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SOME BASIC PROPERTIES OF MULTIDIMENSIONAL PARTIALLY BALANCED DESIGNS

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1. Introduction and summary. An experimental design is called a multidimensional (MD) design, if it involves more than one factor; see e.g. Potthoff (1964a, b). A number of two-, three-, and four-dimensional designs are now in common use. For example, the ordinary balanced and partially balanced incomplete block designs are two-dimensional. The Latin squares, Youden squares, and the designs of Shrikhande (1951) are three-dimensional. Finally, the Graeco–Latin square designs are four-dimensional.

The construction of multidimensional designs involving three or more factors has been discussed by several authors both when additivity is assumed and when interactions are present. To mention a few, we cite Plackett and Burman (1946), Plackett (1946), Potthoff (1964a, b), Agarwal (1966), Anderson (1968), and Causey (1968).

A general class of multidimensional designs, which are partially balanced (PB), has been introduced in Srivastava (1961) and Bose and Srivastava (1964). These designs are called multidimensional partially balanced (MDPB) designs. The (MDPB) designs are useful for economizing on the number of observations to be taken, retaining at the same time a relative ease in analysis. MDPB designs for models containing interaction terms have been considered by Anderson (1968).

The purpose of this paper is to obtain a class of necessary combinatorial conditions satisfied by the parameters of MDPB designs, and to provide a relatively easy method of determining whether a given design is "completely connected". This latter concept, also of a combinatorial nature, is a generalization of the concept of "connected" block-treatment designs. It signifies that for every factor included under the design, the "true" difference between any two factor levels possesses a best linear unbiased estimate. In a succeeding communication, Srivastava and Anderson (1968), general methods of construction of MDPB designs are considered.

2. MDPB designs. Consider an experiment involving factors $F_1, F_2, \cdots, F_m$ where F_r has s_r levels: $F_{r1}, F_{r2}, \cdots, F_{rs_r}$. Denote by S_r the set of levels of factor F_r; $r = 1, 2, \cdots, m$. In what follows, the terminology of Bose and Srivastava (1964), heretofore called paper I, will be freely used. For convenience, however, we recall certain terms: (i) As in paper I, we assume that $S_1, \cdots, S_m$ have MDPB association scheme $\mathscr{A}$ defined over them. (ii) Let $F_{ix} \in S_i$ and $F_{jy} \in S_j$. Then with respect to F_{ix}, the set S_j is partitioned into n_{ij} disjoint sets which are denoted in this paper by $S_j(i, \alpha)$, $(\alpha = 1, 2, \cdots, n_{ij})$, where $S_j(i, \alpha)$ is called the αth associate class. (iii) $n_{ij}^{\alpha} = |S_j(i, \alpha)|$, where $|\Omega|$ denotes the number of elements in any given set Ω. (iv) $p(i, j, \alpha; k, \beta, \gamma) = |S_k(i, \beta) \cap S_k(j, \gamma)|$ where we assume F_{ix} and F_{jy} are αth

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associates of each other. (v) B_{ij}^{α} is an $(s_i \times s_j)$ association matrix corresponding to the αth associate class in S_j induced by the elements in S_i.

Consider any m-dimensional design T. We denote by N the total number of assemblies in T. Usually T is written as an $(m \times N)$ matrix, a column $(j_1, j_2, \cdots, j_m)'$ of which denotes a treatment combination in which F_r occurs at level F_{rj_r}. Now as in paper I, let T be an MDPB design. Recall that μ_r denotes the number of assemblies in T which contain any given level of F_r. Similarly d_{rt}^{α} is the number of assemblies in T in which F_r and F_t occur at level F_{ru} and F_{tv} respectively, where F_{ru} and F_{tv} are αth associates.

The mathematical model for the design T, assuming additivity, may be expressed

(2.1a) $$E(\mathbf{y}) = A'\mathbf{p} = A_1'\mathbf{p}_1 + \cdots + A_m'\mathbf{p}_m, \quad \text{where}$$
$$\mathbf{p}' = (\mathbf{p}_1', \cdots, \mathbf{p}_m'), \qquad \mathbf{p}_i' = (p_i^1, p_i^2, \cdots, p_i^{s_i}),$$

$(i = 1, \cdots, m)$, where p_i^u denotes the "effect" of F_{iu}. Thus $\mathbf{p}(v \times 1)$, where $v = \sum_1^m s_r$, is the vector of unknown parameters. The matrix $A'(N \times v)$ is the design-model matrix, and $A_r'(N \times s_r)$ is the submatrix containing the columns of A' corresponding to F_r. Also $\mathbf{y}(N \times 1)$ is the vector of observations corresponding to the N treatment combinations included in T, and we assume

(2.1b) $$\operatorname{Var}(\mathbf{y}) = \sigma^2 I_n, \sigma^2 \quad \text{unknown}.$$

Let $M_{rt} = A_r A_t'$, then recall from paper I that

(2.2) $$M_{rr} = A_r A_r' = \mu_r I_{s_r}; \qquad M_{rt} = A_r A_t' = \textstyle\sum_{\alpha} d_{rt}^{\alpha} B_{rt}^{\alpha}; \qquad r, t = 1, 2, \cdots, m.$$

The normal equations for estimating $\mathbf{p}$ are

(2.3) $$(AA')\hat{\mathbf{p}} = A\mathbf{y}, \quad \text{or} \quad M\hat{\mathbf{p}} = A\mathbf{y}, \quad \text{where} \quad M \equiv AA' \equiv ((M_{rt})).$$

3. Combinatorial properties of MDPB designs. We now derive a number of new and rather stringent relations connecting the parameters of an MDPB design

THEOREM 3.1. *If T is an* MDPB *design with the* ith *factor at s_i levels, then*

(a) $N = s_1\mu_1 = s_2\mu_2 = \cdots = s_m\mu_m$;

(b) $\sum_{\alpha} n_{ij}^{\alpha} d_{ij}^{\alpha} = \mu_i$; $i \neq j = 1, 2, \cdots, m$;

(c) $\sum_{\alpha} n_{ji}^{\alpha} d_{ij}^{\alpha} = \mu_j$; $i \neq j = 1, \cdots, m$.

PROOF. Part (a) follows immediately since each of the s_i levels of factor F_i appears exactly μ_i times, $i = 1, 2, \cdots, m$. Thus N is an integer multiple of the L.C.M. $(s_1, s_2, \cdots, s_m)$. To show part (b), consider any level of factor F_i. This level appears d_{ij}^{α} times with each of its n_{ij}^{α} αth associates in s_j; hence the indicated sum μ_i is the number of times this level appears. Part (c) follows from (b) by interchanging i and j and noting that $d_{ij}^{\alpha} = d_{ji}^{\alpha}$.

Consider now the matrix $[M_{ij} - \mu_k^{-1} M_{ik} M_{kj}]$, when the M's are given by (2.2). The sums of the different rows of this matrix are the elements of the vector $[M_{ij} - \mu_k^{-1} M_{ik} M_{kj}] J_{s_j 1} = \mathbf{h}(i, j, k)$, say, where J_{uv} is a $(u \times v)$ matrix containing

1 everywhere. But

$$\mathbf{h}(i, j, k) = \mu_i J_{s_i 1} - \mu_k^{-1} M_{ik}(\mu_k J_{s_k 1}) = \mu_i J_{s_i 1} - \mu_i J_{s_i 1} = \mathbf{0}. \tag{3.1}$$

Hence all row (and similarly all column) sums of $[M_{ij} - \mu_k^{-1} M_{ik} M_{kj}]$ are zero. On the other hand, using Lemma 4.2 (paper I), we have

$$\begin{aligned} M_{ij} - \mu_k^{-1} M_{ik} M_{kj} &= \sum_\alpha d_{ij}^\alpha B_{ij}^\alpha - \mu_k^{-1} (\sum_\beta d_{ik}^\beta B_{ik}^\beta)(\sum_\gamma d_{kj}^\gamma B_{kj}^\gamma) \\ &= \sum_\alpha d_{ij}^\alpha B_{ij}^\alpha - \mu_k^{-1} \sum_{\beta,\gamma} d_{ik}^\beta d_{kj}^\gamma [\sum_\alpha p(i, j, \alpha; k, \beta, \gamma) B_{ij}^\alpha] \\ &= \sum_\alpha [d_{ij}^\alpha - \mu_k^{-1} \sum_{\beta,\gamma} d_{ik}^\beta d_{kj}^\gamma p(i, j, \alpha; k, \beta, \gamma)] B_{ij}^\alpha. \end{aligned} \tag{3.2}$$

Thus, since (see paper I) every row sum for B_{ij}^α is n_{ij}^α, we get

THEOREM 3.2. *We must have, for all permissible i, j, and k,*

$$\sum_\alpha d_{ij}^\alpha n_{ij}^\alpha = \mu_k^{-1} \sum_\alpha n_{ij}^\alpha [\sum_{\beta,\gamma} d_{ik}^\beta d_{kj}^\gamma p(i, j, \alpha; k, \beta, \gamma)]. \tag{3.3}$$

In view of Theorem 3.1, (3.3) *is equivalent to*

$$\mu_i \mu_k = \sum_{\beta,\gamma} d_{ik}^\beta d_{kj}^\gamma [\sum_\alpha n_{ij}^\alpha p(i, j, \alpha; k, \beta, \gamma)]. \tag{3.4}$$

The above results are a generalization of well-known identities involving the parameters $\lambda_1, \cdots, \lambda_m$ of a PBIB design, where λ_α is the number of blocks in which two αth associates occur together.

4. Complete connectedness of multidimensional designs. The purpose of this section is to derive necessary and sufficient conditions on an MD design T (whether PB or not) such that all desired linear functions of the parameters are estimable. Recall that a linear function of the parameters is said to be estimable if there exists some linear function of the observations which is an unbiased estimate of this function. The usual requirement on a design T is that, for every factor F_i, all linear contrasts of the parameters $p_i^1, p_i^2, \cdots, p_i^{s_i}$ are estimable.

DEFINITION 4.1. Let T be any MD design. (a) T is said to be a "non-singular" design, if and only if all the linear contrasts

$$\sum_{j=1}^{s_i} a_{ij} p_i^j; \qquad (\sum_{j=1}^{s_i} a_{ij} = 0); \qquad i = 1, 2, \cdots, m; \tag{4.1}$$

are estimable. (b) T is said to be connected with respect to the factor F_i if all linear contrasts of the parameters $p_i^1, p_i^2, \cdots, p_i^{s_i}$ are estimable. (c) T is called completely connected if and only if it is connected with respect to each factor. (d) Thus, the design T is "nonsingular" if and only if it is "completely connected".

THEOREM 4.1. *An* MD *design T* (*whether* PB *or not*) *is completely connected if and only if rank* $(M) = v - m + 1$, *where* $M = (AA')$, *and* $v = s_1 + s_2 + \cdots + s_m$.

PROOF. Suppose first that rank $(M) = v - m + 1$. It is sufficient to show that the normal equations

$$M\hat{\mathbf{p}} = (AA')\hat{\mathbf{p}} = A\mathbf{y} \tag{4.2}$$

subject to the side conditions

(4.3) $$\sum_{j=1}^{s_i} p_i^j = 0; \qquad i = 1, 2, \cdots, m;$$

have a unique solution. Let $\Gamma = ((\Gamma_{ij}))$ be a $v \times v$ matrix, partitioned as M, where

(4.4) $$\Gamma_{ii} = (m-1)J_{s_i s_i}; \qquad i = 1, 2, \cdots, m;$$
$$\Gamma_{ij} = -J_{s_i s_j}; \qquad i \neq j = 1, 2, \cdots, m.$$

The side conditions (4.3) imply that $\Gamma \mathbf{p} = 0_{v1}$, and clearly rank $(\Gamma) = m-1$.

We now show that $(M+\Gamma)$ is nonsingular. To see this we observe that Theorem 3.1 implies that

(4.5) $$M\Gamma = \Gamma M = 0_{vv},$$

the rank of the product being zero. Since M and Γ commute, the roots of any rational function of M and Γ are the same rational functions of their roots, where the roots are arranged in some fixed order. Since rank $(M\Gamma) = 0$, the $(m-1)$ zero roots of M in the ordering must correspond to the $(m-1)$ non-zero roots of Γ. Thus all the roots of $(M+\Gamma)$ must be non-zero and $(M+\Gamma)$ is nonsingular. Now $(M+\Gamma)$ being nonsingular implies that T is completely connected since in this case the normal equations can be uniquely solved. This shows that if rank $(M) = v-m+1$, then T is completely connected. If T is completely connected, then $M+\Gamma$ must be nonsingular. Now rank $(M+\Gamma) \leqq$ rank (M) + rank (Γ). Hence rank $(M) \geqq v-m+1$. On the other hand, rank $(M) \leqq v-m+1$. This completes the proof.

In most cases, the condition given in Theorem 4.1 is computationally very difficult to check, and more practically usable conditions are therefore needed. With this aim, we introduce now a concept called "connectedness of MD designs", which is a generalization of the well-known concept of "connected" block-treatment designs (which are 2-dimensional). Below y_t denotes the observed response corresponding to any assembly $\mathbf{t}$.

DEFINITION 4.2. Two levels of factor F_1, say levels F_{11} and F_{12}, are said to be connected with respect to factors $F_2, F_3, \cdots, F_m$ if there exists a sequence of assemblies in T, say $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_k, \mathbf{t}_{k+1}$, such that

(4.6) $$\delta^{-1}E(\textstyle\sum_{i=1}^{k+1}(-1)^i y_{t_i}) = (p_1^{\,1} - p_1^{\,2}),$$

where $\delta(\neq 0)$ is a constant. The sequence of assemblies $(\mathbf{t}_0, \mathbf{t}_1, \cdots, \mathbf{t}_{k+1})$ is said to be a chain connecting levels 1 and 2 of factor F_1. A similar definition may be made for each factor, and each pair of levels.

THEOREM 4.2. *The sequence of assemblies* $(\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_k, \mathbf{t}_{k+1})$, *with k even, is a chain connecting the levels F_{1u} and F_{1v} of factor F_1 if and only if the following conditions hold. Let* $T_1 = (\mathbf{t}_0, \mathbf{t}_2, \mathbf{t}_4, \cdots, \mathbf{t}_k)$ *and* $T_2 = (\mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_5, \cdots, \mathbf{t}_{k+1})$. *Then*

(a) $\lambda_r^{j_r}(T_1) = \lambda_r^{j_r}(T_2);\ 1 \leqq j_r \leqq s_r;\ 2 \leqq r \leqq m;$

(b) $\lambda_1^{j_1}(T_1) = \lambda_1^{j_1}(T_2);\ j_1 \neq u$ *or* v;

(c) $\lambda_1^{u}(T_1) = \lambda_1^{u}(T_2)+\delta,$ *and* $\lambda_1^{v}(T_1) = \lambda_1^{v}(T_2)-\delta;$

where δ is some non-zero integer, where for any design T^*, $\lambda_r^u(T^*)$ denotes the number of assemblies in T^* in which F_r occurs at level F_{ru}.

PROOF. Suppose conditions (a), (b), and (c) hold, and consider the expression $E(\sum_{i=0}^{k+1}(-1)^i y_{t_i})$. By condition (a), each effect $p_r^{j_r}$ $(r = 2, \cdots, m)$, will appear in this expression the same number of times with a plus sign as with a minus sign. Likewise, by (b), $p_1^{j_1}, j_1 \neq u$ or v, appears in the sum with a plus sign and a minus sign the same number of times. Then by (c) it is seen that

$$E(\textstyle\sum_{i=0}^{k+1}(-1)^i y_{t_i}) = \delta(p_1^u - p_1^v). \tag{4.7}$$

Conversely, suppose that $\mathbf{t}_0, \mathbf{t}_1, \cdots, \mathbf{t}_{k+1}$ is a chain connecting levels u and v of factor F_1. It follows directly from Definition 4.2 that conditions (a), (b), and (c) must hold.

THEOREM 4.3. *The contrast $p_1^u - p_1^v$ is estimable if and only if there is a chain connecting F_{1u} and F_{1v}.*

PROOF. If there is a chain connecting F_{1u} and F_{1v}, then by Definition 4.2, $(p_1^u - p_1^v)$ is estimable. Conversely, if $(p_1^u - p_1^v)$ is estimable, then there exists some linear combination of $E(y_{t_i})$, such that $\sum a_i E(y_{t_i}) = p_1^u - p_1^v$, where the coefficients a_i are rational numbers. To see this, we observe that since $p_1^u - p_1^v$ is estimable, we have that its best linear unbiased estimate is $\hat{p}_1^u - \hat{p}_1^v$ where $\hat{p}_1^u$, $\hat{p}_1^v$ are obtained from the solutions to the normal equations, viz. $\hat{\mathbf{p}} = (M+\Gamma)^{-1}A\mathbf{y}$.

Now the matrices $(M+\Gamma)$ and A have integer elements, and thus the elements of $(M+\Gamma)^{-1}$ are rational. Then we see that, $\hat{p}_1^u$ and $\hat{p}_1^v$ are linear functions of the y's with rational coefficients. Let δ be an integer such that $a_i = c_i\delta^{-1}$, where c_i is an integer for all i. Then

$$\textstyle\sum a_i E(y_{t_i}) = \delta^{-1}\sum c_i E(y_{t_i}) = p_1^u - p_1^v. \tag{4.8}$$

We claim that the sequence of assemblies $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \cdots)$, where the $\mathbf{t}_i$ is included c_i times in this reference, can be rearranged so that it is a chain connecting F_{1u} and F_{1v}. To do this, denote by T_1 the set of assemblies which appears with a plus sign in the above sum, and by T_2 the set appearing with a minus sign. Then the new sequence T^* is obtained by alternately placing in the elements of T_1 and T_2. Finally, it is easily shown that conditions (a), (b), and (c) must hold for this sequence T^*, so that it is a chain connecting F_{1u} and F_{1v}. This completes the proof.

Clearly, as a result of Definition 4.1b and Definition 4.2, and the above theorem, any MD design T is connected with respect to the factor F_i if and only if every pair of levels of F_i is connected. Now, if F_{iu} is connected to both F_{iv} and F_{iw}, then F_{iv} is connected to F_{iw}. Hence it follows that T is connected with respect to factor F_i, if and only if some level of F_i is connected to each other level of F_i. Thus in order to show that T is connected with respect to a factor F_i, one level of F_i may be chosen and chains constructed connecting it with each other level. If m is small this may be accomplished with relative ease. However, if m is large, this may be practically difficult. Also, to show complete connectedness of T it must be shown that T is connected with respect to each factor. The following result is designed to help in this direction.

DEFINITION 4.3. Let T_i be the $(m-i+1)$-dimensional design obtained from T by ignoring the factors $F_1, F_2, \cdots, F_{i-1}$. Then F_i is said to be connected with respect to factors $F_{i+1}, \cdots, F_m$ in the original design T, if T_i is connected with respect to F_i. (Thus here we are introducing the concept of the connectedness of one factor with respect to a proper subset of the remaining factors.)

THEOREM 4.4. *An* MDPB *design T is completely connected if and only if the factor F_i is connected with respect to $F_{i+1}, F_{i+2}, \cdots, F_m$ for $i = 1, 2, \cdots, m-1$.*

PROOF. Suppose that F_1 is connected with respect to $F_2, F_3, \cdots, F_m$. Then for any linear contrast $\sum_i a_{1i} p_1{}^i$ there is a linear function of the observations, say $\mathbf{c}_1'\mathbf{y}$, such that

$$E(\mathbf{c}_1'\mathbf{y}) = \sum a_{1i} p_1{}^i. \tag{4.9}$$

Suppose F_2 is connected with respect to $F_3, F_4, \cdots, F_m$. Then for any pair of levels F_{2u} and F_{2v} there exists a sequence of assemblies $(\mathbf{t}_i)$ in T such that

$$E(\sum(-1)^i y_{t_i}) = \sum a_{2i} p_1{}^i + (p_2{}^u - p_2{}^v)\delta_1, \qquad (\delta_1 \neq 0 \text{ is a constant}), \tag{4.10}$$

with $\sum a_{2i} = 0$, since half the assemblies in the summation occur with a plus sign and the other half with a minus sign. Then

$$E(\sum(-1)^i y_{t_i}\cdot) - E(\mathbf{c}_2'\mathbf{y}) = (p_2{}^u - p_2{}^v)\delta_1, \tag{4.11}$$

where $\mathbf{c}_2'$ corresponds to the a_{2i}. Thus all contrasts of the $p_2{}^i$ are estimable. Likewise if F_3 is connected with respect to $F_4, \cdots, F_m$ then for any pair of levels of F_3 there is a sequence of assemblies in T such that

$$E(\sum(-1)^i y_{t_i}) = \delta_2(p_3{}^u - p_3{}^v) + \sum a_{3i} p_1{}^i + \sum a_{4i} p_2{}^i \tag{4.12}$$

where $\delta_2 \neq 0$ is a constant, and $\sum_i a_{3i} = \sum_i a_{4i} = 0$. Thus since contrasts of the $p_1{}^i$ and $p_2{}^i$ are estimable, the contrast $p_3{}^u - p_3{}^v$ is also estimable. This procedure may obviously be continued to factor F_{m-1}. Hence the condition given in the theorem is sufficient. Also, the necessity part is obvious.

It should be noted that simple pairwise connectedness is not sufficient for complete connectedness.

5. Example. To illustrate the above theory, consider an MDPB design of the $4 \times 6 \times 8 \times 12$ type, with 48 treatments. The values of the different parameters, and the design itself, are exhibited in Table 1.

In Table 1, the levels of the various factors are represented by symbols as follows: $F_1(AG, BH, CE, DF)$, $F_2(x^0, x^1, y^0, y^1, z^0, z^1)$, $F_3(A, B, C, D, E, F, G, H)$. and $F_4(AB, BC, CD$‘ $DA, FF, FG, GH, HE, AE, BF, CG, DH$. These sets of symbols correspond respectively to the diagonals, faces, vertices, and edges of a (3-dimensional) cube.

The idea behind this notation, and the association scheme behind this design is not explained here due to lack of space and to avoid repetition. A more appropriate setting for this discussion would appear in the paper by Srivastava and Anderson (1968), mentioned earlier. It may be remarked here that, for any given factor, the association scheme is quite simple, and would be apparent to the user while attempting to solve the reduced normal equations corresponding to that factor.

TABLE 1

A $4 \times 6 \times 8 \times 12$ *design*

$N = 48$; $\mu_1 = 12$, $\mu_2 = 8$, $\mu_3 = 6$, $\mu_4 = 4$; $\frac{1}{48}$th replicate; $n_e = 21$.

$d_{12}^1 = 2$, $d_{13}^1 = 0$, $d_{13}^2 = 2$, $d_{14}^1 = 0$, $d_{14}^2 = 2$,

$d_{23}^1 = d_{23}^2 = 1$, $d_{24}^1 = 2$, $d_{24}^2 = d_{24}^3 = 0$, $d_{34}^1 = d_{34}^3 = 1$, $d_{34}^2 = 0$.

	Assemblies											
	x^0		x^1		y^0		y^1		z^0		z^1	
AG	*F*	*D*	*D*	*F*	*H*	*B*	*B*	*H*	*C*	*E*	*E*	*C*
	BF	*EF*	*DH*	*CD*	*DH*	*EH*	*BF*	*BC*	*CD*	*BC*	*EF*	*EH*
BH	*E*	*C*	*C*	*E*	*A*	*G*	*G*	*A*	*D*	*F*	*F*	*D*
	AE	*EF*	*CG*	*CD*	*AE*	*AD*	*FG*	*CG*	*CD*	*AD*	*EF*	*FG*
CE	*B*	*H*	*H*	*B*	*D*	*F*	*F*	*D*	*A*	*G*	*G*	*A*
	AB	*BF*	*GH*	*DH*	*AD*	*DH*	*FG*	*BF*	*AD*	*AB*	*GH*	*FG*
DF	*A*	*G*	*G*	*A*	*E*	*C*	*C*	*E*	*B*	*H*	*H*	*B*
	AB	*AE*	*CG*	*GH*	*EH*	*AE*	*BC*	*CG*	*BC*	*AB*	*EH*	*GH*

From Table 1, the different assemblies of the design (like (AG, x^0, F, BF), (BH, x^1, C, CG), etc.) are obtained by combining a pair of levels of F_1 and F_2 with one pair (out of the two pairs) of levels of F_3 and F_4 in the corresponding cell. To check that the design is connected, we first observe that the following three chains show that the level AG of F_1 is connected with the levels BH, CE and DF: (AG, x^0, D, EF), (BH, x^0, C, EF), (AG, z^0, C, CD), (BH, z^0, D, CD); (AG, x^0, F, BF), (CE, x^0, H, BF), (AG, y^0, H, DH), (CE, y^0, F, DH); and (AG, y^0, B, EH), (DF, y^0, E, EH), (AG, z^0, E, BC), (DF, z^0, B, BC). Next ignoring F_1, we show that the level A of F_3 is connected with the levels $B, C, \cdots, H$ respectively. This, however, is obvious from the following seven chains, where each chain consists of just a pair of assemblies: (x^0, A, AB), (x^0, B, AB); (y^0, A, AE), (y^0, C, AE); (z^1, A, FG), (z^1, D, FG); (y^1, A, CG), (y^1, E, CG); (z^0, A, AD), (z^0, F, AD); and (y^1, F, FG), (y^1, G, FG); (x^1, A, GH), (x^1, H, GH). Finally, ignoring F_1 and F_3, the connectedness of F_2 (and hence also of F_4) is obvious. Hence, by Theorem 4.3 and Theorem 4.4, the whole design is completely connected.

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