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CENTERED VARIATIONS OF SAMPLE PATHS OF HOMOGENEOUS PROCESSES[1]

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1. Introduction. Let $X = \{X_t, t \geqq 0\}$ be a homogeneous stochastic process on the probability space $(\Omega, \mathscr{F}, P)$. In other words, the process X is assumed to have stationary independent increments given by the semigroup $\{\mu_t\}$. We wish to consider limits of the sums of the form

$$\sum_{[t_k \in \mathfrak{S}]} f(X_{t_{k+1}} - X_{t_k}) - b \tag{1.1}$$

where f is a certain function on the line and $\mathfrak{S} = \{t_0 < \cdots < t_n\}$ is a partition of $[0, t]$, over a sequence of partitions $\mathfrak{S}$ as the mesh of $\mathfrak{S}$ tends to zero.

The special case $f(x) = x^2$ is of special interest and has received much attention in the literature. It is easy to show, for example, that if X is a Brownian motion with no drift and $EX_t^2 = \sigma^2 t$, then the sum (1.1) converges in $L^2(\Omega, \mathscr{F}, P)$ to $\sigma^2 t$. If one assumes that the partitions are refining, a famous theorem of P. Lévy asserts that the convergence is almost sure.

Convergence in distribution of the sums (1.1) has been considered by Bochner [1] and Loève [3], though the latter paper studied limits where $X_{t_{k+1}} - X_{t_k}$ is replaced by a random variable X_{nk} of a triangular array of u.a.n. variables. In both the above papers, f was assumed to be at least continuous. Almost sure convergence of the sums (1.1) along a refining sequence of partitions was studied by Cogburn and Tucker [2], and they required f to be continuous and have a second derivative at 0, with $f(0) = 0$. In [4], we studied limits of (1.1) in the sense of convergence in probability and in $L^1(\Omega, \mathscr{F}, P)$ in the case where the centering term b vanishes. The function f was of rather general type, but the theorems held for a certain class of processes which included at least the non-Gaussian stable processes. In the present paper, we study limits for the same class of processes, but a different class of functions f, and the convergence in this case is in $L^2(\Omega, \mathscr{F}, P)$ or in probability.

2. Notation. Let $\{\mu_t\}$ be the weakly continuous convolution semigroup of probability measures on $(-\infty, \infty)$ associated with the process $\{X_t\}$. Let ν be the Lévy measure for $\{\mu_t\}$ so that

$$t^{-1}(x^2 \wedge 1)\mu_t(dx) \to \sigma^2 \delta_0(dx) + (x^2 \wedge 1)\nu(dx) \tag{2.1}$$

weakly as $t \to 0$, where σ^2 is the variance of the Gaussian component of X. (Note that in (2.1) of [4] it should be assumed that $f(x) = o(x^2)$ near 0, not $0(x^2)$ as stated.)

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As in [4], the semigroup $\{\mu_t\}$ is called powerfully continuous on the open set $S \subset (-\infty, \infty)$ if

$$\sup \mu_t(\Lambda)/t\nu(\Lambda) < \infty, \tag{2.2}$$

t running over $(0, \infty)$ and Λ running over the Borel subsets of S. We always assume that ν has infinite total mass, and it is then shown in [4] that ν has no atoms in S, and that if $g \geqq 0$ has support in S, then

$$\int g \, d\mu_t \leqq \text{const.}\, t \int g \, d\nu, \quad \text{and} \tag{2.3}$$

$$\int g \, d(t^{-1}\mu_t) \to \int g \, d\nu \qquad \text{as } t \to 0, \quad \text{if } \int g \, d\nu < \infty. \tag{2.4}$$

It follows from (2.1) and (2.4) that if $S = (-\infty, \infty)$, then $\sigma^2 = 0$, so that a semigroup $\{\mu_t\}$ which is powerfully continuous on $(-\infty, \infty)$ can have no Gaussian component. It was shown in [4] that all non-Gaussian stable processes are powerfully continuous on the support of the Lévy measure.

We shall assume that f is Borel measurable and satisfies

$$\nu\{x : |f(x)| > \beta\} < \infty, \quad \text{and} \tag{2.5}$$

$$\int_{\{x : |f(x)| \leqq \beta\}} f^2 \, d\nu < \infty \tag{2.6}$$

for some, and hence all, $\beta > 0$. It may be checked quite easily that (2.5) and (2.6) are necessary and sufficient in order that $\nu \circ f^{-1}$ be a Lévy measure. In [4], we considered $f \geqq 0$ satisfying (2.5) and

$$\int_{\{x : f(x) \leqq \beta\}} f \, d\nu < \infty, \tag{2.7}$$

and in that case, (2.5) and (2.7) are necessary and sufficient in order that $\nu \circ f^{-1}$ be the Lévy measure of a subordinator.

About X, we assume that $X_0 = 0$ and that almost all sample paths are right-continuous and possess left limits everywhere.

For convenience of notation, if $\mathfrak{S} = \{0 = t_0 < \cdots < t_n = t\}$ is a partition of $[0, t]$, we let $t_{n+1} = t$. The symbol const. denotes a constant depending only on the process X, but it is not necessarily the same at each occurrence.

It is convenient to express our results in terms of an integral similar to the Itô integral representation for a homogeneous process. Let $N(t, \Lambda) = N(t, \omega, \Lambda)$ be the number of jumps of the mapping $s \to X_s(\omega)$ of $[0, t]$ into R whose size is in the Borel set Λ. For a fixed Borel set Λ with $\nu(\Lambda) < \infty$, $N(t, \Lambda)$ is a Poisson process with Lévy measure $\nu(\Lambda)\delta_1$, and for disjoint X/Λ_i the processes $N(t, \Lambda_i)$ are mutually independent.

LEMMA 2.1. *If $\int |f| \, d\nu < \infty$, and Λ is a Borel set bounded away from 0 and ∞, then*

$$E \int_\Lambda f(x) N(t, dx) = t \int_\Lambda f(x) \nu(dx), \quad \textit{and}$$

$$V \int_\Lambda f(x) N(t, dx) = t \int_\Lambda f^2(x) \nu(dx).$$

PROOF. If f is an indicator function, the results hold because $N(t, B)$ has a Poisson distribution, and if f is an infinite linear combination of indicators of disjoint events, the result follows by independence of the $N(t, B_i)$ for disjoint B_i. For general f, if $\varepsilon > 0$ is given, we can find such an infinite sum g such that $|f-g| \leqq \varepsilon$ on Λ and $\int g^2\,dv < \infty$. Letting $h = f-g$, we find that

$$E\left|\int_\Lambda h(x)N(t, dx)\right| \leqq \varepsilon EN(t, \Lambda) = \varepsilon t v(\Lambda)$$

and

$$E\left|\int_\Lambda h(x)N(t, dx)\right|^2 \leqq E\int_\Lambda h^2(x)N(t, dx)\cdot\int_\Lambda N(t, dx)$$
$$\leqq E\varepsilon^2 N(t, \Lambda)^2 = \varepsilon^2(tv(\Lambda)+t^2v^2(\Lambda))$$

and the general result follows. $\square$

THEOREM 2.1. *If f satisfies* (2.5) *and* (2.6), *then*

$$\lim_{n\to\infty}\int_{1/n<|x|<n}[f(x)N(t, dx)-\theta(f(x))tv(dx)]$$

exists a.s., *and in L^2 if, in addition, $\int f^2\,dv < \infty$. The function θ is defined by*

$$\begin{aligned}\theta(y) &= y &&\textit{if}\quad |y| < 1;\\ &= 1 &&\textit{if}\quad y > 1;\\ &= -1 &&\textit{if}\quad y < -1.\end{aligned}$$

PROOF. Suppose firstly that $\int f^2\,dv < \infty$. Then $\int_{\{|f|\geqq 1\}}|f|\,dv \leqq \int_{\{|f|\geqq 1\}} f^2\,dv < \infty$, and so $\int|f-\theta(f)|\,dv < \infty$. It suffices, therefore, to prove that

$$\lim_{n\to\infty}\int_{1/n<|x|<n} f(x)[N(t, dx)-tv(dx)]$$

exists a.s. and in L^2. But this is actually a sum of independent random variables

$$Y_n = \int_{\{1/n+1<|x|\leqq 1/n\}\cup\{n\leqq|x|<n+1\}} f(x)[N(t, dx)-tv(dx)]$$

and by Lemma 1,

$$EY_n = 0, \qquad EY_n^2 = \int_{\{1/n+1<|x|\leqq 1/n\}\cup\{n\leqq|x|<n+1\}} f^2 tv(dx).$$

Therefore, $\sum V(Y_n) = t\int f^2 v(dx) < \infty$, and by a well-known result, the series $\sum Y_n$ converges a.s. and in L^2.

To complete the proof, we note simply that the integral with f replaced by $f1_{\{|f|\leqq\beta\}}$ differs from the original for some n only on $\{\omega: N(t, \omega, \{|f| > \beta\}) > 0\} = \Omega_\beta$, and by (2.6) we obtain convergence a.s. on $\Omega_\beta{}^c$ for every $\beta > 0$; however, Ω_β contracts to a null event as $\beta\to\infty$, so we have convergence a.s. on Ω. $\square$

Let us denote by $C-\int f(x)N(t, dx)$ the expression

$$\lim_{n\to\infty}\int_{1/n<|x|<n}[f(x)N(t, dx)-\theta(f(x))tv(dx)]$$

and call it the compensated integral of f relative to N. In case f satisfies (2.5) and $\int_{\{|f|\leqq\beta\}}|f|\,dv < \infty$, we showed in [4] that $\int f(x)N(t, dx)$ exists a.s., because it is then simply the sum $\sum_{s\leqq t} f(J_s)$ where $J_s(\omega)$ is the jump of the sample path $X_u(\omega)$ at s.

In any case $Y_t = C - \int f(x)N(t, dx)$ certainly is a homogeneous process whose Lévy measure is $\nu \circ f^{-1}$. If f satisfies (2.5) and $\int_{\{|f| \leqq \beta\}} |f| \, d\nu < \infty$, then we have $\int_{-\beta}^{\beta} |x| \, \nu \circ f^{-1}(dx) < \infty$, and the compensating terms can be dispensed with, modifying the process by a translation if necessary.

LEMMA 2.2. *If $\{\mu_t\}$, powerfully continuous on S, has Lévy measure ν, if $f = 0$ off S, if $\int |f| \, d\nu < \infty$, and if $\{\mathfrak{S}_n\}$ is a sequence of partitions of $[0, t]$ with mesh $\mathfrak{S}_n \to 0$, then $\sum_{t_k \in \mathfrak{S}_n} \int f \, d\mu_{t_{k+1}-t_k} \to t \int f \, d\nu$ as $n \to \infty$.*

PROOF.

$$\begin{aligned} &\left|\sum_{t_k \in \mathfrak{S}_n} \int f \, d_{\mu t_{k+1}-t_k} - t \int f \, d\nu\right| \\ &\leqq \sup_{t_k \in \mathfrak{S}_n} \left| \int f \frac{d\mu_{t_{k+1}-t_k}}{t_{k+1}-t_k} - \int f \, d\nu \right| \cdot \sum_{t_k \in \mathfrak{S}_n} (t_{k+1} - t_k) \\ &\to 0 \quad \text{as} \quad n \to \infty, \text{ by (2.4).} \quad \square \end{aligned}$$

3. The main results.

THEOREM 3.1. *Let X be a real-valued homogeneous process whose semigroup $\{\mu_t\}$ is powerfully continuous on the open set S. Suppose f has support in S and satisfies (2.5) and (2.6). If t is fixed and $\{\mathfrak{S}_n\}$ is a sequence of partitions of $[0, t]$ such that mesh $(\mathfrak{S}_n) \to 0$ as $n \to \infty$, we have*

$$\sum_{\{t_k \in \mathfrak{S}_n\}} [f(X_{t_{k+1}} - X_{t_k}) - \int \theta(f(x)\mu_{t_{k+1}-t_k}(dx)] \to C - \int f(x)N(t, dx) \quad \text{as} \quad n \to \infty,$$

in probability, and in $L^2(\Omega, \mathscr{F}, P)$ if $\int f^2 \, d\nu < \infty$. In the latter case, the function θ can be replaced on both sides by the identity function.

PROOF. Suppose firstly that $\int f^2 \, d\nu < \infty$. Since $|f| \leqq f^2$ on $\{|f| > 1\}$, we have

$$\sum_{\{t_k \in \mathfrak{S}_n\}} \int f 1_{\{|f|>1\}} \, d\mu_{t_{k+1}-t_k} \to t \int f 1_{\{|f|>1\}} \, d\nu$$

as $n \to \infty$, by Lemma 2.2. Hence it suffices to show

$$\begin{aligned} (3.1) \qquad &\sum_{\{t_k \in \mathfrak{S}_n\}} [f(X_{t_{k+1}} - X_{t_k}) - Ef(X_{t_{k+1}} - X_{t_k})] \\ &\qquad \to \lim_{m \to \infty} \int_{1/m < |x| < m} f(x)[N(t, dx) - t\nu(dx)] \end{aligned}$$

in $L^2(\Omega, \mathscr{F}, P)$ as $n \to \infty$, the right side converging because the mth term differs from $C - \int f(x)N(t, dx)$ by

$$\int_{1/m < |x| < m} f(x) 1_{\{|f|>1\}}(x) t\nu(dx) \to \int f 1_{\{|f|>1\}} t \, d\nu.$$

Let $\varepsilon > 0$ be given: choose m_0 so large that $h = f - f 1_{(1/m_0, m_0)}$ satisfies

$$(3.2) \qquad \int h^2 \, d\nu < \varepsilon.$$

Each side of (3.1) is linear in f, and

$$\begin{aligned} &E\left|\int_{1/m < |x| < m} h(x)[N(t, dx) - t\nu(dx)]\right|^2 \\ &= V \int_{1/m < |x| < m} h(x)N(t, dx) = t \int_{1/m < |x| < m} h^2 \, d\nu, \qquad \text{(by Lemma 2.1)} \\ &< t\varepsilon \qquad \text{for all} \quad m \geqq 1. \end{aligned}$$

Also

$$E\left|\sum_{t_k\in\mathfrak{S}_n} h(X_{t_{k+1}}-X_{t_k})-Eh(X_{t_{k+1}}-X_{t_k})\right|^2$$

$$= V\sum_{t_k\in\mathfrak{S}_n} h(X_{t_{k+1}}-X_{t_k}) \leqq \sum_{t_k\in\mathfrak{S}_n} h^2\, d\mu_{t_{k+1}-t_k}$$

$$\leqq \text{const.}\, t\int h^2\, dv < \text{const.}\, t\varepsilon, \qquad \text{by (3.2).}$$

These last two estimates show that in proving (3.1), one may assume that f is zero in a neighborhood of 0 and ∞; say f is supported in $1/m_0 < |x| < m_0$. Note now that each side of (3.1) has zero expectation so the problem is to prove that

$$(3.3) \qquad V\left[\sum_{t_k\in\mathfrak{S}_n} f(X_{t_{k+1}}-X_{t_k})-\int_{1/m_0<|x|<m_0} f(x)N(t, dx)\right] \to 0 \qquad \text{as} \quad n\to\infty.$$

Suppose $f = 1_G$, where G is an open interval in $(1/m_0, m_0)$. By a simple argument, written down in detail in [4] (Lemma 4.3), one has

$$Z_n(\omega) = \sum_{t_k\in\mathfrak{S}_n} 1_G(X_{t_{k+1}}(\omega)-X_{t_k}(\omega)) \geqq N(t, \omega, G) = \int 1_G(x)N(t, dx)$$

for all large n (depending on ω). Thus $\liminf_{n\to\infty} Z_n \geqq N(t, G)$, and

$$V(Z_n - N(t, G)) = E[(Z_n - N(t, G))^2] + [E(Z_n - N(t, G))]^2.$$

Now

$$EZ_n = \sum_{t_k\in\mathfrak{S}_n} \mu_{t_{k+1}-t_k}(G) \to tv(G) = EN(t, G)$$

so the second term above tends to zero. On the other hand,

$$E[(Z_n - N(t, G))^2] = EZ_n^{\,2} - 2E[Z_n N(t, G)] + E[N(t, G)^2]. \qquad \text{But}$$

$$EZ_n^{\,2} = V(Z_n)+(EZ_n)^2 = \sum_{t_k\in\mathfrak{S}_n} \mu_{t_{k+1}-t_k}(G)(1-\mu_{t_{k+1}-t_k}(G)) + \left(\sum_{t_k\in\mathfrak{S}_n} \mu_{t_{k+1}-t_k}(G)\right)^2$$

$$\to tv(G) + t^2v^2(G) \qquad \text{as} \quad n\to\infty,$$

Also,

$$E[N(t, G)^2] = V(N(t, G)) + (EN(t, G))^2$$

$$= tv(G) + t^2v^2(G)$$

and so

$$\limsup_{n\to\infty} E[(Z_n - N(t, G))^2] = 2tv(G) + 2t^2v^2(G) - 2\liminf_{n\to\infty} E[Z_n\cdot N(t, G)],$$

and $\qquad \liminf_{n\to\infty} E[Z_n\cdot N(t, G)]$

$$\geqq E[\liminf_{n\to\infty} E[Z_n\cdot N(t, G)] \geqq E[\liminf_{n\to\infty} Z_n\cdot N(t, G)]$$

$$\geqq E[N(t, G)^2] = tv(G) + t^2v^2(G).$$

Hence $V(Z_n - N(t, G)) \to 0$ as $n\to\infty$, and the theorem is proven in case $f = 1_G$.

If Λ is a Borel set bounded away from 0 and ∞, we may choose Λ_0, a finite union of open intervals bounded away from 0, such that $v(\Lambda\Delta\Lambda_0) < \varepsilon$, and we then have $N(t, \Lambda)-N(t, \Lambda_0)=N(t, \Lambda-\Lambda_0)-N(t, \Lambda_0-\Lambda)$ and since $\Lambda-\Lambda_0$ and $\Lambda_0-\Lambda$ are disjoint,

$$\begin{aligned} E(N(t, \Lambda)-N(t, \Lambda_0))^2 &= V(N(t, \Lambda-\Lambda_0)-N(t, \Lambda_0-\Lambda)) \\ &\quad +[tv(\Lambda-\Lambda_0)-tv(\Lambda_0-\Lambda)]^2 \\ &\leqq tv(\Lambda \Delta \Lambda_0)+t^2v^2(\Lambda \Delta \Lambda_0) < t\varepsilon+t^2\varepsilon^2. \end{aligned}$$

Also,

$$\begin{aligned} &E[(\textstyle\sum_{t_k\in\mathfrak{S}_n} 1_\Lambda(X_{t_{k+1}}-X_{t_k})-\sum_{t_k\in\mathfrak{S}_n} 1_{\Lambda_0}(X_{t_{k+1}}-X_{t_k}))^2] \\ &\quad \leqq E[(\textstyle\sum_{t_k\in\mathfrak{S}_n} 1_{\Lambda\Delta\Lambda_0}(X_{t_{k+1}}-X_{t_k}))^2] \\ &\quad = \textstyle\sum_{t_k\in\mathfrak{S}_n} \mu_{t_{k+1}-t_k}(\Lambda\Delta\Lambda_0)[1-\mu_{t_{k+1}-t_k}(\Lambda\Delta\Lambda_0)] \\ &\qquad\quad +[\textstyle\sum_{t_k\in\mathfrak{S}_n} \mu_{t_{k+1}-t_k}(\Lambda\Delta\Lambda_0)]^2 \\ &\quad \leqq \text{const.}\,(t\varepsilon+t^2\varepsilon^2) \end{aligned}$$

while

$$\begin{aligned} &\left|E(\textstyle\sum_{t_k\in\mathfrak{S}_n} 1_\Lambda(X_{t_{k+1}}-X_{t_k})-\sum_{t_k\in\mathfrak{S}_n} 1_{\Lambda_0}(X_{t_{k+1}}-X_{t_k}))\right| \\ &\quad \leqq \textstyle\sum_{t_k\in\mathfrak{S}_n} \mu_{t_{k+1}-t_k}(\Lambda\Delta\Lambda_0) \leqq \text{const.}\, t\varepsilon. \end{aligned}$$

The result is therefore proven in the case $f=1_\Lambda$, where Λ is a Borel set bounded away from zero and infinity. The extension to simple functions is immediate. For a general f whose support is bounded away from zero and infinity, given any $\varepsilon>0$, we may choose a simple function g whose support is bounded away from zero and infinity such that $\int (f-g)^2\, dv < \varepsilon$.

Then $V\int (f-g)(x)[N(t, dx)-tv(dx)] = \int (f-g)^2\, dv$ and

$$\begin{aligned} V[\textstyle\sum_{t_k\in\mathfrak{S}_n} (f-g)(X_{t_{k+1}}-X_{t_k})] &\leqq \textstyle\sum_{t_k\in\mathfrak{S}_n} \int (f-g)^2 \mu_{t_{k+1}-t_k} \\ &\leqq \text{const.}\, t \int (f-g)^2\, dv \\ &\leqq \text{const.}\, t\varepsilon. \end{aligned}$$

and an obvious estimate completes the proof in the case $\int f^2\, dv < \infty$.

In case $\int f^2\, dv = \infty$, we observe that if $g = f\cdot 1_{\{|f|\leqq\beta\}}$, then $\int g^2\, dv < \infty$, by (2.6) and that the terms $\sum_{t_k\in\mathfrak{S}_n} f(X_{t_{k+1}}-X_{t_k})$ and $\sum_{t_k\in\mathfrak{S}_n} g(X_{t_{k+1}}-X_{t_k})$ differ only on $\{\omega: X_{t_{k+1}}(\omega)-X_{t_k}(\omega)\in\{|f|>\beta\}$ for some $t_k\in\mathfrak{S}_n\}$ and the probability of this event is

$$\begin{aligned} &1-P\{X_{t_{k+1}}-X_{t_k}\in\{|f|\leqq\beta\}\ \forall t_k\in\mathfrak{S}_n\} \\ &\quad = 1-\textstyle\prod_k \mu_{t_{k+1}-t_k}(|f|\leqq\beta) \\ &\quad \leqq 1-\textstyle\prod_k (1-\mu_{t_{k+1}-t_k}(|f|>\beta)) \\ &\quad \leqq 1-\textstyle\prod_k (1-\text{const.}\,(t_{k+1}-t_k)v(|f|>\beta)) \\ &\quad \leqq 1-\exp[-\text{const.}\, tv(|f|>\beta)] \end{aligned}$$

when mesh $\mathfrak{S}_n$ is fine. By taking β sufficiently large, we guarantee that

$\sum_{t_k \in \mathfrak{S}_n} f(X_{t_{k+1}} - X_{t_k}) = \sum_{t_k \in \mathfrak{S}_n} g(X_{t_{k+1}} - X_{t_k})$ except on a set with arbitrarily small probability, when n is large. The terms $\theta(f(x))$ and $\theta(g(x))$ are identical if $\beta \geqq 1$ and we imagine β so chosen. Note finally that $\int_{1/n<|x|<n} f(x)N(t, dx)$ and $\int_{1/n<|x|<n} g(x)N(t, dx)$ differ only on $\{\omega : N(t, \omega, |f| > \beta) > 0\}$ and because of the Poissonian character of $N(t, |f| > \beta)$, the probability of this event is $1 - \exp\{-t\nu(|f| > \beta)\}$ and this can be made as small as desired by taking β sufficiently large.

Thus, the result on convergence in probability follows immediately from L^2 convergence for $f 1_{\{|f| \leqq \beta\}}$. □

It is natural to ask about the possible distributions of the limits $C - \int f(x)N(t, dx)$ for a fixed Lévy measure ν, as f is allowed to vary.

THEOREM 3.2. *Let ν be a Lévy measure on $(-\infty, \infty)$ with the property that there is a Borel set S such that $\nu(S) = \infty$ and ν has no atoms in S. Then, given any Lévy measure λ on $(-\infty, \infty)$. There is a function f with support in S, and satisfying the conditions* (2.5) *and* (2.6), *and such that $\lambda = \nu \circ f^{-1}$.*

COROLLARY. *Under the conditions of Theorem* 3.1, *any infinitely divisible distribution without a Gaussian component may appear as the distribution of the limit variable $C - \int f(x)N(t, dx)$.*

PROOF OF THEOREM 3.2. Write $\lambda = \lambda_1 + \lambda_2$ where λ_1 is concentrated on $(0, \infty)$ and λ_2 is concentrated on $(-\infty, 0)$. Let $S = S_1 \cup S_2$ where S_1 and S_2 are disjoint and $\nu(S_1) = \nu(S_2) = \infty$. Define $\nu_i(dx) = 1_{S_i}(x)\nu(dx)$. We shall construct a function f_1 with support in S_1 such that $\lambda_1 = \nu_1 \circ f_1^{-1}$, and a completely analogous construction will give f_2 with support in S_2 such that $\lambda_2 = \nu_2 \circ f_2^{-1}$. Setting $f = f_1 + f_2$, we obtain $\lambda = \nu \circ f^{-1}$. (Recall that, by convention, $\nu\{0\} = 0$.)

To construct f_1, we begin by letting $F(x) = \nu_1\{(x, \infty)\}$, $x > 0$, and $G(x) = \lambda_1\{(x, \infty)\}$, $x > 0$. Since ν has no atoms in S, F is a non-increasing continuous function on $(0, \infty)$ and G is a non-increasing right-continuous function on $(0, \infty)$. Define $\Phi(y) = \sup\{x : F(x) \geqq y\}$. It is easily checked that Φ is a left-continuous non-increasing function on $(0, \infty)$ and that consequently $\phi(x) = \Phi(G(x))$ is non-decreasing and right-continuous. Notice that $F(\phi(x)) = G(x)$ because of the continuity of F. We let h be the right-continuous inverse of ϕ, namely $h(y) = \inf\{x : \phi(x) > y\}$ so that $\phi(x) = \inf\{y : h(y) > x\}$, and therefore, $(\phi(x), \infty) \subset h^{-1}[x, \infty) \subset [\phi(x-0), \infty)$ for all $x > 0$. Therefore, if x is a continuity point of ϕ and x is not an atom of λ_1, we have, since ν_1 has no atoms,

$$\nu_1 \circ h^{-1}[x, \infty) = \nu_1\{(\phi(x), \infty)\} = F(\phi(x)) = G(x) = \lambda_1\{(x, \infty)\} = \lambda\{[x, \infty)\}.$$

This being true for a dense set of x in $(0, \infty)$ we must have $\nu_1 \circ h^{-1} = \lambda_1$. Since ν_1 is concentrated on S_1, we can take $f_1 = h \cdot 1_{S_1}$ and obtain $\nu_1 \circ f_1^{-1} = \lambda_1$, as desired. □

It is, of course, also true that the Lévy measure of any subordinator may be represented in the form $\nu \circ f^{-1}$, where $f \geqq 0$ satisfies (2.5) and (2.7). The same proof applies essentially.

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