2.1) can therefore be rewritten as:

(2.5) $$P_{N+1}(D) = (1+P_N)\int_{-\infty}^{\infty} D(N+1, r, x)[(f(r, r-x))^N - (S(N))^N] \, dH(x) + NP_N/(N+1)$$

where

(2.6) $$S(N) = (P_N/(1+P_N))^{1/N}.$$

Maximizing (2.5) with respect to the decision function D, one obtains the optimum decision function:

(2.7) $$\begin{aligned} D^*(N+1, r, x) &= 1 \quad \text{if} \quad H(r+|r-x|) - H(r-|r-x|) \leqq 1 - S(N) \\ &= 0 \quad \text{if} \quad H(r+|r-x|) - H(r-|r-x|) > 1 - S(N) \end{aligned}$$

for $N = 0, 1, 2, \cdots$ if one defines $S(0) = 0$.

Using the result (A.4) in the appendix, one can evaluate (2.5) when $D = D^*$. By definition $P_N(D^*) = P_N$, hence one obtains:

(2.8) $$\begin{aligned} P_{N+1} &= (1+P_N)[J(N, N+1, 1) - (S(N))^N J(N, 1, 1)] + NP_N/(N+1) \\ &= (1+NS(N)P_N)/(N+1). \end{aligned}$$

Surprisingly, this result is independent of both the distribution $H(x)$ and the real number r. Using (2.8) one can compute $S(N)$, $N = 1, 2, \cdots$, from which the optimum decision function can be obtained. Table 1 presents a tabulation of the optimum decision function for $1 \leqq N \leqq 1000$.

The asymptotic value of P_N has previously been shown by Enns (1969) to equal $\lim_{N\to\infty} P_N = 0.4659$.

3. The order statistic accepted. If the sample is of size N and the optimum decision function (2.7) is followed, then the probability of accepting the order statistic y_i can be written as:

(3.1) $$\begin{aligned} P_N(i) = &\int_{-\infty}^{\infty} D^*(N, r, x) P\{x_1 = y_i \mid x_1 = x\} \, dH(x) \\ &+ P_{N-1}(i-1)\int_{-\infty}^{\infty} (1 - D^*(N, r, x)) \textstyle\sum_{k=1}^{i-1} P\{x_1 = y_k \mid x_1 = x\} \, dH(x) \\ &+ P_{N-1}(i)\int_{-\infty}^{\infty} (1 - D^*(N, r, x)) \textstyle\sum_{k=i+1}^{N} P\{x_1 = y_k \mid x_1 = x\} \, dH(x) \end{aligned}$$

where

(3.2) $$P\{x_1 = y_i \mid x_1 = x\} = \binom{N-1}{i-1}(1 - f(r, r-x))^{i-1}(f(r, r-x))^{N-i}.$$

TABLE 1

The optimum decision function for* $1 \leqq N \leqq 1000$

N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$
1	1.00000000	51	.02230311	101	.01130388	151	.00757057	201	.00569104	251	.00455916	301	.00380283	351	.00326174	401	.00285544	451	.00253916
2	.50000000	52	.02187731	102	.01119348	152	.00752089	202	.00566293	252	.00454110	302	.00379026	352	.00325248	402	.00284835	452	.00253354
3	.34534633	53	.02146748	103	.01108522	153	.00747186	203	.00563508	253	.00452318	303	.00377776	353	.00324328	403	.00284129	453	.00252796
4	.26452861	54	.02107272	104	.01097903	154	.00742347	204	.00560751	254	.00450540	304	.00376535	354	.00323412	404	.00283426	454	.00252239
5	.21456097	55	.02069222	105	.01087485	155	.00737570	205	.00558021	255	.00448776	305	.00375303	355	.00322503	405	.00282727	455	.00251685
6	.18054385	56	.02032522	106	.01077263	156	.00732854	206	.00555318	256	.00447026	306	.00374078	356	.00321598	406	.00282031	456	.00251134
7	.15586951	57	.01997101	107	.01067232	157	.00728197	207	.00552640	257	.00445289	307	.00372861	357	.00320698	407	.00281339	457	.00250585
8	.13714514	58	.01962893	108	.01057386	158	.00723600	208	.00549988	258	.00443566	308	.00371652	358	.00319803	408	.00280650	458	.00250038
9	.12244621	59	.01929838	109	.01047720	159	.00719060	209	.00547362	259	.00441856	309	.00370451	359	.00318913	409	.00279965	459	.00249494
10	.11059860	60	.01897878	110	.01038229	160	.00714577	210	.00544760	260	.00440159	310	.00369257	360	.00318028	410	.00279282	460	.00248952
11	.10084490	61	.01866960	111	.01028909	161	.00710150	211	.00542183	261	.00438475	311	.00368071	361	.00317148	411	.00278604	461	.00248413
12	.09267444	62	.01837032	112	.01019754	162	.00705777	212	.00539631	262	.00436804	312	.00366893	362	.00316273	412	.00277928	462	.00247875
13	.08573027	63	.01808050	113	.01010761	163	.00701457	213	.00537012	263	.00435146	313	.00365722	363	.00315403	413	.00277256	463	.00247340
14	.07975532	64	.01779967	114	.01001925	164	.00697191	214	.00534597	264	.00433500	314	.00364559	364	.00314537	414	.00276587	464	.00246808
15	.07455976	65	.01752744	115	.00993242	165	.00692975	215	.00532115	265	.00431866	315	.00363403	365	.00313677	415	.00275921	465	.00246278
16	.07000031	66	.01726341	116	.00984708	166	.00688810	216	.00529656	266	.00430245	316	.00362255	366	.00312820	416	.00275258	466	.00245750
17	.06596680	67	.01700722	117	.00976320	167	.00684696	217	.00527220	267	.00428636	317	.00361113	367	.00311969	417	.00274599	467	.00245224
18	.06237315	68	.01675852	118	.00968073	168	.00680630	218	.00524805	268	.00427039	318	.00359979	368	.00311122	418	.00273942	468	.00244700
19	.05915107	69	.01651699	119	.00959965	169	.00676612	219	.00522413	269	.00425454	319	.00358852	369	.00310280	419	.00273289	469	.00244179
20	.05624574	70	.01628232	120	.00951991	170	.00672641	220	.00520043	270	.00423881	320	.00357732	370	.00309442	420	.00272639	470	.00243660
21	.05361262	71	.01605423	121	.00944149	171	.00668716	221	.00517694	271	.00422319	321	.00356619	371	.00308609	421	.00271992	471	.00243143
22	.05121515	72	.01583244	122	.00936435	172	.00664838	222	.00515366	272	.00420769	322	.00355513	372	.00307781	422	.00271348	472	.00242628
23	.04902303	73	.01561670	123	.00928846	173	.00661003	223	.00513060	273	.00419230	323	.00354414	373	.00306956	423	.00270707	473	.00242116
24	.04701095	74	.01540676	124	.00921379	174	.00657213	224	.00510773	274	.00417702	324	.00353321	374	.00306136	424	.00270070	474	.00241605
25	.04515760	75	.01520239	125	.00914031	175	.00653466	225	.00508057	275	.00416185	325	.00352235	375	.00305321	425	.00269435	475	.00241097
26	.04344491	76	.01500337	126	.00906799	176	.00649761	226	.00506261	276	.00414679	326	.00351156	376	.00304510	426	.00268803	476	.00240591
27	.04185743	77	.01480949	127	.00899681	177	.00646099	227	.00504035	277	.00413184	327	.00350084	377	.00303703	427	.00268174	477	.00240087
28	.04038193	78	.01462056	128	.00892674	178	.00642477	228	.00501828	278	.00411700	328	.00349018	378	.00302900	428	.00267548	478	.00239585
29	.03900694	79	.01443640	129	.00885775	179	.00638896	229	.00499640	279	.00410227	329	.00347958	379	.00302102	429	.00266925	479	.00239086
30	.03772254	80	.01425681	130	.00878982	180	.00635354	230	.00497472	280	.00408764	330	.00346905	380	.00301308	430	.00266305	480	.00238588
31	.03652005	81	.01408164	131	.00872292	181	.00631851	231	.00495322	281	.00407311	331	.00345858	381	.00300518	431	.00265687	481	.00238092
32	.03539189	82	.01391072	132	.00865703	182	.00628337	232	.00493190	282	.00405869	332	.00344818	382	.00299732	432	.00265073	482	.00237599
33	.03433135	83	.01374390	133	.00859213	183	.00624961	233	.00491077	283.	00404437	333	.00343783	383	.00298950	433	.00264461	483	.00237107
34	.03333255	84	.01358103	134	.00852820	184	.00621572	234	.00488982	284	.00403015	334	.00342755	384	.00298172	434	.00263853	484	.00236618
35	.03239024	85	.01342198	135	.00846521	185	.00618219	235	.00486905	285	.00401603	335	.00341733	385	.00297399	435	.00263247	485	.00236130
36	.03149975	86	.01326661	136	.00840315	186	.00614902	236	.00484845	286	.00400200	336	.00340718	386	.00296629	436	.00262643	486	.00235645
37	.03065693	87	.01311480	137	.00834199	187	.00611621	237	.00482803	287	.00398808	337	.00339708	387	.00295863	437	.00262043	487	.00235161
38	.02985805	88	.01296642	138	.00828171	188	.00608375	238	.00480778	288	.00397425	338	.00338704	388	.00295102	438	.00261445	488	.00234680
39	.02909975	89	.01282136	139	.00822230	189	.00605162	239	.00478769	289	.00396052	339	.00337706	389	.00294344	439	.00260850	489	.00234200
40	.02837903	90	.01267952	140	.00816373	190	.00601984	240	.00476778	290	.00394688	340	.00336714	390	.00293590	440	.00260258	490	.00233723
41	.02769315	91	.01254077	141	.00810599	191	.00598839	241	.00474803	291	.00393333	341	.00335728	391	.00292840	441	.00259668	491	.00233247
42	.02703965	92	.01240503	142	.00804907	192	.00595726	242	.00472844	292	.00391988	342	.00334747	392	.00292093	442	.00259081	492	.00232773
43	.02641629	93	.01227220	143	.00799293	193	.00592646	243	.00470901	293	.00390652	343	.00333772	393	.00291351	443	.00258497	493	.00232302
44	.02582102	94	.01214219	144	.00793758	194	.00589597	244	.00468975	294	.00389325	344	.00332803	394	.00290612	444	.00257915	494	.00231832
45	.02525200	95	.01201490	145	.00728298	195	.00586580	245	.00467064	295	.00388007	345	.00331840	395	.00289877	445	.00257336	495	.00231364
46	.02470753	96	.01189024	146	.00788914	196	.00583593	246	.00465168	296	.00386698	346	.00330882	396	.00289146	446	.00256760	496	.00230898
47	.02418603	97	.01176816	147	.00777602	197	.00580637	247	.00463288	297	.00385398	347	.00329929	397	.00288418	447	.00256186	497	.00230434
48	.02368611	98	.01164855	148	.00772362	198	.00577710	248	.00461423	298	.00384106	348	.00328982	398	.00287694	448	.00255615	498	.00229971
49	.02320643	99	.01153135	149	.00767192	199	.00574813	249	.00459573	299	.00382824	349	.00328041	399	.00286974	449	.00255046	499	.00229511
50	.02274580	100	.01141648	150	.00762091	200	.00571944	250	.00457737	300	.00381549	350	.00327105	400	.00286258	450	.00254480	500	.00229052

* The optimum decision function: $D^*(N, r, x) = 1$ if $H(r+|r-x|) - H(r-|r-x|) \leqq 1 - S(N-1)$

$= 0$ if $H(r+|r-x|) - H(r-|r-x|) > 1 - S(N-1)$

TABLE 1—*continued*

N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$	N	$1-S(N-1)$
501	.00228595	551	.00207867	601	.00190585	651	.00175956	701	.00163413	751	.00152539	801	.00143022	851	.00134623	901	.00127156	951	.00120473
502	.00228140	552	.00207490	602	.00190269	652	.00175686	702	.00163180	752	.00152337	802	.00142844	852	.00134465	902	.00127015	952	.00120347
503	.00227687	553	.00207115	603	.00189953	653	.00175418	703	.00162948	753	.00152134	803	.00142666	853	.00134308	903	.00126874	953	.00120220
504	.00227236	554	.00206742	604	.00189639	654	.00175150	704	.00162717	754	.00151933	804	.00142489	854	.00134151	904	.00126734	954	.00120094
505	.00226786	555	.00206370	605	.00189326	655	.00174882	705	.00162486	755	.00151732	805	.00142312	855	.00133993	905	.00126594	955	.00119969
506	.00226338	556	.00205999	606	.00189014	656	.00174616	706	.00162256	756	.00151531	806	.00142135	856	.00133837	906	.00126454	956	.00119843
507	.00225892	557	.00205629	607	.00188702	657	.00174350	707	.00162027	757	.00151331	807	.00141959	857	.00133681	907	.00126315	957	.00119718
508	.00225448	558	.00205261	608	.00188392	658	.00174086	708	.00161798	758	.00151131	808	.00141784	858	.00133525	908	.00126176	958	.00119593
509	.00225005	559	.00204894	609	.00188083	659	.00173822	709	.00161570	759	.00150932	809	.00141609	859	.00133370	909	.00126037	959	.00119469
510	.00224564	560	.00204529	610	.00187775	660	.00173558	710	.00161343	760	.00150734	810	.00141434	860	.00133215	910	.00125899	960	.00119344
511	.00224125	561	.00204164	611	.00187468	661	.00173296	711	.00161116	761	.00150536	811	.00141260	861	.00133060	911	.00125761	961	.00119220
512	.00223688	562	.00203801	612	.00187162	662	.00173034	712	.00160890	762	.00150338	812	.00141086	862	.00132906	912	.00125623	962	.00119096
513	.00223252	563	.00203439	613	.00186856	663	.00172773	713	.00160664	763	.00150141	813	.00140912	863	.00132752	913	.00125485	963	.00118973
514	.00222818	564	.00203079	614	.00186552	664	.00172513	714	.00160440	764	.00149945	814	.00140739	864	.00132598	914	.00125348	964	.00118849
515	.00222386	565	.00202720	615	.00186249	665	.00172254	715	.00160215	765	.00149749	815	.00140567	865	.00132445	915	.00125211	965	.00118726
516	.00221955	566	.00202362	616	.00185947	666	.00171996	716	.00159992	766	.00149554	816	.00140395	866	.00132292	916	.00125074	966	.00118603
517	.00221526	567	.00202005	617	.00185646	667	.00171738	717	.00159768	767	.00149359	817	.00140223	867	.00132140	917	.00124938	967	.00118481
518	.00221099	568	.00201650	618	.00185346	668	.00171481	718	.00159546	768	.00149165	818	.00140051	868	.00131988	918	.00124802	968	.00118358
519	.00220673	569	.00201296	619	.00185047	669	.00171225	719	.00159324	769	.00148971	819	.00139880	869	.00131836	919	.00124666	969	.00118236
520	.00220249	570	.00200943	620	.00184748	670	.00170970	720	.00159103	770	.00148777	820	.00139710	870	.00131684	920	.00125431	970	.00118114
521	.00219827	571	.00200591	621	.00184451	671	.00170715	721	.00158883	771	.00148584	821	.00139540	871	.00131533	921	.00124396	971	.00117993
522	.00219406	572	.00200241	622	.00184155	672	.00170461	722	.00158663	772	.00148392	822	.00139370	872	.00131383	922	.00124261	972	.00117871
523	.00218987	573	.00199891	623	.00183859	673	.00170208	723	.00158443	773	.00148200	823	.00139201	873	.00131232	923	.00124126	973	.00117750
524	.00218569	574	.00199543	624	.00183565	674	.00169955	724	.00158225	774	.00148009	824	.00139032	874	.00131082	924	.00123992	974	.00117629
525	.00218153	575	.00199197	625	.00183271	675	.00169704	725	.00158007	775	.00147818	825	.00138864	875	.00130932	925	.00123858	975	.00117509
526	.00217739	576	.00198851	626	.00182979	676	.00169453	726	.00157789	776	.00147628	826	.00138696	876	.00130783	926	.00123724	976	.00117389
527	.00217326	577	.00198507	627	.00182687	677	.00169203	727	.00157572	777	.00147438	827	.00138528	877	.00130634	927	.00123591	977	.00117268
528	.00216915	578	.00198163	628	.00182396	678	.00168953	728	.00157356	778	.00147248	828	.00138361	878	.00130485	928	.00123458	978	.00117149
529	.00216505	579	.00197821	629	.00182106	679	.00168705	729	.00157140	779	.00147059	829	.00138194	879	.00130337	929	.00123325	979	.00117029
530	.00216097	580	.00197481	630	.00181818	680	.00168457	730	.00156925	780	.00146871	830	.00138028	880	.00130189	930	.00123192	980	.00116910
531	.00215690	581	.00197141	631	.00181530	681	.00168209	731	.00156710	781	.00146683	831	.00137862	881	.00130041	931	.00123060	981	.00116790
532	.00215285	582	.00196802	632	.00181243	682	.00167963	732	.00156496	782	.00146495	832	.00137696	882	.00129894	932	.00122928	982	.00116672
533	.00214881	583	.00196465	633	.00180957	683	.00167717	733	.00156283	783	.00146308	833	.00137531	883	.00129747	933	.00122796	983	.00116553
534	.00214479	584	.00196129	634	.00180671	684	.00167472	734	.00156070	784	.00146122	834	.00137366	884	.00129600	934	.00122665	984	.00116435
535	.00214079	585	.00195794	635	.00180387	685	.00167228	735	.00155858	785	.00145936	835	.00137201	885	.00129454	935	.00122534	985	.00116316
536	.00213679	586	.00195460	636	.00180103	686	.00166984	736	.00155646	786	.00145750	836	.00137037	886	.00129308	936	.00122403	986	.00116198
537	.00213282	587	.00195127	637	.00179821	687	.00166741	737	.00155435	787	.00145565	837	.00136874	887	.00129162	937	.00122272	987	.00116081
538	.00212886	588	.00194796	638	.00179539	688	.00166499	738	.00155225	788	.00145381	838	.00136710	888	.00129016	938	.00122142	988	.00115963
539	.00212491	589	.00194465	639	.00179258	689	.00166257	739	.00155015	789	.00145196	839	.00136548	889	.00128871	939	.00122012	989	.00115846
540	.00212098	590	.00194136	640	.00178979	690	.00166017	740	.00154805	790	.00145013	840	.00136385	890	.00128727	940	.00121882	990	.00115729
541	.00211706	591	.00193808	641	.00178700	691	.00165777	741	.00154597	791	.00144830	841	.00136223	891	.00128582	941	.00121753	991	.00115613
542	.00211316	592	.00193480	642	.00178421	692	.00165537	742	.00154388	792	.00144647	842	.00136061	892	.00128438	942	.00121624	992	.00115496
543	.00210927	593	.00193154	643	.00178144	693	.00165298	743	.00154181	793	.00144464	843	.00135900	893	.00128294	943	.00121495	993	.00115380
544	.00210540	594	.00192829	644	.00177867	694	.00165060	744	.00153974	794	.00144283	844	.00135739	894	.00128151	944	.00121366	994	.00115264
545	.00210154	595	.00192505	645	.00177592	695	.00164823	745	.00153767	795	.00144101	845	.00135579	895	.00128008	945	.00121238	995	.00115148
546	.00209769	596	.00192183	646	.00177317	696	.00164586	746	.00153561	796	.00143920	846	.00135418	896	.00127865	946	.00121110	996	.00115032
547	.00209386	597	.00191861	647	.00177043	697	.00164350	747	.00153356	797	.00143740	847	.00135258	897	.00127722	947	.00120982	997	.00114917
548	.00209004	598	.00191540	648	.00176770	698	.00164115	748	.00153151	798	.00143560	848	.00135099	898	.00127580	948	.00120854	998	.00114802
549	.00208623	599	.00191221	649	.00176498	699	.00163880	749	.00152946	799	.00143380	849	.00134940	899	.00127438	949	.00120727	999	.00114629
550	.00208244	600	.00190902	650	.00176227	700	.00163646	750	.00152742	800	.00143201	850	.00134781	900	.00127297	950	.00120600	1000	.00114572

Then using

$$\int_{-\infty}^{\infty} P\{x_1 = y_i \mid x_1 = x\}\, dH(x) = P\{x_1 = y_i\} = 1/N$$

and the result (A.4) in the appendix, (3.1) can be rewritten as:

$$(3.3)\quad P_N(i) = J(N-1, N, i) - P_{N-1}(i-1)\Big[\sum_{k=1}^{i-1} J(N-1, N, k) - (i-1)/N\Big] - P_{N-1}(i)\Big[\sum_{k=i+1}^{N} J(N-1, N, k) - (N-i)/N\Big].$$

Let $nJ(N, n, i) = 1 - B(N, n, i)$ where $B(N, n, i)$ has the binomial form:

$$(3.4)\quad B(N, n, i) = \sum_{k=0}^{i-1} \binom{n}{k} (1 - S(N))^k (S(N))^{n-k}.$$

Employing the identity:

$$(3.5)\quad \sum_{k=1}^{i} B(N, n, k) = iB(N, n, i+1) - n(1 - S(N))B(N, n-1, i),$$

(3.3) can be finally written as:

$$(3.6)\quad NP_N(i) = (1 - B(N-1, N, i)) + P_{N-1}(i-1)[(i-1)B(N-1, N, i) - N(1 - S(N-1)) \cdot B(N-1, N-1, i-1)] + P_{N-1}(i)[NS(N-1) - iB(N-1, N, i) + N(1 - S(N-1)) \cdot B(N-1, N-1, i-1)].$$

It is readily shown that $\sum_{i=1}^{N} P_N(i) = 1$ and that $P_N(i)$ is monotonically decreasing for $i = 1, 2, \cdots, N$.

4. A moment of interest. If x is the value of the rv chosen when following the optimum decision function on a sample of size N, then let:

$$(4.1)\quad y(N, r, x) = H(r + |r - x|) - H(r - |r - x|) \geqq 0.$$

$y(N, r, x)$ is therefore the probability that an arbitrary rv selected from the distribution $H(x)$ will be closer to the real number r than the rv x chosen by the sequential decision process.

The expected value of $y(N, r, x)$ is therefore:

$$(4.2)\quad \begin{aligned} E(y(N, r, x)) &= \int_{-\infty}^{\infty} D^*(N, r, x) y(N, r, x)\, dH(x) \\ &\quad + E(y(N-1, r, x)) \int_{-\infty}^{\infty} (1 - D^*(N, r, x))\, dH(x) \\ &= J(N-1, 2, 2) + E(y(N-1, r, x))[1 - J(N-1, 1, 1)] \\ &= S(N-1)E(y(N-1, r, x)) + \tfrac{1}{2}(1 - S(N-1))^2 \end{aligned}$$

for $N \geqq 2$ and $E(y(1, r, x)) = \frac{1}{2}$.

This recursive relation is easily solved; however it is more informative in the above form.

APPENDIX

When following the optimum decision function, the following integral often arises.

(A.1) $$J(N,n,i) = \binom{n-1}{i-1}\int_{-\infty}^{\infty} D^*(N+1,r,x)(1-f(r,r-x))^{i-1}(f(r,r-x))^{n-i}\,dH(x)$$

where

$$D^*(N+1, r, x) = 1 \quad \text{if} \quad f(r, r-x) \geqq S(N)$$
$$= 0 \quad \text{if} \quad f(r, r-x) < S(N)$$

and $f(r, x) = f(r, -x) = 1 + H(r-|x|) - H(r+|x|)$ is a monotonically decreasing function of $x \geqq 0$. (A.1) can therefore be written as:

(A.2) $$J(N, n, i) = \binom{n-1}{i-1}\int_a^{2r-a}(1-f(r, r-x))^{i-1}(f(r, r-x))^{n-i}\,dH(x)$$

with $a < r$ and $f(r, r-a) = S(N)$. Thus

(A.3) $$J(N, n, i) = -\binom{n-1}{i-1}\int_0^{r-a}(1-f(r, u))^{i-1}(f(r, u))^{n-i}\,df(r, u).$$

Successive integration by parts yields:

(A.4) $$J(N, n, i) = J(N, n, i+1) + \binom{n}{i}(1-S(N))^i(S(N))^{n-i}/n$$
$$= \sum_{k=i}^{n}\binom{n}{k}(1-S(N))^k(S(N))^{n-k}/n.$$

REFERENCES

[1] ENNS, E. G. (1970). The optimum strategy for choosing the maximum of independent random variables. *Unternehmensforschung* **14** 89–96.

[2] KARLIN, S. (1962). Stochastic models and optimal policy for selling an asset. *Studies in Applied Probability and Management Science*. Stanford Univ. Press.