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BOUNDARY CROSSING PROBABILITIES FOR THE WIENER PROCESS AND SAMPLE SUMS[1]

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1. Introduction and summary. Let $W(t)$ denote a standard Wiener process for $0 \leqq t < \infty$. We compute the probability that $W(t) \geqq g(t)$ for some $t \geqq \tau > 0$ (or for some $t > 0$) for a certain class of functions $g(t)$, including functions which are $\sim (2t \log\log t)^{\frac{1}{2}}$ as $t \to \infty$. We also prove an invariance theorem which states that this probability is the limit as $m \to \infty$ of the probability that $S_n \geqq m^{\frac{1}{2}} g(n/m)$ for some $n \geqq \tau m$ (or for some $n \geqq 1$), where S_n is the nth partial sum of any sequence $x_1, x_2, \cdots$ of independent and identically distributed (i.i.d.) random variables with mean 0 and variance 1.

The main results were announced in [19]. Some aspects of the invariance theorem were considered independently by Müller [14], who also studied the rate of convergence to the limiting distribution. Statistical applications of these ideas are indicated in [3] and [18].

In Section 2 we state the general theorems and give several examples. Sections 3–5 are devoted to the proof of these results. In Section 6 we indicate the applicability of our methods to stochastic processes other than the Wiener process. Of particular interest in this regard is the analogue of Theorem 1 for Bessel diffusion processes. Section 7 raises questions which will be treated in a subsequent paper.

2. Statement of Theorems and examples. Let F denote any measure on $(0, \infty)$ which is finite on bounded intervals, and define for $-\infty < x < \infty$, $-\infty < t < \infty$, $0 < \varepsilon < \infty$

$$0 < f(x, t) = \int_0^\infty \exp(xy - y^2 t/2)\, dF(y) \leqq \infty,$$
$$-\infty \leqq A(t, \varepsilon) = \inf\{x : f(x, t) \geqq \varepsilon\} < \infty.$$

It is easily seen that

$$(1) \qquad x < A(t, \varepsilon) \Rightarrow f(x, t) < \varepsilon, \qquad f(x, t) < \varepsilon \Rightarrow x \leqq A(t, \varepsilon),$$

and that if for some $b, h\, f(b, h) < \infty$ then for each $t > h$ the equation $f(x, t) = \varepsilon$ has the unique solution $x = A(t, \varepsilon)$. The function $A(t, \varepsilon)$ is continuous and increasing in t for $h \leqq t < \infty$, and for $t > h\, f(x, t) \geqq \varepsilon$ if and only if $x \geqq A(t, \varepsilon)$. Set

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2), \qquad \Phi(x) = \int_{-\infty}^x \varphi(y)\, dy.$$

THEOREM 1. (i) *For any* b, h, ε *such that* $f(b, h) < \varepsilon$,

$$(2) \qquad P\{W(t) \geqq A(t+h, \varepsilon) - b \text{ for some } t > 0\} = f(b, h)/\varepsilon.$$

(ii) *For any* b, h, ε *and* $\tau > 0$,

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(3) $P\{W(t) \geqq A(t+h, \varepsilon)-b \text{ for some } t \geqq \tau\}$

$$= 1-\Phi\left(\frac{A(\tau+h, \varepsilon)-b}{\tau^{\frac{1}{2}}}\right)+\varepsilon^{-1}\int_0^\infty \exp(by-y^2h/2)\Phi\left(\frac{A(\tau+h, \varepsilon)-b}{\tau^{\frac{1}{2}}}-y\tau^{\frac{1}{2}}\right)dF(y).$$

THEOREM 2. (i) *Suppose that $g(t)$ is continuous for $t \geqq \tau > 0$, that $t^{-\frac{1}{2}}g(t)$ is ultimately non-decreasing as $t \to \infty$, and that*

(4) $$\int_\tau^\infty \frac{g(t)}{t^{3/2}}\exp(-g^2(t)/2t)\,dt < \infty.$$

Then

(5) $$\lim_{m\to\infty} P\{S_n \geqq m^{\frac{1}{2}}g(n/m) \text{ for some } n \geqq \tau m\}$$
$$= P\{W(t) \geqq g(t) \text{ for some } t \geqq \tau\},$$

where $S_n = x_1 + \cdots + x_n$ and the x_i are any i.i.d. *random variables having mean* 0 *and variance* 1.

(ii) *Suppose that in addition to the hypotheses of* (i) *g is continuous for $t > 0$, that $t^{-\frac{1}{2}}g(t)$ is non-increasing for t sufficiently small, and that*

$$\int_{0+}^1 \frac{g(t)}{t^{3/2}}\exp(-g^2(t)/2t)\,dt < \infty.$$

Then (5) *continues to hold with $n \geqq \tau m$ replaced by $n \geqq 1$ and $t \geqq \tau$ by $t > 0$.*

REMARKS ON THEOREM 2. (a) The same relations are valid if S_n, $W(t)$ are replaced by $|S_n|$, $|W(t)|$.

(b) Instead of assuming that the continuous function g satisfies the indicated growth conditions, it is sufficient to assume that it majorizes some function which does.

(c) If $g(t)$ is continuous for $0 < t \leqq \tau < \infty$ and the growth conditions of (ii) hold for t sufficiently small, then

$$\lim_{m\to\infty} P\{S_n \geqq m^{\frac{1}{2}}g(n/m) \text{ for some } 1 \leqq n \leqq \tau m\}$$
$$= P\{W(t) \geqq g(t) \text{ for some } 0 < t \leqq \tau\}.$$

In discussing the following examples, we use the fact (cf. [2], page 266) that the process $W^*(t)$ defined by

(6) $$W^*(t) = tW(t^{-1}) \qquad (t > 0), \quad W^*(0) \equiv 0$$

is also a standard Wiener process.

EXAMPLE 1. Let F be the degenerate measure which puts unit mass at some point $2a > 0$, and let $\varepsilon = 1$. Then $A(t, \varepsilon) = at$ and (3) with $h = 0$ (together with (6)) and

Theorem 2 give the result that

$$\begin{aligned}
&\lim_{m\to\infty} P\{\max_{n\geqq \tau m}(S_n - an/m^{\frac{1}{2}}) \geqq bm^{\frac{1}{2}}\}\\
&\quad = \lim_{m\to\infty} P\{\max_{1\leqq n\leqq m/\tau}(S_n - bn/m^{\frac{1}{2}}) \geqq am^{\frac{1}{2}}\}\\
&\quad = P\{\max_{t\geqq\tau}(W(t)-at)\geqq b\}\\
&\quad = P\{\max_{0<t\leqq\tau^{-1}} W((t)-bt)\geqq a\}\\
&\quad = 1-\Phi(b/\tau^{\frac{1}{2}}+a\tau^{\frac{1}{2}})+\exp(-2ab)\Phi(b/\tau^{\frac{1}{2}}-a\tau^{\frac{1}{2}})\\
&\qquad\qquad (\tau>0,\ a>0,\ -\infty<b<\infty).
\end{aligned}$$

The special case $b = 0$ yields the relation

$$\begin{aligned}
&\lim_{m\to\infty} P\{\max_{n\geqq \tau m} S_n/n \geqq a/m^{\frac{1}{2}}\}\\
&\quad = \lim_{m\to\infty} P\{\max_{1\leqq n\leqq m/\tau} S_n \geqq am^{\frac{1}{2}}\}\\
&\quad = P\{\max_{t\geqq\tau} W(t)/t\geqq a\}\\
&\quad = P\{\max_{0<t\leqq\tau^{-1}} W(t)\geqq a\} = 2(1-\Phi(a\tau^{\frac{1}{2}})) \qquad (\tau>0,\ a>0).
\end{aligned} \tag{7}$$

For any $\varepsilon > 0$ define as in [20]

$$M = M(\varepsilon) = \sup\{n: S_n \geqq n\varepsilon\}.$$

By the strong law of large numbers, $P\{M<\infty\} = 1$. Since $\{M\geqq m\} = \{\max_{n\geqq m} S_n/n \geqq \varepsilon\}$, letting $m\to\infty$, $\varepsilon\to 0$ in such a way that $\varepsilon m^{\frac{1}{2}} = a > 0$, we obtain from (7) the result

$$\lim_{\varepsilon\to 0} P\{\varepsilon^2 M \geqq a^2\} = \lim_{m\to\infty} P\{m^{\frac{1}{2}}\max_{n\geqq m} S_n/n \geqq a\} = 2(1-\Phi(a));$$

i.e., as $\varepsilon\to 0$ the random variable $\varepsilon^2 M$ converges in law to the chi-square distribution with one degree of freedom.

Using (2) instead of (3) we obtain from Theorem 1 and Theorem 2

$$\begin{aligned}
&\lim_{m\to\infty} P\{S_n \geqq an/m^{\frac{1}{2}} + bm^{\frac{1}{2}} \quad\text{for some}\quad n\geqq 1\}\\
&\quad = P\{\max_{t>0}(W(t)-at)\geqq b\} = \exp(-2ab) \qquad (a>0,\ b>0).
\end{aligned}$$

Equation (2) with $h = 1$ also gives certain probabilities associated with the "tied down" Wiener process:

$$\begin{aligned}
&P\{\max_{0<t\leqq 1} W(t)\geqq a \mid W(1) = b\}\\
&\quad = P\{\max_{t\geqq 1} W(t)/t \geqq a \mid W(1) = b\}\\
&\quad = P\left\{\max_{t>0}\frac{b+W(t)}{1+t}\geqq a\right\} = \exp(-2a(a-b)) \qquad (a>0,\ b<a).
\end{aligned}$$

EXAMPLE 2. Let $dF(y) = (2/\pi)^{\frac{1}{2}}\, dy/y^{\gamma}$ for $0<y<\infty, \gamma<1$. Then

$$A(t,\varepsilon) = t^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log t + 2\log\tfrac{1}{2}\varepsilon),$$

where we have set

$$\alpha(x) = x^2 + 2\log \int_0^\infty \varphi(y-x)\,dy/y^\gamma \sim x^2 \quad \text{as} \quad x \to \infty,$$

and (2) with $b = 0$ and Theorem 2 imply that

$$\lim_{m\to\infty} P\{S_m \geqq (n+hm)^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log(n/m+h)+2\log \varepsilon/2) \text{ for some } n \geqq 1\}$$

(8) $$= P\{W(t) \geqq (t+h)^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log(t+h)+2\log\varepsilon/2) \text{ for some } t > 0\}$$

$$= \frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^{(\gamma+2)/2}\pi^{\frac{1}{2}}h^{(1-\gamma)/2}\varepsilon}, \qquad \left(h > 0,\ \varepsilon > \frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^{(\gamma+2)/2}\pi^{\frac{1}{2}}h^{(1-\gamma)/2}}\right).$$

For $\gamma = 0$, $\alpha(x) = x^2 + 2\log\Phi(x)$, and the right-hand side of (8) becomes $(2h^{\frac{1}{2}}\varepsilon)^{-1}$.

Setting $b = h = 0$, $\varepsilon = 2\exp(\frac{1}{2}\alpha(a))$, we obtain from (3) and Theorem 2 for any $\tau > 0$ the result

$$\lim_{m\to\infty} P\{S_n \geqq n^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log n/m + \alpha(a)) \text{ for some } n \geqq \tau m\}$$

(9) $$= P\{W(t) \geqq t^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log t + \alpha(a)) \text{ for some } t \geqq \tau\}$$

$$= 1 - \Phi(\alpha^{-1}((1-\gamma)\log\tau + \alpha(a))) + \varphi(a)\frac{\int_0^\infty \Phi(\alpha^{-1}((1-\gamma)\log\tau+\alpha(a))-y)\frac{dy}{y^\gamma}}{\tau^{(1-\gamma)/2}\int_0^\infty \varphi(a-y)\frac{dy}{y^\gamma}}.$$

For $\tau = 1$ the right-hand side of (9) simplifies to

$$1-\Phi(a)+\frac{\varphi(a)\int_0^\infty \Phi(a-y)\frac{dy}{y^\gamma}}{\int_0^\infty \varphi(a-y)\frac{dy}{y^\gamma}},$$

which for $\gamma = 0$ becomes

$$1-\Phi(a)+\varphi(a)\left(a+\frac{\varphi(a)}{\Phi(a)}\right).$$

Finally, for $\tau = h = 0$, $b > 0$, $\varepsilon > (2/\pi)^{\frac{1}{2}}\Gamma(1-\gamma)/b^{1-\gamma}$, we have

$$\lim_{m\to\infty} P\{S_n \geqq n^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log n/m + 2\log\varepsilon/2)+b \text{ for some } n \geqq 1\}$$
$$= P\{W(t) \geqq t^{\frac{1}{2}}\alpha^{-1}((1-\gamma)\log t + 2\log\varepsilon/2)+b \text{ for some } t > 0\}$$
$$= (2/\pi)^{\frac{1}{2}}\Gamma(1-\gamma)/(\varepsilon b^{1-\gamma}).$$

EXAMPLE 3. Theorem 1 generalizes in an obvious manner to the case in which F is a measure on $(-\infty, \infty)$ which assigns measure 0 to $\{0\}$. For $dF = dy/(2\pi)^{\frac{1}{2}}$,

$b = 0, \varepsilon = e^{\frac{1}{2}a^2}$ the results are particularly elegant:

$$\lim_{m\to\infty} P\{|S_n| \geqq (n(a^2+\log n/m))^{\frac{1}{2}} \text{ for some } n \geqq \tau m\}$$
$$= P\{|W(t)| \geqq (t(a^2+\log t))^{\frac{1}{2}} \text{ for some } t \geqq \tau\}$$
$$= 2[1-\Phi((a^2+\log\tau)^{\frac{1}{2}})+((a^2+\log\tau)/\tau)^{\frac{1}{2}}\varphi(a)], \quad (\tau > e^{-a^2});$$
$$\lim_{m\to\infty} P\{|S_n| \geqq [(n+hm)(a^2+\log(n/m+h))]^{\frac{1}{2}} \text{ for some } n \geqq 1\}$$
$$= P\{|W(t)| \geqq [(t+h)(a^2+\log(t+h))]^{\frac{1}{2}} \text{ for some } t > 0\}.$$
$$= h^{-\frac{1}{2}}e^{-\frac{1}{2}a^2} \qquad (h > e^{-a^2}).$$

EXAMPLE 4. For $\delta > 0$ let

$$dF(y) = dy/y(\log 1/y)(\log_2 1/y)\cdots(\log_n 1/y)^{1+\delta} \quad \text{for} \quad 0 < y < 1/e_n$$
$$= 0 \text{ elsewhere,}$$

where we write $\log_2 x = \log(\log x)$, $e_2 = e^e$, etc. It will be shown in Section 4 that as $t \to \infty$, $A(t, \varepsilon) \sim (2t\log_2 t)^{\frac{1}{2}}$; in fact for $n \geqq 3$

$$(10) \qquad A(t,\varepsilon) = [2t(\log_2 t+3/2\log_3 t+\textstyle\sum_{k=4}^{n}\log_k t+(1+\delta)\log_{n+1} t + \log\frac{1}{2}\varepsilon/\pi^{\frac{1}{2}}+o(1))]^{\frac{1}{2}}$$

while for $n=2$,

$$A(t,\ \varepsilon)=[2t\,(\log_2 t+(3/2+\delta)\log_3 t+\log \varepsilon/2\pi^{\frac{1}{2}}+o(1))]^{\frac{1}{2}},$$

so Theorem 1 and Theorem 2 give a deeper content to the "easy" half of the law of the iterated logarithm. For example, for $b = h = 0$ we have

$$\lim_{m\to\infty} P\{S_n \geqq m^{\frac{1}{2}}A(n/m,\varepsilon) \text{ for some } n \geqq 1\}$$
$$= P\{W(t) \geqq A(t,\varepsilon) \text{ for some } t > 0\} = 1/(\delta\varepsilon) \quad (\varepsilon > 1/\delta).$$

EXAMPLE 5. For $\delta > 0$ let

$$dF(y) = dy/y(\log y)(\log_2 y)\cdots(\log_n y)^{1+\delta} \quad \text{for} \quad y > e_n$$
$$= 0 \quad \text{elsewhere.}$$

In this case $f(0, 0) = \delta^{-1}$, but $f(x, 0) = \infty$ for each $x > 0$, so that $A(0, \varepsilon) = 0$ for each $\varepsilon > \delta^{-1}$. An argument similar to that leading to (10) shows that for any $\varepsilon > \delta^{-1}$, as $t \to 0$,

$$A(t,\varepsilon) = [2t(\log_2 t^{-1}+3/2\log_3 t^{-1}+\textstyle\sum_4^n \log_k t^{-1} + (1+\delta)\log_{n+1} t^{-1}+\log\frac{1}{2}(\varepsilon-\delta^{-1})/\pi^{\frac{1}{2}}+o(1))].$$

EXAMPLE 6. It will be shown in Section 4 that if

$$dF(y) = \tfrac{1}{2}(3/\pi)^{\frac{1}{2}}\exp(-16/27y^{-1})\,dy \qquad (0 < y < \infty),$$

then as $t \to \infty$

$$A(t, \varepsilon) = t^{\frac{2}{3}} + 4^{-1} t^{\frac{1}{3}}(\tfrac{1}{2}\log t + \log \varepsilon + o(1)). \tag{11}$$

More generally, if

$$dF(y) = \gamma \exp(-\alpha y^{-\beta})\, dy,$$

for any $\beta > 0$ and appropriate values $\alpha = \alpha(\beta) > 0$ and $\gamma = \gamma(\beta) > 0$, then

$$A(t, \varepsilon) \sim t^{(1+\beta)/(2+\beta)}$$

and an expansion similar to (11) may be obtained. We omit the details.

3. Proof of Theorem 1. The proof of Theorem 1 is an application of Lemma 1 below to certain martingales defined in terms of the function $f(x, t)$. Since (cf., e.g., [21]) $\{\exp(yW(t) - \frac{1}{2}y^2 t, \mathscr{B}(W(s), s \leqq t), t \geqq 0\}$ is a martingale for each fixed y, it follows from Fubini's theorem that for any real numbers b and h

$$\{z(t), \mathscr{F}(t), t \geqq 0\} = \{f(b + W(t), t + h), \mathscr{B}(W(s), s \leqq t), t \geqq 0\}$$

is also a martingale except that $Ez(t)$ may be ∞. Although the definition of a martingale usually includes the assumption that $E|z(t)| < \infty$ (cf. [16], page 131), the proof of Lemma 1 does not actually require this hypothesis, and our departure from customary usage permits applications such as Example 2 of Section 2.

LEMMA 1. *Let ε be any positive constant and $\{z(t), \mathscr{F}(t), t \geqq \tau\}$ a nonnegative martingale. If $z(t)$ has continuous sample paths on $\{z(\tau) < \varepsilon\}$ and converges to 0 in probability on $\{\sup_{t>\tau} z(t) < \varepsilon\}$, then*

$$P\{\sup_{t>\tau} z(t) \geqq \varepsilon \mid \mathscr{F}(\tau)\} = \varepsilon^{-1} z(\tau) \quad on \quad \{z(\tau) < \varepsilon\}. \tag{12}$$

PROOF. Define $T = \inf\{t: t \geqq \tau, z(t) \geqq \varepsilon\}$, where the inf of the empty set is taken to be $+\infty$. It is well known (e.g. [16], page 142) that $\{z(T \wedge t), \mathscr{F}(t), t \geqq \tau\}$ is a martingale. Hence for any $A \in \mathscr{F}(\tau)$ and $t \geqq \tau$,

$$\begin{aligned}\int_{A\{z(\tau)<\varepsilon\}} z(\tau)\, dP &= \int_{A\{z(\tau)<\varepsilon\}} z(T \wedge t)\, dP \\ &= \varepsilon P(A\{z(\tau) < \varepsilon,\ T \leqq t\}) + \int_{A\{z(\tau)<\varepsilon, T>t\}} z(t)\, dP.\end{aligned}$$

Since $I_{\{T>t\}} z(t) \leqq \varepsilon I_{\{t<T<\infty\}} + I_{\{T=\infty\}} z(t) \leqq \varepsilon$ and converges to 0 in probability as $t \to \infty$, we have by the dominated convergence theorem

$$\begin{aligned}\int_{A\{z(\tau)<\varepsilon\}} z(\tau)\, dP &= \varepsilon P(A\{z(\tau) < \varepsilon,\ T < \infty\}) \\ &= \varepsilon \int_{A\{z(\tau)<\varepsilon\}} P\{T < \infty \mid \mathscr{F}(\tau)\}\, dP,\end{aligned}$$

which proves (12).

From (12) it follows directly that

$$P\{\sup_{t\geqq\tau} z(t) \geqq \varepsilon\} = P\{z(\tau) \geqq \varepsilon\} + \varepsilon^{-1} \int_{\{z(\tau)<\varepsilon\}} z(\tau)\, dP. \tag{13}$$

LEMMA 2. *If $f(x, \tau) < \infty$ for some x, τ, then for any b, h $f(b + W(t), h + t) \to 0$ in probability as $t \to \infty$.*

PROOF. By replacing τ by $\tau+1$ if necessary, we may assume without loss of generality that $f(x, \tau) < \infty$ for all x. Suppose first that $h \geqq \tau$. For any $c > 0$

$$f(b+ct^{\frac{1}{2}}, h+t) = (\varphi(c))^{-1} \int_0^\infty \varphi(c-yt^{\frac{1}{2}}) \exp(by-\tfrac{1}{2}hy^2)\, dF(y) \to 0$$

as $t \to \infty$ by the dominated convergence theorem. Hence for any $\varepsilon > 0$, for all t sufficiently large

$$P\{f(b+W(t), h+t) \geqq \varepsilon\} \leqq P\{W(t) \geqq ct^{\frac{1}{2}}\} = 1-\Phi(c),$$

which can be made arbitrarily small by taking c sufficiently large. Now suppose that $h < \tau$. Then for any $t > \tau - h$ and $\varepsilon > 0$

$$P\{f(b+W(t), h+t) \geqq \varepsilon\}$$
$$= \int_{-\infty}^{\infty} P\{f(b+(\tau-h)^{\frac{1}{2}}x + W(t-\tau+h), \tau+(t-\tau+h)) \geqq \varepsilon\} \cdot \varphi(x)\, dx \to 0$$

as $t \to \infty$ by the first part of the proof and the dominated convergence theorem.

PROOF OF THEOREM 1. (ii) Let b, h, ε be arbitrary and $\tau > 0$. We may assume that $f(x, \tau+h) < \infty$ for some x, since otherwise the theorem is trivially true. For each $t \geqq \tau$ let $z(t) = f(b+W(t), t+h)$, $\mathscr{F}(t) = \mathscr{B}(W(s), s \leqq t)$, and set

$$B_1 = \{W(\tau) < A(\tau+h, \varepsilon) - b\}, \qquad B_2 = \{f(b+W(\tau), \tau+h) < \varepsilon\}.$$

If $x < A(\tau+h, \varepsilon)$ then for some $\delta > 0$ $x+\delta < A(\tau+h, \varepsilon)$ and hence $f(x+\delta, \tau+h) < \varepsilon$. Thus from the continuity of the sample paths of $W(t)$ and the dominated convergence theorem it follows that the martingale $\{z(t), \mathscr{F}(t), t \geqq \tau\}$ has continuous sample paths on B_1. But $B_1 = B_2$ a.s. by (1) and the continuity of the distribution function of $W(\tau)$, and hence by Lemma 2 and (12)

$$P\{W(t) \geqq A(t+h, \varepsilon) - b \ \text{ for some } \ t \geqq \tau\}$$
$$= 1 - P(B_1) + \int_{B_1} P\{W(t) \geqq A(t+h, \varepsilon) - b \ \text{ for some } \ t > \tau \mid \mathscr{F}(\tau)\}\, dP$$
$$= 1 - \Phi\left(\frac{A(\tau+h, \varepsilon) - b}{\tau^{\frac{1}{2}}}\right) + \int_{B_2} P\{\sup_{t>\tau} f(b+W(t), t+h) \geqq \varepsilon \mid \mathscr{F}(\tau)\}\, dP$$
$$= 1 - \Phi\left(\frac{A(\tau+h, \varepsilon) - b}{\tau^{\frac{1}{2}}}\right) + \varepsilon^{-1} \int_{B_1} f(b+W(\tau), \tau+h)\, dP$$
$$= 1 - \Phi\left(\frac{A(\tau+h, \varepsilon) - b}{\tau^{\frac{1}{2}}}\right) + \varepsilon^{-1} \int_0^\infty \exp(by - \tfrac{1}{2}y^2 h) \Phi\left(\frac{A(\tau+h, \varepsilon) - b}{\tau^{\frac{1}{2}}} - y\tau^{\frac{1}{2}}\right) dF(y).$$

(i) By absorbing the factor $\exp(by - \frac{1}{2}y^2 h)$ into the measure F and dividing by a constant, we may without loss of generality assume that $b = h = 0$ and $f(0, 0) = 1 < \varepsilon$. Now (3) may be written

$$(14) \qquad P\{f(W(t), t) \geqq \varepsilon \quad \text{for some} \quad t \geqq \tau\}$$
$$= P\{f(W(\tau), \tau) \geqq \varepsilon\} + \varepsilon^{-1} \int_{\{f(W(\tau), \tau) < \varepsilon\}} f(W(\tau), \tau)\, dP,$$

valid for any $\tau > 0$. An argument similar to that of Lemma 2 shows that $f(W(\tau), \tau) \to 1$ in probability as $\tau \to 0$, and hence (2) follows from (14) and the dominated convergence theorem on letting $\tau \to 0$.

REMARKS. (a) Although (2) is a special case of (12), we chose to prove it as the limit of (3). The reason for this chicanery is that under the sole assumption that $f(0, 0) = 1$ it is not immediately obvious that the martingale $\{f(W(t), t), \mathscr{F}(t), t \geqq 0\}$ has sample paths which are continuous from the right at $t = 0$, a condition that is required in order that (12) apply. (That this is in fact the case follows from our argument that $f(W(t), t) \to 1 = f(0, 0)$ in probability as $t \to 0$ and the martingale convergence theorem, which asserts that this convergence takes place with probability one.)

(b) An additional argument shows that if $\varepsilon \leqq f(b, h) \leqq \infty$, then the probability on the left-hand side of (2) is 1. Hence we obtain instead of (2) the completely general statement that *for any* b, h, ε

(2′) $$P\{W(t) \geqq A(t+h, \varepsilon) - b \text{ for some } t > 0) = \min(1, f(b, h)/\varepsilon).$$

(c) Part (ii) of Theorem 1 remains valid if we replace "for some $t \geqq \tau$" by "for some $t > \tau$". In contrast, part (i) with "for some $t \geqq 0$" replacing "for some $t > 0$" is false when $A(h, \varepsilon) = b$ and $f(b, h) < \varepsilon$ (see Example 5 with $b = h = 0$ and $\varepsilon > \delta^{-1}$).

(d) We shall say that a function $\psi: [\tau, \infty) \to (-\infty, \infty)$ has the property (*) if for every $\tau' > \tau$ such that $\psi(\tau) < \psi(\tau')$ and every $c \in (\psi(\tau), \psi(\tau'))$ there exists a smallest $t \in (\tau, \tau')$ such that $\psi(t) = c$. It is clear that Lemma 1 still holds if "continuous sample paths" is replaced by "sample paths having the property (*)". This remark will be used toward the end of Section 6.

4. Asymptotic expansions for $A(t, \varepsilon)$**.** In this section we obtain asymptotic expansions for the functions $A(t, \varepsilon)$ associated with the measures F of Examples 4, 5 and 6 of Section 2.

Suppose first that F is as in Example 4 with $n = 2$. (The case of general n requires only minor modifications.) Let $f = F'$, and for fixed $\varepsilon > 0$ let $B = B(t) = A(t, \varepsilon)/t^{\frac{1}{2}}$ be defined by the equation

$$\varepsilon = \int_0^\infty \exp(Byt^{\frac{1}{2}} - \tfrac{1}{2}y^2 t) f(y)\, dy = (\varphi(B))^{-1} \int_0^\infty \varphi(yt^{\frac{1}{2}} - B) f(y)\, dy.$$

It is easily verified that $B \to \infty$, $B = o(t^{\frac{1}{2}})$ as $t \to \infty$. Let $\gamma > 1$. Since f is decreasing in $(0, \varepsilon')$ for some $\varepsilon' > 0$ we have for all t sufficiently large

$$\begin{aligned} \varepsilon &\geqq \frac{1}{\varphi(B)} \int_0^{\gamma B/t^{1/2}} \varphi(yt^{\frac{1}{2}} - B) f(y)\, dy \\ &\geqq \frac{f(\gamma B/t^{\frac{1}{2}})}{\varphi(B)} \int_0^{\gamma B/t^{1/2}} \varphi(yt^{\frac{1}{2}} - B)\, dy \\ &= \frac{\Phi((\gamma-1)B) - \Phi(-B)}{\gamma B \varphi(B) \log t^{\frac{1}{2}}/\gamma B\, (\log_2 t^{\frac{1}{2}}/\gamma B)^{1+\delta}}. \end{aligned}$$

Letting $t \to \infty$, then $\gamma \to 1$, we obtain

$$\limsup_{t\to\infty} \frac{1}{B\varphi(B)\log t^{\frac{1}{2}}/B\,(\log_2 t^{\frac{1}{2}}/B)^{1+\delta}} \leqq \varepsilon. \tag{15}$$

Now let $0 < \alpha < \gamma < 1$. Then for t sufficiently large

$$\varepsilon = \frac{1}{\varphi(B)}\int_0^\infty \varphi(yt^{\frac{1}{2}}-B)f(y)\,dy = \frac{1}{\varphi(B)}\left(\int_0^{\alpha B/t^{1/2}} + \int_{\alpha B/t^{1/2}}^{\gamma B/t^{1/2}} + \int_{\gamma B/t^{1/2}}^{\infty}\right)\varphi(yt^{\frac{1}{2}}-B)f(y)\,dy$$

$$\leqq \frac{1}{\varphi(B)}\left[\varphi((\alpha-1)B)F(\alpha B/t^{\frac{1}{2}}) + f(\alpha B/t^{\frac{1}{2}})\int_{-\infty}^{\gamma B/t^{1/2}} \varphi(yt^{\frac{1}{2}}-B)\,dy \right.$$

$$\left. + f(\gamma B/t^{\frac{1}{2}})\int_{\gamma B/t^{1/2}}^{\infty} \varphi(yt^{\frac{1}{2}}-B)\,dy\right]$$

$$\leqq \frac{1}{\varphi(B)}\left[\frac{\varphi((\alpha-1)B)}{\delta(\log_2 t^{\frac{1}{2}}/\alpha B)^{\delta}} + f(\alpha B/t^{\frac{1}{2}})(1/t^{\frac{1}{2}})\Phi((\gamma-1)B) + f(\gamma B/t^{\frac{1}{2}})(1/t^{\frac{1}{2}})(1-\Phi((\gamma-1)B)\right].$$

It follows easily from (15) that $B^2 = O(\log_2 t)$. Hence, setting $\alpha = (\log_2 t)^{-1}$ and using the inequality

$$\Phi(x) \leqq |x|^{-1}\varphi(x) \qquad (x<0),$$

we obtain

$$\varepsilon \leqq o(1) + \frac{(\log_2 t)\exp[(\frac{1}{2}(B^2-(1-\gamma)^2B^2)]}{(1-\gamma)B^2\log t^{\frac{1}{2}}/B\,(\log_2 t^{\frac{1}{2}}/B)^{1+\delta}} + \frac{(2\pi)^{\frac{1}{2}}\,e^{\frac{1}{2}B^2}}{\gamma B\log t^{\frac{1}{2}}/B\,(\log_2 t^{\frac{1}{2}}/B)^{1+\delta}}. \tag{16}$$

From (16) we have

$$\varepsilon \leqq \text{const.}\,\frac{(\log_2 t)\,e^{\frac{1}{2}B^2}}{\log t^{\frac{1}{2}}/B} + o(1),$$

from which it follows that $\log_2 t = O(B^2)$. Hence from (15) the second term on the right-hand side of (16) is $o(1)$ as $t \to \infty$ and thus

$$\frac{(2\pi)^{\frac{1}{2}}\,e^{\frac{1}{2}B^2}}{B\log t^{\frac{1}{2}}/B\,(\log_2 t^{\frac{1}{2}}/B)^{1+\delta}} \geqq \gamma\varepsilon + o(1).$$

Letting $t \to \infty$, then $\gamma \to 1$, we have

$$\liminf_{t\to\infty} \frac{(2\pi)^{\frac{1}{2}}\,e^{\frac{1}{2}B^2}}{B\log t^{\frac{1}{2}}/B\,(\log_2 t^{\frac{1}{2}}/B)^{1+\delta}} \geqq \varepsilon. \tag{17}$$

From (15) and (17) it follows that

$$B^2 = 2\log_2 t^{\frac{1}{2}}/B + \log B^2 + 2(1+\delta)\log_3 t^{\frac{1}{2}}/B + 2\log\varepsilon - \log 2\pi + o(1) \quad \text{as} \quad t\to\infty. \tag{18}$$

Now $\log_2 t^{\frac{1}{2}}/B = \log(\frac{1}{2}\log t - \log B) = \log(\frac{1}{2}\log t(1+o(1))) = \log_2 t - \log 2 + o(1)$, and it follows from (18) that

$$B^2 \sim 2\log_2 t.$$

Hence $\log B^2 = \log_3 t + \log 2 + o(1)$, and thus as $t \to \infty$ (18) can be simplified to

$$B^2 = 2[\log_2 t + (3/2+\delta)\log_3 t + \log \tfrac{1}{2}\varepsilon\pi^{-\frac{1}{2}} + o(1)],$$

which is equivalent to (10).

The expansion as $t \to 0$ given in Example 5 may be obtained by a similar argument, with the following important difference. Whereas the behavior of F near 0 completely determines the asymptotic behavior (as described by (10)) of $A(t, \varepsilon)$ in Example 4, and we rightfully expect the behavior of F near infinity to play an analogous role in Example 5; nevertheless, in this case the measure that F assigns to bounded sets cannot be neglected. To be precise, for all K sufficiently large, we have by the dominated convergence theorem as $t \to 0$

$$\int_0^K \exp(Ay - \tfrac{1}{2}y^2 t)\, dF(y) \to \frac{1}{\delta}\left(1 - \frac{1}{(\log_n K)^\delta}\right).$$

An argument similar to that leading to (15) allows us to infer that $B \equiv t^{-\frac{1}{2}}A = O((\log_2 t^{-1})^{\frac{1}{2}})$ and hence

$$\frac{1}{\delta} \leqq \lim_{t\to 0} \frac{1}{\varphi(B)} \int_0^{(t^{1/2}\log_2 t^{-1})^{-1}} \varphi(yt^{\frac{1}{2}} - B)\, dF(y)$$

$$\leqq \limsup_{t\to 0} \left[\frac{\varphi\left(\dfrac{1}{\log_2 t^{-1}} - B\right)}{\varphi(B)}\right] \cdot \frac{1}{\delta} = \frac{1}{\delta}.$$

The remainder of the argument follows as before.

Next, let F be as in Example 6, and for a given $\varepsilon > 0$ let $A = A(t, \varepsilon)$ be defined by the equation

$$f(A, t) = \varepsilon.$$

For $x > 0$ let

(19) $$h(y) = xy - y^2 t/2 - \alpha/y \qquad (\alpha = 16/27),$$

so that

(20) $$h'(y) = x - yt + \alpha/y^2 \qquad \text{and}$$

(21) $$h''(y) = -t - 2\alpha/y^3.$$

Fix $\delta > 0$ and let $y^* = ct^{-\frac{1}{3}}$, $x = bt^{\frac{2}{3}}$, where $b = b(t)$ and $c = c(t)$ satisfy

(22) $$h'(y^*) = 0 \qquad \text{and}$$

(23) $$h(y^*) = \log(\varepsilon(1+\delta)t^{\frac{1}{2}}).$$

From (19), (20), (22), and (23), we obtain

$$b = c - \alpha/c^2 \tag{24}$$

and

$$(b - c/2 - \alpha/c^2)ct^{\frac{1}{3}} = \log(\varepsilon(1+\delta)t^{\frac{1}{2}}). \tag{25}$$

It follows from (24) and (25) that

$$c = 4/3 + \tfrac{1}{2}t^{-\frac{1}{3}}\log(\varepsilon(1+\delta)t^{\frac{1}{2}}) + o(t^{-\frac{1}{3}}). \tag{26}$$

For any $\xi \geqq y^* - t^{-5/12}$, we have from (21)

$$h''(\xi) \geqq -t - \frac{2\alpha}{(y^* - t^{-5/12})^3} = -t\left(1 + \frac{2\alpha}{(c - t^{-1/12})^3}\right). \tag{27}$$

Since by (22) $h(y) = h(y^*) + \frac{1}{2}(y - y^*)^2\, h''(\xi)$ for some ξ in the interval between y and y^*, we have by (27) and (23)

$$\begin{aligned} f(bt^{\frac{2}{3}}, t) &\geqq \tfrac{1}{2}(3/\pi)^{\frac{1}{2}} \int_{y^* - t^{-5/12}}^{\infty} e^{h(y)}\, dy \\ &\geqq \tfrac{1}{2}(3t/\pi)^{\frac{1}{2}}\varepsilon(1+\delta) \int_{-t^{-5/12}}^{\infty} \exp\left[-\tfrac{1}{2}y^2 t\left(1 + \frac{2\alpha}{(c - t^{-1/12})^3}\right)\right] dy \\ &\geqq \frac{\varepsilon(1+\delta)(3/2)^{\frac{1}{2}}}{\left(1 + \frac{2\alpha}{(c - t^{-1/12})^3}\right)^{\frac{1}{2}}}\,[1 - \Phi(-t^{1/12})]. \end{aligned}$$

Hence from (26) we obtain

$$\liminf_{t\to\infty} f(bt^{\frac{2}{3}}, t) \geqq \varepsilon(1+\delta),$$

and it follows that

$$A(t) \leqq bt^{\frac{2}{3}} \tag{28}$$

for all sufficiently large t. After some calculation we see by (24), (26), and (28) that

$$A(t, \varepsilon) \leqq t^{\frac{2}{3}} + 4^{-1}t^{\frac{1}{3}}\log(\varepsilon(1+\delta)t^{\frac{1}{2}}) + o(t^{\frac{1}{3}}),$$

and since δ is arbitrary

$$A(t, \varepsilon) \leqq t^{\frac{2}{3}} + 4^{-1}t^{\frac{1}{3}}(\tfrac{1}{2}\log t + \log\varepsilon + o(1)).$$

A similar argument proves the reverse inequality.

5. Proof of Theorem 2. For any $0 < \tau < c < \infty$ we have

$$\begin{aligned} &P\{S_n \geqq m^{\frac{1}{2}}g(n/m) \text{ for some } \tau m \leqq n \leqq cm\} \\ &\quad \leqq P\{S_n \geqq m^{\frac{1}{2}}g(n/m) \text{ for some } n \geqq \tau m\} \\ &\quad \leqq P\{S_n \geqq m^{\frac{1}{2}}g(n/m) \text{ for some } \tau m \leqq n \leqq cm\} \\ &\qquad + P\{S_n \geqq m^{\frac{1}{2}}g(n/m) \text{ for some } n > cm\}. \end{aligned} \tag{29}$$

Part (i) follows from (29) and Lemma 4 and Lemma 5 below by first letting $m \to \infty$ and then letting $c \to \infty$. The proof of (ii) is similar and is omitted.

LEMMA 3. *For any* $0 < \tau < c < \infty$

$$P\{\max_{\tau \leqq t \leqq c}(W(t) - g(t)) = 0\} = 0.$$

PROOF. This result has been obtained by Ylvisaker [21]. It also follows from Theorem 7 of Doob [4] and the strong Markov property.

LEMMA 4. *For any* $0 < \tau < c < \infty$

$$\lim_{m \to \infty} P\{S_n \geqq m^{\frac{1}{2}} g(n/m) \text{ for some } \tau m \leqq n \leqq cm\}$$
$$= P\{W(t) \geqq g(t) \text{ for some } \tau \leqq t \leqq c\}.$$

PROOF. This result is easily deduced from Donsker's invariance principle, Lemma 3, and (for example) Theorem 4.1 of [1]. Alternatively, given Lemma 3, it may be proved in an elementary way by the method of Erdös and Kac [6].

Let $\psi(t) = t^{-\frac{1}{2}} g(t)$, and assume that ψ is ultimately non-decreasing and that (4) holds, or, what is more convenient, that

$$\sum_1^\infty \frac{\psi(n)}{n} e^{-\frac{1}{2}\psi^2(n)} < \infty. \tag{30}$$

By passing to $\min(\psi(n), 2(\log_2 n)^{\frac{1}{2}})$, we may assume without loss of generality that

$$\psi^2(n) \leqq 4 \log_2 n. \tag{31}$$

From the eventual monotonicity of ψ and (30) it follows that for all sufficiently large n

$$\psi^2(n) \geqq 2 \log_2 n. \tag{32}$$

In fact, we have

$$\sum_{n^{1/2}}^n \frac{\psi(k)\exp(-\frac{1}{2}\psi^2(k))}{k} \geqq \frac{\psi(n)}{\log n} \sum_{n^{1/2}}^n 1/k \geqq \tfrac{1}{4}\psi(n) \to \infty$$

along any subsequence of integers n for which $\psi^2(n) < 2 \log_2 n$. It follows from (30) (31), and (32) that if v_k denotes $\exp(k/\log k)$, then

$$\sum_2^\infty \frac{1}{\psi(v_k)} \exp(-\tfrac{1}{2}\psi^2(v_k)) < \infty. \tag{33}$$

LEMMA 5. *Suppose that* $\psi(t)$ *is eventually non-decreasing as* $t \to \infty$ *and satisfies* (30) (*and hence by the preceding remarks* (31)–(33) *as well*). *Then*

$$\liminf_{c \to \infty} \limsup_{m \to \infty} P\{S_n \geqq n^{\frac{1}{2}} \psi(n/m) \text{ for some } n > cm\} = 0.$$

PROOF. We shall use the following notation:

$$n_k = \exp[k/\log(k+\log c)], \qquad \bar{n}_k = cmn_k \qquad (k = 0, 1, \cdots),$$

$K_1, K_2, \cdots$ numerical constants not depending on c nor m (provided c is sufficiently large),

$$U(x) = \max(1, \log x), \qquad U_2(x) = U(U(x)),$$
$$H(x) = P\{x_k \leqq x\}.$$

Let $\Psi = \{\psi^*: \psi^*$ is eventually non-decreasing and satisfies (30)–(33)$\}$, and define

$$a_n = a_n(m) = (nU_2(n/m))^{\frac{1}{2}}$$
$$x_n' = x_n I_{\{x_n \leqq a_n\}}, \qquad x_n'' = x_n - x_n',$$
$$S_n' = \sum_1^n x_k', \qquad S_n'' = \sum_1^n x_k''.$$

Then

$$\begin{aligned} &P\{S_n \geqq n^{\frac{1}{2}}\psi(n/m) \text{ for some } n > cm\} \\ &\quad \leqq P\left\{S_n' \geqq n^{\frac{1}{2}}\left(\psi(n/m) - \frac{1}{U_2^{\frac{1}{2}}(n/m)}\right) \text{ for some } n > cm\right\} \\ &\qquad + P\{S_n'' \geqq (n/U_2(n/m))^{\frac{1}{2}} \text{ for some } n > cm\} \\ &\quad = p_1 + p_2, \qquad \text{say.} \end{aligned}$$

It will be shown in Lemma 6 below that $p_2 \to 0$ as $m \to \infty$. Hence to complete the proof it suffices to show that

$$\liminf_{c\to\infty} \limsup_{m\to\infty} p_1 = 0. \tag{34}$$

Define $\psi_1 = \psi - 2/\psi$. It is easily verified that $\psi_1 \in \Psi$. Furthermore, by (31)

$$p_1 \leqq \sum_{k=0}^{\infty} P\{\max_{\bar{n}_k \leqq n < \bar{n}_{k+1}} S_n' \geqq \bar{n}_k^{\frac{1}{2}}\psi_1(\bar{n}_k/m)\}. \tag{35}$$

For each $n = 1, 2, \cdots$ define

$$x_i^{(n)} = x_i I_{\{|x_i| \leqq a_n\}} \qquad (i = 1, 2, \cdots n).$$

Then $x_1^{(n)}, \cdots, x_n^{(n)}$ are i.i.d., and since $x_i^{(n)} \geqq x_i'$ $(i = 1, 2, \cdots, n)$ we have

$$P\{\max_{\bar{n}_k \leqq n < \bar{n}_{k+1}} S_n' \geqq \bar{n}_k^{\frac{1}{2}}\psi_1(cn_k)\} \leqq P\{\max_{\bar{n}_k \leqq n < \bar{n}_{k+1}} S_n^{(\bar{n}_{k+1})} \geqq \bar{n}_k^{\frac{1}{2}}\psi_1(cn_k)\} \tag{36}$$

for all $k = 0, 1, \cdots$. Since $n|Ex_1^{(n)}| \leqq n\int_{|x|>a_n}|x|\,dH \leqq (n/U_2(n/m))^{\frac{1}{2}}$, we may as above define a function $\psi_2 \in \Psi$ such that the right-hand side of (36) is majorized by

$$P\{\max_{\bar{n}_k \leqq n < \bar{n}_{k+1}}(S_n^{(\bar{n}_{k+1})} - ES_n^{(\bar{n}_{k+1})}) \geqq \bar{n}_k^{\frac{1}{2}}\psi_2(cn_k)\},$$

which by Lemma 7 below is in turn

$$\leqq 2P\{S_{\bar{n}_{k+1}}^{(\bar{n}_{k+1})} - ES_{\bar{n}_{k+1}}^{(\bar{n}_{k+1})} \geqq \bar{n}_k^{\frac{1}{2}}(\psi_2(cn_k) - [2(n_{k+1}/n_k - 1)]^{\frac{1}{2}})\}. \tag{37}$$

It is easily verified that

$$\frac{n_{k+1}}{n_k}-1 \leqq \exp\left(\frac{1}{\log(k+\log c)}\right)-1 \leqq \frac{K_1}{\log(k+\log c)},$$

and hence by (31) that

$$\psi_2(cn_k)-[2(n_{k+1}/n_k-1)]^{\frac{1}{2}} \geqq \psi_2(cn_k)-K_2/\psi_2(cn_k).$$

Letting $\psi_3 = \psi_2 - K_2/\psi_2$, we see that $\psi_3 \in \Psi$, and from (35)–(37) we obtain

$$(38) \qquad p_1 \leqq \sum_{k=0}^{\infty} P\{S_{\bar{n}_{k+1}}^{(\bar{n}_{k+1})} - ES_{\bar{n}_{k+1}}^{(\bar{n}_{k+1})} \geqq \bar{n}_k^{\frac{1}{2}}\psi_3(cn_k)\}.$$

Since $\operatorname{Var} x_1^{(n)} \leqq 1$, it follows from (31) and Lemma 8 below that

$$(39) \quad P\{S_{\bar{n}_{k+1}}^{(\bar{n}_{k+1})} - ES_{\bar{n}_{k+1}}^{(\bar{n}_{k+1})} \geqq \bar{n}_k^{\frac{1}{2}}\psi_3(cn_k)\}$$
$$\leqq 1-\Phi((n_k/n_{k+1})^{\frac{1}{2}}\psi_3(cn_k))+K_3\,\bar{n}_{k+1}^{-\frac{1}{2}}(U_2(cn_{k+1}))^{-\frac{3}{2}}\int_{|x|\leqq a_{\bar{n}_{k+1}}}|x|^3\,dH.$$

By Lemma 9 below the series of which the second term on the right-hand side of (39) is the kth summand converges to 0 as $m \to \infty$. The series

$$\sum_{k=0}^{\infty}(1-\Phi((n_k/n_{k+1})^{\frac{1}{2}}\psi_3(cn_k)))$$

does not depend on m, and by Lemma 10 below it converges to 0 as $c \to \infty$ through the values $\exp(i/\log i)$. This proves (34) and hence the lemma.

LEMMA 6.

$$\lim_{m\to\infty} P\{S_n'' \geqq (n/U_2(n/m))^{\frac{1}{2}} \textit{ for some } n \geqq m\} = 0.$$

PROOF. By the Markov inequality

$$P\{S_n'' \geqq (n/U_2(n/m))^{\frac{1}{2}} \text{ for some } n \geqq m\} \leqq E\left[\max_{n\geqq m}\frac{S_n''}{(n/U_2(n/m))^{\frac{1}{2}}}\right]$$
$$\leqq E(S_m''/m^{\frac{1}{2}})+E(\sum_{k=m+1}^{\infty} x_k''(U_2(k/m)/k)^{\frac{1}{2}}).$$

Now

$$\begin{aligned}\sum_{k=m}^{\infty}(U_2(k/m)/k)^{\frac{1}{2}}Ex_k'' &= \sum_{k=m}^{\infty}(U_2(k/m)/k)^{\frac{1}{2}}\sum_{i=k}^{\infty}\int_{a_i<x\leqq a_{i+1}} x\,dH\\ &= \sum_{i=m}^{\infty}\sum_{k=m}^{i}(U_2(k/m)/k)^{\frac{1}{2}}\int_{a_i<x\leqq a_{i+1}} x\,dH\\ &\leqq K_4\sum_{i=m}^{\infty}(iU_2(i/m))^{\frac{1}{2}}\int_{a_i<x\leqq a_{i+1}} x\,dH\\ &\leqq K_4\int_{x>a_m} x^2\,dH \to 0 \quad \text{as} \quad m\to\infty.\end{aligned}$$

Also

$$\begin{aligned}m^{-\frac{1}{2}}ES_m'' &\leqq m^{-\frac{1}{2}}\sum_{n=1}^{m}\sum_{k=n}^{\infty}\int_{a_k<x\leqq a_{k+1}} x\,dH\\ &\leqq m^{-\frac{1}{2}}\sum_{k=1}^{\infty}\min(k,m)\int_{a_k<x\leqq a_{k+1}} x\,dH\\ &\leqq m^{-\frac{1}{2}}\sum_{k=1}^{m} k\int_{a_k<x\leqq a_{k+1}} x\,dH + m^{\frac{1}{2}}\int_{x>a_{m+1}} x\,dH\\ &\leqq m^{-\frac{1}{2}}(\sum_{k=1}^{[\varepsilon m]}+\sum_{k=[\varepsilon m]+1}^{m})k^{\frac{1}{2}}\int_{a_k<x\leqq a_{k+1}} x^2\,dH+\int_{x>a_{m+1}} x^2\,dH\\ &\leqq \varepsilon^{\frac{1}{2}}Ex_1^{\,2}+\int_{x>a_{[\varepsilon m]}} x^2\,dH \to 0,\end{aligned}$$

as first $m \to \infty$, then $\varepsilon \to 0$. This completes the proof.

LEMMA 7. *Let $z_1, z_2, \cdots$ be independent random variables with $Ez_n = 0$, $Ez_n^2 \leqq \sigma^2$ $(n = 1, 2, \cdots)$. For any $a > 0, n = 1, 2, \cdots, m = 1, 2, \cdots, n-1$,*

$$P\{\max_{m \leqq k \leqq n} \sum_{i=1}^{k} z_i \geqq a\} \leqq 2P\{\sum_{i=1}^{n} z_i \geqq a - \sigma(2(n-m))^{\frac{1}{2}}\}.$$

PROOF. This result is well known when $m = 0$. The proof given, for example, by Lamperti ([12], page 45) works for general m as well.

The following result was proved by Nagaev [15].

LEMMA 8. *Let $z_1, z_2, \cdots$ be* i.i.d. *with $Ez_1 = 0$, $Ez_1^2 = \sigma^2$, $E|z_1|^3 = \beta < \infty$. There exists a universal constant L such that*

$$\left|P\left\{\sum_1^n z_i \leqq x\sigma n^{\frac{1}{2}}\right\} - \Phi(x)\right| \leqq \frac{L\beta}{n^{\frac{1}{2}}\sigma^3(1+|x|^3)}.$$

LEMMA 9.

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} \bar{n}_k^{-\frac{1}{2}}(U_2(cn_k))^{-\frac{3}{2}} \int_{|x| \leqq a_{n_k}} |x|^3 \, dH(x) = 0.$$

PROOF. In the following proof $K_1', K_2', \cdots$ denote constants which may depend on the fixed value of c, but not on m. Let

$$Q(m) = \sum_{k=1}^{\infty} \bar{n}_k^{-\frac{1}{2}}(U_2(cn_k))^{-\frac{3}{2}} \int_{|x| \leqq a_{n_k}} |x|^3 \, dH(x).$$

Then

(40) $$Q(m) \leqq m^{-\frac{1}{2}} \int_{0<|x|<\infty} |x|^3 \left(\sum_{k \geqq 1; cn_k U_2(cn_k) \geqq x^2/m} (cn_k)^{-\frac{1}{2}}(U_2(cn_k))^{-\frac{3}{2}}\right) dH(x).$$

Letting $g(z)$ denote $c \exp(z/\log(z+\log c))$, making the change of variable $u = g(z)$, and letting R denote the inverse of the function $x \to xU_2(x)$, we see that the series appearing in (40) is majorized by

$$K_1' \int_{u \geqq \max(c, R(x^2/m))} u^{-\frac{3}{2}} (U_2(u))^{-\frac{1}{2}} \, du.$$

Hence

$$Q(m) \leqq K_1' m^{-\frac{1}{2}}\left(\int_{|x| \leqq K_2' m^{1/2}} |x|^3 \, dH + \int_{|x| > K_2' m^{1/2}} |x|^3 \left(\int_{R(x^2/m)}^{\infty} u^{-\frac{3}{2}} (U_2(u))^{-\frac{1}{2}} \, du\right) dH(x)\right)$$
$$= K_1'(Q_1 + Q_2), \qquad \text{say.}$$

Now

$$Q_1 \leqq \varepsilon + K_2' \int_{\varepsilon m^{1/2} < |x| \leqq K_2' m^{1/2}} |x|^2 \, dH(x) \to 0$$

as first $m \to \infty$, then $\varepsilon \to 0$. Also since $R(x) \sim x/U_2(x)$, for all $x \geqq K_2' m^{\frac{1}{2}}$ we have

$$\int_{R(x^2/m)}^{\infty} u^{-\frac{3}{2}} (U_2(u))^{-\frac{1}{2}} \, du \leqq K_4'(m^{\frac{1}{2}}/|x|),$$

and thus

$$Q_2 \leqq \int_{|x| > K_2' m^{1/2}} |x|^2 \, dH \to 0$$

as $m \to \infty$.

LEMMA 10. *If* $c_i = \exp(i/\log i)$, *then for any* $\psi \in \Psi$,

$$\lim_{i\to\infty} \sum_{k=0}^{\infty} (1-\Phi((n_k/n_{k+1})^{\frac{1}{2}}\psi(c_i n_k))) = 0.$$

PROOF. Since $1-\Phi(x) \leqq Kx^{-1} e^{-\frac{1}{2}x^2}$ $(x \geqq 1)$, it suffices to show that

$$\sum_{k=0}^{\infty} \frac{1}{\psi(c_i n_k)} \exp\left(-\tfrac{1}{2}\frac{n_k}{n_{k+1}}\psi^2(c_i n_k)\right) \to 0$$

as $i \to \infty$. It is easily seen from (31) that $(1-n_k/n_{k+1})\psi^2(cn_k)$ is bounded in c and k. Hence

$$\frac{1}{\psi(cn_k)} \exp\left(-\tfrac{1}{2}\frac{n_k}{n_{k+1}}\psi^2(cn_k)\right) \leqq \frac{K_7}{\psi(cn_k)} \exp(-\tfrac{1}{2}\psi^2(cn_k)).$$

Set $c_i = \exp(i/\log i)$ and note that for large i

$$c_i n_k = \exp\left(\frac{i}{\log i} + \frac{k}{\log(k+i/\log i)}\right) \geqq \exp\left(\frac{i+k}{\log(i+k)}\right),$$

and hence by the monotonicity of ψ and by (33)

$$\sum_{k=0}^{\infty} \frac{1}{\psi(c_i n_k)} \exp(-\tfrac{1}{2}\psi^2(c_i n_k)) \leqq K_8 \sum_{k=0}^{\infty} \frac{1}{\psi(n_{i+k})} \exp(-\tfrac{1}{2}\psi^2(n_{i+k})) \to 0 \text{ as } i \to \infty.$$

REMARK. The original law of the iterated logarithm for i.i.d. random variables with mean 0 and variance 1 states that

$$P\{\limsup_{n\to\infty} S_n/(2n \log_2 n)^{\frac{1}{2}} = 1\} = 1.$$

It was proved by Khintchin [10] in the Bernoulli case and by Hartman and Wintner [8], who relied heavily on the results of Kolmogorov [11], in the general case. The more difficult problem of deciding for an arbitrary ultimately non-decreasing function ψ whether

$$P\{S_n < n^{\frac{1}{2}}\psi(n) \text{ for all sufficiently large } n\} \tag{41}$$

is 0 or 1 was posed by P. Lévy and has been studied by several authors.

Erdös [5] proved that in the symmetric Bernoulli case the probability (41) is 1 or 0 according as

$$\int^{\infty} \psi/t \exp(-\tfrac{1}{2}\psi^2/2)\, dt \tag{42}$$

converges or diverges. In the case of a standard Wiener process Itô and McKean [9] give a simple proof that the convergence of (42) implies that

$$P\{W(t) < t^{\frac{1}{2}}\psi(t) \text{ for all sufficiently large } t\} \tag{41'}$$

equals 1 and, following Motoo [13], prove that (41′) equals 0 if (42) diverges. In the context of the first boundary value problem for the heat equation this result had been discovered earlier by Petrovski [17].

The relation between (41) and (42) for general sums of i.i.d. random variables is not so adequately treated in the literature. *It follows from Lemma* 5 *that if* (42) *converges the probability* (41) *is* 1. This conclusion is implicit in a paper by Feller [7], but we are unable to justify the steps in his argument.

It may be worth noting that if a function $g(t)$ satisfying some mild regularity conditions is such that (4) holds, then for any $\delta > 0$ we can find a finite measure F on $(0, \infty)$ and an $\varepsilon > 0$ such that $g(t) \geqq A(t, \varepsilon)$ for all sufficiently large t and

$$P\{W(t) \geqq A(t, \varepsilon) \text{ for some } t > 0\} = f(0, 0)/\varepsilon < \delta.$$

6. Other stochastic processes. The idea underlying Theorem 1 is applicable to stochastic processes other than the Wiener process. For example, let $R(t)$, $t \geqq 0$, denote the distance of 3-dimensional Brownian motion from the origin. It may be checked by direct calculation that

$$\frac{\sinh yR(t)}{yR(t)} \exp(-\tfrac{1}{2}y^2 t) \qquad (t > 0) \tag{43}$$

is a martingale for each fixed $y > 0$. Hence if we define

$$f_1(x, t) = x^{-1} \textstyle\int_0^\infty y^{-1} \sinh xy \exp(-\tfrac{1}{2}y^2 t)\, dF(y),$$

where F is any measure on $(0, \infty)$ such that $f(x, 1) < \infty$ for all $x > 0$, then

$$f_1(R(t), t) \qquad (t \geqq 1)$$

is a martingale. As in Section 2, let $A_1(t, \varepsilon)$ denote the solution of

$$f_1(x, t) = \varepsilon \qquad (0 < \varepsilon, 1 \leqq t < \infty).$$

Since $y^{-1} \sinh y \leqq e^y$ for all $y \geqq 0$, an argument similar to the proof of Lemma 2 shows that $f_1(R(t), t) \to 0$ in probability as $t \to \infty$. Hence from Lemma 1 we obtain

THEOREM 3. *For any $a > 0$ and $\varepsilon = f(a, 1)$*

$$P\{R(t) \geqq A_1(t, \varepsilon) \textit{ for some } t \geqq 1\} = 2(1 - \Phi(a)) + f(a, 1)^{-1} \textstyle\int_0^\infty [\Phi(a-y) - \Phi(a+y) + 2\Phi(y) - 1]\, dF(y).$$

For the measure F of Example 4 of Section 2 it may be shown by methods similar to those of Section 4 that as $t \to \infty$

$$A_1(t, \varepsilon) = [2t(\log_2 t + 5/2 \log_3 t + \textstyle\sum_4^n \log_k t + (1+\delta)\log_{n+1} t + \log 2\varepsilon/\pi^{\frac{1}{2}} + o(1))]^{\frac{1}{2}}. \tag{44}$$

(To see that this is the "right" result, compare (44) with equation (14) of [9], page 163.)

Again, let $X(t)$ be a one-sided stable process of index $\frac{1}{2}$, i.e., let $X(t)$ be a process having stationary independent increments and Laplace transform

$$Ee^{-\lambda X(t)} = \exp(-(2\lambda)^{\frac{1}{2}} t) \qquad (t \geqq 0, \lambda \geqq 0). \tag{45}$$

Without loss of generality we may assume that the sample paths of $X(t)$ are non-decreasing, right-continuous, and increase only by jumps (e.g. [2], page 317). It follows from (45) that for each $y > 0$

$$\{\exp(-\tfrac{1}{2}y^2X(t)+yt),\ t \geqq 0\}$$

is a martingale. Let

$$f_2(x, t) = \textstyle\int_0^\infty \exp(-\tfrac{1}{2}y^2x+yt)\,dF(y)$$

for any measure F on $(0, \infty)$ such that $f_2(0, 1) < \infty$. Then for any $x \geqq 0$, $f_2(x+X(t), t)$ $(t \geqq 1)$ is a martingale, and by the sample path properties of $X(t)$ and the dominated convergence theorem it is easy to see that the sample paths of $f(x+X(t), t)$ $(t \geqq 1)$ have the property (*) (see Remark (d) at the end of Section 3).

From (45) it follows that for each $t > 0$, $t^{-2}X(t)$ and $X(1)$ have the same distribution. Moreover, for any $x \geqq 0$ and $\delta > 0$

$$f_2(x+\delta t^2, 1+t) = (\varphi(\delta^{-\frac{1}{2}}))^{-1}\textstyle\int_0^\infty \varphi(\delta^{\frac{1}{2}}yt-\delta^{-\frac{1}{2}})\exp(-\tfrac{1}{2}xy^2+y)\,dF(y) \to 0$$

as $t \to \infty$ by the dominated convergence theorem, and it follows by the argument of Lemma 2 that $f_2(x+X(t), t) \to 0$ in probability as $t \to \infty$. Letting $A_2(t, \varepsilon)$ denote the solution of the equation $f_2(x, t) = \varepsilon$, we obtain

THEOREM 4. *For any $a > 0$ and $\varepsilon = f_2(a, 1)$*

$$P\{X(t) \leqq A_2(t, \varepsilon) \text{ for some } t \geqq 1\}$$

$$= 2\left(1-\Phi\left(\frac{1}{a^{\frac{1}{2}}}\right)\right)+\frac{\varphi\left(\frac{1}{a^{\frac{1}{2}}}\right)\int_0^\infty\left[\Phi\left(ya^{\frac{1}{2}}+\frac{1}{a^{\frac{1}{2}}}\right)+e^{-2y}\Phi\left(ya^{\frac{1}{2}}-\frac{1}{a^{\frac{1}{2}}}\right)\right]dF(y)}{\int_0^\infty \varphi\left(ya^{\frac{1}{2}}-\frac{1}{a^{\frac{1}{2}}}\right)dF(y)}.$$

For the measure F of Example 4 of Section 2, since $f_2(x, t) = f(t, x)$, we obtain by inversion of (10) that

$$A_2(t, a) = t^2/2(\log_2 t+3/2\log_3 t+\textstyle\sum_{k=4}^n \log_k t+(1+\delta)\log_{n+1} t+\log \varepsilon/\pi^{\frac{1}{2}}+o(1))$$

as $t \to \infty$.

7. Final remarks. The examples of Section 6 were chosen for computational simplicity. A closer look at them suggests many questions which will be treated in a subsequent publication. (a) What is the origin of the martingale (43), and how can analogous martingales be found for other diffusion processes? (b) Since $R(t) \to \infty$ with probability one as $t \to \infty$, there exist functions $g(t)$ which are $o(t^{\frac{1}{2}})$ such that

$$P\{R(t) \leqq g(t) \text{ for some } t \geqq 1\} < 1.$$

Does our method permit us to calculate these probabilities exactly?

(c) For the Wiener process itself, if for some $0 < \alpha \leqq \beta < \infty$, F attributes positive measure to the interval $[\alpha, \beta]$, then

$$f(x, t) \geqq \exp(x\alpha-\beta^2t/2)F[\alpha, \beta],$$

so $f(g(t), t) \to \infty$ if $g(t)/t \to \infty$. Hence the method of Theorem 1 can only generate boundaries $g(t)$ which are $O(t)$. What is the class of martingales suitable for computing $P\{W(t) \geqq g(t)$ for some $t \geqq 1\}$ for *arbitrary continuous functions* g, and what is the relation of the class of martingales we have obtained to this much larger class? This question is closely connected with the study of certain boundary value problems for partial differential equations involving the generator $\partial/\partial t + \frac{1}{2}\partial^2/\partial x^2$ of the space-time Wiener process. We shall briefly indicate the nature of the connection.

Suppose for simplicity that $f(b, 0) < \infty$ for all b. Equation (3) implies that for any $h \geqq 0$, $-\infty < b < \infty$, and any $\varepsilon > f(b, h)$,

$$P\{W(t) \geqq A(t, \varepsilon) \text{ for some } t \geqq h \mid W(h) = b\} = f(b, h)/\varepsilon. \tag{46}$$

If we let $P(b, h)$ denote the left-hand side of (46) we have

$$\begin{aligned} P(b, h) &= f(b, h)/\varepsilon \qquad (h \geqq 0,\, b < A(h, \varepsilon)) \\ &= 1 \qquad\qquad\quad (h \geqq 0,\, b \geqq A(h, \varepsilon)). \end{aligned} \tag{47}$$

It follows from (47) that

$$\frac{\partial P}{\partial h} + \frac{1}{2}\frac{\partial^2 P}{\partial b^2} = 0 \qquad (h \geqq 0,\, b < g(h)), \tag{48}$$

where we have put $g(t) = A(t, \varepsilon)$.

Now let $g(t)$ be any positive, continuous, and increasing function of $t \geqq 0$, not necessarily of the form $A(t, \varepsilon)$. Then the left-hand side of (46) still defines a function $P(b, h)$ for $h \geqq 0$ and $-\infty < b < \infty$ which is 1 for $b \geqq g(h)$, and we may ask whether (48) continues to hold. Conversely, if P is any function defined for $h \geqq 0$ and $b \leqq g(h)$ which satisfies (48) and is 1 for $b = g(h)$, we may ask whether it is necessarily equal to the left-hand side of (46).

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