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AN APPLICATION OF EXTREME VALUE THEORY TO RELIABILITY THEORY[1]

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0. Introduction. The limiting distribution of the maximum term in a sequence of independent, identically distributed random variables was completely analysed in a series of works by many writers, culminating in the comprehensive work of Gnedenko [4]. Results for order statistics of fixed and increasing rank were obtained by Smirnov [10], who completely characterized the limiting types and their domains of attraction. Generalizations of these results for the maximum term have been made by several writers; Juncosa [7] dropped the assumption of a common distribution, Watson [11] proved that under slight restrictions the limiting distribution of the maximum term in a stationary sequence of m-dependent random variables is the same as in the independent case, and Berman [1] studied exchangeable random variables and samples of random size. A bibliography and discussion of applications is contained in the book by Gumbel [6].

This paper extends the classical theory by introducing a model from reliability theory—essentially a series system with replaceable components. It is shown that the asymptotic distribution of system lifetime can belong to one of two types when the number of spares is fixed or of a smaller order than the total number n of components, as n becomes infinite, and that these limiting distributions are the same as those obtained by Gnedenko, Chibisov [2] and Smirnov.

1. Notation and classical results. Throughout this paper, the distribution function of a random variable X will be denoted by $P\{X \leqq x\} = F(x)$, and the tail of the distribution by $P\{X > x\} = \bar{F}(x)$. The abbreviation "df" will be used for distribution function. A df will be called *proper* if:

$$\lim_{x\to\infty} F(x) = 1, \qquad \lim_{x\to-\infty} F(x) = 0$$

and not all its mass is concentrated at one point. Two df's $F_1(x)$ and $F_2(x)$ are said to be of the *same type* if there exist constants $A > 0$ and B such that: $F_1(Ax+B) = F_2(x)$ for all values of x. Unless otherwise stated, all df's will be assumed proper and all limiting df's should be taken to mean limiting types of df's. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables with common distribution $F(x)$, and let $\xi_n = \min(X_1, X_2, \cdots, X_n)$. Then the limiting df of ξ_n belongs to exactly one of three types [4]; that is to say, if there exist sequences of normalizing constants $\{a_n > 0\}$ and $(b_n\}$ and a df $G(x)$ such that:

$$\lim_{n\to\infty} P\{a_n^{-1}(\xi_n - b_n) \leqq x\} = G(x)$$

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at each continuity point of $G(x)$, then $G(x)$ belongs to one of the following types:

$$(1.1)\qquad \begin{aligned} \Phi_{(1)}(x) &= 0 && \text{for} \quad x \leqq 0 \\ &= 1-\exp[-x^{\alpha}] && \text{for} \quad x > 0, \alpha > 0 \\ \Phi_{(2)}(x) &= 1-\exp[-(-x)^{-\alpha}] && \text{for} \quad x < 0, \alpha > 0 \\ &= 1 && \text{for} \quad x \geqq 0 \\ \Phi_{(3)}(x) &= 1-\exp[-\exp x] && -\infty < x < \infty. \end{aligned}$$

The *domain of attraction* of a limiting df $G(x)$ is the set of all df's $F(x)$ such that for suitable choice of normalizing constants $\{a_n > 0\}$ and $\{b_n\}$

$$(1.2)\qquad \lim_{n\to\infty} \bar{F}^n(a_n x + b_n) = \bar{G}(x).$$

By a well-known theorem of Khintchine (e.g., see [5] page 40), each df can belong to at most one domain of attraction. Necessary and sufficient conditions were given by Gnedenko [4] for a df to belong to the domain of attraction of $\Phi_{(1)}(x)$, $\Phi_{(2)}(x)$ or $\Phi_{(3)}(x)$. For example, $F(x)$ is in the domain of attraction of $\Phi_{(1)}(x)$ if and only if $\exists x_0$ such that $F(x_0) = 0$, $F(x_0+\varepsilon) > 0$ for each $\varepsilon > 0$ and

$$(1.3)\qquad \lim_{x\to 0+} F(x_0+tx)/F(x_0+x) = t^{\alpha} \qquad \text{for all} \quad t > 0.$$

The kth smallest variable from $(X_1, X_2, \cdots, X_n)$ will be denoted by $\xi_n^{(k)}$, so that $\xi_n^{(1)} = \xi_n$; limiting df's for these random variables as obtained by Smirnov and Chibisov will be introduced as needed.

2. Structures with replacement. The problem that is investigated here is the following: a system consists of n identical and independent components in series, with m inactive spare components available which instantaneously replace the components as they fail, until there are no more spares, whereupon the system fails. The system lifetime will be denoted by $\eta_n^{(m+1)}$, $(m+1)$ being the total number of component failures which must occur before system failure. The investigation is in two parts, corresponding to the cases when $m = m(n)$ is of a smaller order than n or of the same order as n, and a third subsection describes how some of the results may be carried over to more general types of systems. It is assumed in this section that $F(0-) = 0$.

Extreme terms. Let $G_{nm}^*(x) = P\{\eta_n^{(m+1)} \leqq x\}$. Then it is shown that the class of limiting df's for the system lifetime as $n \to \infty$, with appropriate linear norming constants, is the same as the limiting df's of the corresponding order statistics provided that m is finite or of smaller order than $n^{\frac{1}{2}}$, as in the following two theorems.

THEOREM 2.1. *The limit laws for sequences $G_{nm}^*(a_n x + b_n)$ of system lifetime* df's, *with m fixed, are exhausted by the following two types*:

$$\Phi^{(m)}_{(1)}(x) = 0 \qquad \textit{for} \quad x \leqq 0$$

(2.1)
$$= \frac{1}{(m-1)!}\int_0^{x^\alpha} e^{-y}y^{m-1}\,dy \qquad \textit{for} \quad x > 0, \alpha > 0$$

$$\Phi^{(m)}_{(3)}(x) = \frac{1}{(m-1)!}\int_0^{e^x} e^{-y}y^{m-1}\,dy \qquad -\infty < x < \infty.$$

THEOREM 2.2. *If $m \sim cn^\alpha$, with $c > 0$, $0 < \alpha < \frac{1}{2}$, then the only possible limit* df'*s for the sequence $G^*_{nm}(a_n x + b_n)$ are*:

(2.2)
$$G_{(1)}(x) = \Phi(x)$$
$$G_{(2)}(x) = 0 \qquad \textit{for} \quad x \leqq 0$$
$$= \Phi(\beta \log x) \qquad \textit{for} \quad x > 0, \beta > 0.$$

Notice that $G_{(2)}(x)$ is the log normal df.

Some preliminary results are needed before the proofs of Theorem 2.1 and Theorem 2.2 can be given:

(2.3)
$$\bar{G}^*_{nm}(x) = \textstyle\sum_{j=0}^m \sum_{i_1+\cdots+i_n=j} \prod_{k=1}^n \{F^{(i_k)}(x) - F^{(i_k+1)}(x)\}.$$

Where $F^{(k)}(x)$ is the k-fold convolution of the df $F(x)$ and the inner summation is over all nonnegative combinations of $(i_1, i_2, \cdots, i_n)$ which sum to j. This formula follows from the superposition of n identical renewal processes.

The df $F(x)$ will be assumed to be concentrated on the nonnegative real axis in this section since the concept of component lifetime is meaningful only in this case. Use will be made of the inequality

(2.4)
$$F^{(k)}(x) \leqq \{F(x)\}^k, \qquad \forall k \geqq 1, \forall x \geqq 0.$$

It is convenient to speak of n "sockets" in series, each of which must contain a working component for the system to work. When m is not too large, a key step in the proofs will be to show that the probability of two or more failures in any socket is negligible as $n \to \infty$. Define

(2.5)
$$\bar{G}_{nm}(x) = \textstyle\sum_{j=0}^m \binom{n}{j} \bar{F}^{n-j}(x) F^j(x)$$

i.e., the survival probability of an $(m+1)$-out-of-n system.

THEOREM 2.3. *If $m = o(n^{\frac{1}{2}})$ as $n \to \infty$, and if $\{a_n > 0\}$ and $\{b_n\}$ are sequences of normalizing constants such that*

(2.6)
$$F(a_n x + b_n) = o(n^{-\frac{1}{2}}) \qquad \textit{as} \quad n \to \infty, \forall x \geqq 0, \quad \textit{then}$$

$$\lim_{n\to\infty} |\bar{G}_{nm}(a_n x + b_n) - \bar{G}^*_{nm}(a_n x + b_n)| = 0, \qquad \forall x \geqq 0.$$

The proof of this theorem will depend on the following lemmas.

LEMMA 2.1.

(i) *The number of ways in which j failures can occur, in such a way that at most one failure occurs in each socket, is* $\binom{n}{j}$.

(ii) *The total number of ways in which j failures can occur, the number of failures in any socket being arbitrary, is* $\binom{n+j-1}{j}$.

The proof of this lemma is from elementary probability. Assertion (ii) appears in Feller [3] page 38.

LEMMA 2.2. *If* $0 \leqq j \leqq m$, *and* $m = o(n^{\frac{1}{2}})$ *as* $n \to \infty$, *then* $\binom{n}{j}/\binom{n+j-1}{j} \to 1$, *as* $n \to \infty$.

PROOF. By Stirling's formula or elementary calculations.

PROOF OF THEOREM 2.3. Define the following notation:

(i) $A_{nj} = \binom{n+j-1}{j} - \binom{n}{j}$

(ii) $u_{nj}(x) = \bar{F}^{n-j}(x)F^j(x)$

(iii) $v_{nj}(x) = \bar{F}^{n-j}(x)\{F(x) - F^{(2)}(x)\}^j$

(iv) $w_{nj}(x)$ will be used for all terms of the form: $\prod_{k=1}^{n} \{F^{(i_k)}(x) - F^{(i_k+1)}(x)\}$, where $i_1 + \cdots + i_n = j$ and at least one of the $i_k \geqq 2$. Notice from (2.4) that:

$$0 \leqq v_{nj}(x) \leqq u_{nj}(x), \qquad 0 \leqq w_{nj}(x) \leqq u_{nj}(x). \tag{2.7}$$

Now: $\bar{G}^*_{nm}(x)/\bar{G}_{nm}(x) = \{\sum_{j=0}^{m} [\binom{n}{j} v_{nj}(x) + A_{nj} w_{nj}(x)]\} / \sum_{j=0}^{m} \binom{n}{j} u_{nj}(x)$.

But

$$\begin{aligned} 0 &\leqq \textstyle\sum_{j=0}^{m} A_{nj} w_{nj}(x) / \sum_{j=0}^{m} \binom{n}{j} u_{nj}(x) \\ &\leqq \max_{j=0,\cdots,m} A_{nj} w_{nj}(x)/\binom{n}{j} u_{nj}(x) \\ &\to 0, \qquad \text{by (2.7) and Lemma 2.2.} \end{aligned}$$

Also

$$\begin{aligned} 0 &\leqq 1 - \textstyle\sum_{j=0}^{m} \binom{n}{j} v_{nj}(x) / \sum_{j=0}^{m} \binom{n}{j} u_{nj}(x) \\ &= \textstyle\sum_{j=0}^{m} \binom{n}{j} [u_{nj}(x) - v_{nj}(x)] / \sum_{j=0}^{m} \binom{n}{j} u_{nj}(x) \\ &\leqq \max_{j=0,\cdots,m} [u_{nj}(x) - v_{nj}(x)]/u_{nj}(x) \\ &= 1 - [1 - F^{(2)}(x)/F(x)]^m \\ &\leqq 1 - [1 - F(x)]^m, \qquad \text{by use of (2.4).} \end{aligned}$$

Now if x is replaced by $(a_n x + b_n)$ and the second assumption of the theorem used, it is seen that the last term approaches zero as $n \to \infty$. Combining results:

$$|\bar{G}^*_{nm}(a_n x + b_n)/\bar{G}_{nm}(a_n x + b_n) - 1| \to 0, \qquad \text{as} \quad n \to \infty,$$

and since df's are bounded, the theorem is proved.

PROOF OF THEOREM 2.1 AND THEOREM 2.2. To examine the possible normalizing sequences $\{a_n\}$ and $\{b_n\}$ which satisfy the conditions of Theorem 2.3, it is necessary to consider separately the cases where m remains finite or $m \to \infty$. Suppose first that m remains finite. Then Smirnov [10] has shown that in order for

$$G_{nm}(a_n x + b_n) \to G(x) \tag{2.8}$$

for suitable choice of normalizing constants, where $G(x)$ is a proper df, it is necessary and sufficient that

$$v_n(x) = nF(a_n x + b_n) \to v(x) \tag{2.9}$$

where $v(x)$ is a nondecreasing nonnegative function defined by:

$$\frac{1}{(m-1)!}\int_0^{v(x)} e^{-y} y^{m-1}\, dy = G(x). \tag{2.10}$$

Furthermore, he proved that (up to a linear transformation) the function $v(x)$ must be one of the three forms x^α, $(-x)^{-\alpha}$ or e^x, where α is an arbitrary positive constant. The domain of attraction corresponding to the second form for $v(x)$ consists of df's which are unbounded below, so that on using Theorem 2.3 and the nonnegativity assumption on the $\{X_i\}$, Theorem 2.1 is proved.

Now suppose that $m \sim cn^\alpha$, where $c > 0$, $0 < \alpha < \frac{1}{2}$. Chibisov [2] has shown that $G_{nm}(a_n x + b_n) \to G(x)$ if and only if

$$u_n(x) = m^{-\frac{1}{2}}\{nF(a_n x + b_n) - m\} \to u(x) \tag{2.11}$$

where $u(x)$ is defined by the equation

$$G(x) = \Phi(u(x)) \tag{2.12}$$

and Φ is the normal (0, 1) df. The function $u(x)$ must be of the same type as one of x, $\beta \log x$ or $-\beta \log|x|$, where $\beta > 0$ is an arbitrary constant, and the domain of attraction corresponding to the third form contains only df's which are unbounded below. For a normalizing sequence which satisfies (2.11), it is clear that $F(a_n x + b_n) = O(n^{\alpha-1}) = o(n^{-\frac{1}{2}})$; thus the conditions of Theorem 2.3 are satisfied and Theorem 2.2 is proved.

Similarly, characterizations of the domains of attraction of these limit df's may be made. Note also that one might wish to restrict the limiting law itself to correspond to a nonnegative random variable, thus eliminating one of the types in Theorem 2.1 and Theorem 2.2.

The assumption that the spares have the same lifetime df as the original components is unnecessary; any df $F^*(x)$ such that $F^*(a_n x + b_n) = o(n^{-\frac{1}{2}})$ will suffice. The appropriate modifications to the proof of Theorem 2.3 present no difficulty.

It would be desirable to relax the restriction $\alpha < \frac{1}{2}$ which appears in the conditions of Theorem 2.2. Results may be obtained for $\alpha < \frac{2}{3}$ as described by Lemma 2.3 and Lemma 2.4, but the more general case $\alpha < 1$ does not seem amenable to analysis and a counter-intuitive reason for this is given in Lemma 2.5 and Lemma 2.6.

Let the symbol "$\geqq_{st}$" stand for "stochastically greater than."

LEMMA 2.3. *For independent, identically distributed nonnegative component random variables*

$$\xi_n^{(m)} \geqq_{\mathrm{st}} \eta_n^{(m)} \geqq_{\mathrm{st}} \xi_{n+m}^{(m)}$$

where, as before, $\xi_n^{(m)}$ *is the* mth *smallest order statistic from a sample of size* n $(m \leqq n)$.

PROOF. The first part of the inequality follows by observing the replacements themselves may fail, thus giving rise to more failures; the second part by observing that time to system failure decreases if the spares are subject to failure from the initial instant.

LEMMA 2.4. *If* $m \sim cn^{\alpha}$, *with* $c > 0$, $\frac{1}{2} \leqq \alpha < \frac{2}{3}$, *then the limit* df's (2.2) *are possible for the sequence* $G_{nm}^*(a_n x + b_n)$.

PROOF. Suppose that F and $\{a_n > 0\}$, $\{b_n\}$ are such that (2.11) and (2.12) hold, so that

$$F(a_n x + b_n) = m/n + u(x)m^{\frac{1}{2}}/n + o(m^{\frac{1}{2}}/n). \tag{2.13}$$

Then

$$[(n+m)F(a_n x + b_n) - m]/m^{\frac{1}{2}} = u(x) + O(m^{\frac{3}{2}}/n) \to u(x). \tag{2.14}$$

Thus both $a_n^{-1}(\xi_n^{(m)} - b_n)$ and $a_n^{-1}(\xi_{n+m}^{(m)} - b_n)$ have the same limiting df and hence by Lemma 2.3 so does $a_n^{-1}(\eta_n^{(m)} - b_n)$. Thus with Chibisov's results, the lemma is proved.

It should be noted that although Lemma 2.4 shows that the limiting df's (2.2) are possible, it does not rule out other limiting df's, in contrast to the results of Theorem 2.1 and Theorem 2.2.

The following lemma is related to the classical occupancy problem [3] page 101.

LEMMA 2.5. *The number of ways that* m *failures can occur in* n *sockets with at most* r *failures per socket is*

$$c(n, m, r) = \sum_{i=0}^{[m/(r+1)]} (-1)^i \binom{n}{i} \binom{n+m-ri-i-1}{n-1},$$

where $[x]$ *denotes the largest integer less than or equal to* x.

PROOF. The form of $c(n, m, r)$ follows by observing that it is the coefficient of Z^m in:

$$(1 + Z + Z^2 + \cdots + Z^r)^n = (1 - Z^{r+1})^n (1 - Z)^{-n}.$$

Let $c(n, m) = \binom{n+m-1}{m}$—the total number of ways that m failures can occur in n sockets.

LEMMA 2.6. *If* $m \sim cn^{\alpha}$, *where* $c > 0$, $0 < \alpha < 1$, *and* r *is fixed, then* $c(n, m, r)/c(n, m) \to 1$ *as* $n \to \infty$ *provided* $r + 1 > (1 - \alpha)^{-1}$.

PROOF. Write $c(n, m, r) = a_0 - a_1 + a_2 - \cdots (-)^s a_s$, where $a_i = \binom{n}{i}\binom{n+m-ri-i-1}{n-1}$ and $s = [m/(r+1)]$. Then:

$$\frac{a_i}{a_{i+1}} = \frac{i+1}{n-i} \cdot \frac{(n+m-ri-i-1)!}{(m-ri-i)!} \cdot \frac{(m-ri-r-i-1)!}{(n-m-ri-r-i-2)!}$$
$$\geqq \frac{i+1}{n-i}\left\{1+\frac{n}{m-ri-i}\right\}^{r+1}$$
$$\geqq \frac{1}{n}\left\{1+\frac{n}{m}\right\}^{r+1} = \frac{1}{\rho(n)}, \quad \text{say.}$$

Then $\rho(n) \to 0$ as $n \to \infty$ provided $(r+1) > (1-\alpha)^{-1}$. Now

$$\begin{aligned} |c(n, m, r)/c(n, m) - 1| &= |(a_0 - a_1 + a_2 - \cdots (-)^s a_s)/a_0 - 1| \\ &\leqq |(a_1 + a_2 + \cdots + a_s)/a_0| \\ &\leqq \rho + \rho^2 + \cdots + \rho^s \\ &= \rho(1-\rho^s)/(1-\rho) \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned}$$

When $r = 1$, it may also be shown that $c(n, m, 1)/c(n, m) \to 1$ as $n \to \infty$ only if $\alpha < \frac{1}{2}$. Thus when $\alpha \geqq \frac{1}{2}$, the proportion of possible ways of system failure which involve more than one failure in at least one socket is not negligible. This makes it unlikely that the proof of Theorem 2.3 can be generalized to the case $\alpha \geqq \frac{1}{2}$.

Central terms. The results obtained in the first part of this section are for the limiting df's of extreme terms in which the number of spares is of a smaller order than the number of components in the system; this part treats the central terms where the numbers of spares and components are of the same order. It is shown in Theorem 2.5 that under fairly weak conditions the limiting df of $a_n^{-1}(\eta_n^{(n)} - b_n)$, for appropriate choice of $a_n > 0$ and b_n, is the normal df. For simplicity of notation, it is assumed that $m = n-1$ although it is obvious that Theorem 2.4 and Theorem 2.5 hold with slight modifications when $m = m(n)$ is such that $m(n)/n \to \lambda$, $0 < \lambda < \infty$.

DEFINITION. Following Kolmogorov [8] and Smirnov [10], a sequence $\{X_n\}$ of random variables is said to be *stable* if $\exists$ constants a_n such that $P\{|X_n - a_n| < \varepsilon\} \to 1$, as $n \to \infty$, for each fixed $\varepsilon > 0$.

Theorem 2.4 demonstrates the stability of the sequence of system lifetime $\{\eta_n^{(n)}\}$ under mild restrictions. Some additional notation is needed; let $N_i(t)$ denote the number of component failures in the ith socket up to and including time t, $1 \leqq i \leqq n$, $S_n(t) = \sum_{i=1}^n N_i(t)$ the total number of failures. Set $\mu(t) = E\{N_i(t)\}$ and $\sigma^2(t) = \text{Var}\{N_i(t)\}$ as the mean and variance of $N_i(t)$. It is well known that renewal counting functions $N_i(t)$ have finite moments of all orders for each fixed t so that the existence of $\mu(t)$ and $\sigma(t)$ is guaranteed.

THEOREM 2.4. *If $\mu(t)$ is increasing in a neighborhood of $t = \mu^{-1}(1)$, then the sequence $\{\eta_n^{(n)}\}$ is stable.*

PROOF. Fix $\varepsilon > 0$ and let t_1 be the unique t such that $\mu(t) = 1$. Then

$$\{t_1 - \varepsilon < \eta_n^{(n)} < t_1 + \varepsilon\} \Leftrightarrow \{S_n(t_1+\varepsilon)/n \geqq 1 > S_n(t_1 - \varepsilon)/n\}.$$

For arbitrary $\varepsilon^* > 0$, $P\{|S_n(t)/n - \mu(t)| < \varepsilon^*\} \to 1$, as $n \to \infty$, for all finite t, by the weak law of large numbers. Thus $P\{S_n(t_1+\varepsilon)/n > \mu(t_1+\varepsilon) - \varepsilon^*\} \to 1$, and by choosing ε^* sufficiently small, it is clear that $\mu(t_1+\varepsilon)-\varepsilon^* \geqq 1$ and so $P\{S_n(t_1+\varepsilon)/n > 1\} \to 1$. Similarly, $P\{S_n(t_1-\varepsilon)/n < 1\} \to 1$, so that finally

$$P\{|\eta_n^{(n)} - t_1| < \varepsilon\} \to 1. \tag{2.15}$$

In fact, Theorem 2.4 can be replaced by a stronger result that is analogous to the strong law of large numbers, viz. $P\{\lim \eta_n^{(n)} = t_1\} = 1$. The proof of this is similar to that of Theorem 2.4 with the strong law of large numbers applied to the sum $S_n(t)$.

THEOREM 2.5. *If $\mu(t)$ has a positive first derivative $\mu'(t)$ at t_1 then $n^{\frac{1}{2}}(\eta_n^{(n)} - t_1)$ has a limiting normal* df *with mean zero and variance $\{\sigma(t_1)/\mu'(t_1)\}^2$.*

The proof of this theorem depends on the well-known result:

LEMMA 2.7. *If $\mu(t)$ is continuous at some point t_1, then $\sigma(t)$ is continuous at t_1.*

PROOF OF THEOREM 2.5. For fixed x

$$\begin{aligned}\{n^{\frac{1}{2}}(\eta_n^{(n)} - t_1) > x\} &\Leftrightarrow \{S_n(t_1 + x/n^{\frac{1}{2}}) < n\} \\ &\Leftrightarrow \left\{\frac{S_n(t_1+x/n^{\frac{1}{2}}) - n\mu(t_1+x/n^{\frac{1}{2}})}{n^{\frac{1}{2}}\sigma(t_1+x/n^{\frac{1}{2}})} < \frac{n - n\mu(t_1+x/n^{\frac{1}{2}})}{n^{\frac{1}{2}}\sigma(t_1+x/n^{\frac{1}{2}})}\right\}.\end{aligned} \tag{2.16}$$

Now $S_n(t_1+x/n^{\frac{1}{2}})$ may be written in the form $\sum_{k=1}^n X_{nk}$, where $X_{nk} = N_k(t_1+x/n^{\frac{1}{2}})$; it is clear that the $\{X_{nk}\}$ are independent, identically distributed and have finite moments of all orders. Thus, a modification of Liapunov's version of the central limit theorem (see [9] page 277) may be applied to give:

$$P\left\{\frac{S_n(t_1+x/n^{\frac{1}{2}}) - n\mu(t_1+x/n^{\frac{1}{2}})}{n^{\frac{1}{2}}\sigma(t_1+x/n^{\frac{1}{2}})} \leqq u\right\} \to \Phi(u) \tag{2.17}$$

where, as before, Φ is the normal (0, 1) df.

Now $\mu(t_1+x/n^{\frac{1}{2}})$ may be written in the form

$$\mu(t_1+x/n^{\frac{1}{2}}) = \mu(t_1) + (x/n^{\frac{1}{2}})\mu'(t_1) + o(1/n^{\frac{1}{2}}), \tag{2.18}$$

as $n \to \infty$. Also, from Lemma 2.7,

$$\sigma(t_1+x/n^{\frac{1}{2}}) \to \sigma(t_1). \tag{2.19}$$

Combining (2.18) and (2.19)

$$\frac{n - n\mu(t_1+x/n^{\frac{1}{2}})}{n^{\frac{1}{2}}\sigma(t_1+x/n^{\frac{1}{2}})} \to -\left(\frac{x\mu'(t_1)}{\sigma(t_1)}\right). \tag{2.20}$$

Since the normal df is continuous, the conclusion of the theorem follows by substituting (2.17) and (2.20) into (2.16).

It should be noted that the proof of the theorem is not sensitive to the assumption of common lifetime df for each of the original and spare components. All that is needed is a central limit theorem to hold for the sum $S_n(t_1+x/n^{\frac{1}{2}})$ and convergence of the appropriate sequence of constants as in (2.20).

EXAMPLE 1. Suppose that $F(t)=1-e^{-\lambda t}$, so that $\mu(t)=\sigma^2(t)=\lambda t$. Then $n^{\frac{1}{2}}(\eta_n{}^{(n)}-\lambda^{-1})$ has a limiting normal df with mean zero and variance λ^{-2}. In fact, this result can be obtained quite simply by observing that the times between consecutive failures are independent, identically distributed exponential random variables.

EXAMPLE 2. Nonidentical components. Suppose that the original components have lifetime df $F_e(t)=\lambda\int_0^t \bar{F}(x)\,dx$ and the spares have lifetime df $F(t)$, where $F(t)=1-(1+2\lambda t)e^{-2\lambda t}$ is a gamma df. Thus, the sequence of failures in each socket corresponds to an equilibrium renewal process, so that $\mu(t)=\lambda t$ and $\sigma^2(t)=\lambda t/2+\frac{1}{8}e^{-4\lambda t}$. Then $n^{\frac{1}{2}}(\eta_n{}^{(n)}-\lambda^{-1})$ has a limiting normal df with mean zero and variance $(\frac{1}{2}+\frac{1}{8}e^{-4})\lambda^{-2}$.

k-out-of-n structures. The methods of this section can be applied to more general types of systems with replaceable components. For example, consider an $(n-k+1)$-out-of-n system with m spares where, as before, the component lifetimes are assumed to be independent and identically distributed. As components fail, they are immediately detected and replaced by new components until m replacements have been made; the system fails when k additional failures have occurred, i.e., $k+m$ in all. Let the system lifetime be denoted by $\zeta_{nk}^{(m)}$.

LEMMA 2.8. *If* $k+m \leqq n$, *then* $\xi_n{}^{(k+m)} \geqq_{\text{st}} \zeta_{nk}^{(m)} \geqq_{\text{st}} \xi_n{}^{(k)}$.

LEMMA 2.9. *If* $k+m \leqq n$, *then* $\xi_n{}^{(k+m)} \geqq_{\text{st}} \zeta_{nk}^{(m)} \geqq_{\text{st}} \eta_n{}^{(k+m)}$.

PROOF. The first part of Lemma 2.8 is proved as in Lemma 2.3, and the second part by observing that a system with spares survives longer than a system without spares.

The second part of Lemma 2.9 follows by noting that between the mth and $(k+m)$th failures there are fewer than n components liable to failure and so system failure is stochastically larger than in the case where replacements are continually available.

Making certain assumptions about the behaviour of $k=k(n)$ and $m=m(n)$ as $n\to\infty$ enables some deductions to be made concerning the limiting df's of $a_n{}^{-1}(\zeta_{nk}^{(m)}-b_n)$. For example, consider the two cases:

(i) $m/n\to 0$, $k/n\to\lambda$, $0<\lambda<1$. By using Lemma 2.8 and the results of Smirnov [10] concerning limit df's of central order statistics, the limiting df's of system lifetime may be completely characterised.

(ii) $(k+m)$ finite or $(k+m)\sim cn^{\alpha}$, $c>0$, $0<\alpha<\frac{1}{2}$. Then Lemma 2.9 and Theorem 2.1 and Theorem 2.2 enable one to describe completely the possible limiting df's.

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