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ON THE TAIL σ-ALGEBRA OF THE FINITE INHOMOGENEOUS MARKOV CHAINS

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Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\{X_n : n = 1, 2, \cdots\}$ an inhomogeneous Markov chain assuming a finite number of states $I = \{a_1, \cdots, a_s\}$, $\mathscr{F}_n$ the σ-algebra generated by the random variable X_n and $\mathscr{F}_m{}^n$ the σ-algebra generated by the random variables $X_m, X_{m+1}, \cdots, X_n$. The σ-algebra $\bigcap_{n=1}^{\infty} \mathscr{F}_n{}^{\infty}$ will be denoted by $\mathscr{T}$ and called the tail σ-algebra of the sequence of random variables $\{X_n : n = 1, 2, \cdots\}$.

The aim of this paper is to prove the following:

THEOREM. *The tail σ-algebra of any finite inhomogeneous Markov chain is atomic.*

PROOF. Let us notice firstly that if $M_1{}^n, \cdots, M_s{}^n$ stands for the partition corresponding to the random variable X_n, that is $M_i{}^n = \{\omega : X_n(\omega) = a_i\}$ for $i = 1, \cdots, s$, then we have:

$$P\{A \mid \mathscr{F}_n\} - P\{A\} \leqq 1 - \min_{1 \leqq i \leqq s} P\{M_i{}^n\} \tag{1}$$

for any event A, in virtue of the fact that (1) is equivalent to

$$P\{A \cap (\textstyle\bigcup_{i \in \Lambda} M_i{}^n)\} - P\{A\} P\{\bigcup_{i \in \Lambda} M_i{}^n\} \leqq (1 - \min P\{M_i{}^n\}) P\{\bigcup_{i \in \Lambda} M_i{}^n\} \tag{2}$$

Λ being an arbitrary subset of the set $\{1, \cdots, s\}$ (see [1]). Furthermore, let $\gamma = \liminf_{n \to \infty} (\min_{1 \leqq i \leqq s} P\{M_i{}^n\})$. If $\gamma > 0$, by (1) we get

$$\max_{A \in \mathscr{F}_{n+1}} (P\{A \mid \mathscr{F}_n\} - P\{A\}) \leqq \delta \tag{3}$$

with $\delta < 1$. If $\gamma = 0$, we shall prove that we can find a sequence of positive integers $\{n_k : k \geqq 1\}$ such that

$$\limsup_{k \to \infty} \max_{A \in \mathscr{F}_{n_{k+1}}} (P\{A \mid \mathscr{F}_{n_k}\} - P\{A\}) \leqq \delta \tag{4}$$

almost surely.

Indeed, in this case we can extract from the initial sequence of random variables, a subsequence $\{X_{n_k} : k = 1, 2, \cdots\}$ such that for some states a_i, with $i \in \Gamma$, $\alpha = \liminf_{n \to \infty} \min_{1 \leqq i \leqq s} P\{M_i{}^n\} > 0$ and $\lim_{n \to \infty} P\{\bigcup_{i \in \Gamma'} M_i{}^n\} = 0$, Γ' being the complementary set of Γ with respect to the set $\{1, \cdots, s\}$. If k is sufficiently large, then $\min_{i \in \Gamma} P\{M_i{}^{n_k}\} > \alpha - \varepsilon$ for a certain ε chosen so that $\alpha - \varepsilon > 0$. Therefore, for any M_{n_k} and $M_{n_{k+1}}$ which are arbitrary unions of $M_i{}^{n_k}$ or $M_i{}^{n_{k+1}}$ respectively, with $i \in \Gamma$, we have

$$P\{M_{n_{k+1}} M_{n_k}\} - P\{M_{n_{k+1}}\} P\{M_{n_k}\} \leqq \delta P\{M_{n_k}\}$$

with $\delta = 1 - \alpha + \varepsilon$.

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But if we take an event $R_{n_{k+1}}$ formed by an arbitrary union of the events $M_i^{n_k}$ with $i \in \Gamma'$, then we have

$$P\{M_{n_k} R_{n_{k+1}}\} - P\{M_{n_k}\}P\{R_{n_{k+1}}\} < P\{M_{n_k} R_{n_{k+1}}\} = \alpha_n P\{M_{n_k}\}$$

where $\alpha_n = P\{R_{n_{k+1}}\}/P\{M_{n_k}\} \to 0$ as $n \to \infty$, by virtue of the above considerations. Therefore $P\{A \mid \mathscr{F}_{n_k}\} - P\{A\} \leqq \delta$ holds for all ω except for the set $U_{i \in \Gamma'} M_i^{n_k}$. Hence (4) is proved.

Besides, if we consider an arbitrary N-dimensional Borelian set B, and write $A = \{(X_{n_{k+2}}, \cdots, X_{n_{k+N+1}}) \in B\}$, we get

$$(5) \qquad P\{A \mid X_{n_k} = a_i\} = \textstyle\sum_{j=1}^{s} P\{A \mid X_{n_{k+1}} = a_j\} P\{X_{n_{k+1}} = a_j \mid X_{n_k} = a_i\}$$

and

$$(6) \qquad P\{A\} = \textstyle\sum_{j=1}^{s} P\{A \mid X_{n_{k+1}} = a_j\} P\{X_{n_k} = a_j\}.$$

Subtracting (6) from (5), we obtain

$$(7) \qquad P\{A \mid X_{n_k} = a_i\} - P\{A\} \leqq \delta.$$

However $P\{A \mid X_{n_k}\} = P\{A \mid \mathscr{F}_1{}^{n_k}\}$ and by a usual approximation argument, we get

$$P\{A \mid \mathscr{F}_1{}^{n_k}\} - P\{A\} \leqq \delta$$

for any $A \in \mathscr{F}_{n_{k+2}}^{\infty}$ except, perhaps, for the set $\bigcup_{i \in \Gamma'} M_i^{n_k}$ (see [1], [2]). From this, we deduce immediately that for any $T \in \mathscr{T}$, we have

$$(8) \qquad P\{T \mid \mathscr{F}_1{}^{\infty}\} - P\{T\} \leqq \delta$$

almost surely whatever may be ω. Integrating (8) on the set T, we obtain

$$P\{T\} - P^2\{T\} \leqq \delta P\{T\}$$

wherefrom we get that either $P\{T\} = 0$ or $P\{T\} \geqq 1 - \delta$ and the proof is concluded.

REMARK. We can find an upper bound for the number of the atoms comprised in the tail σ-algebra. This number is smaller than $[\sup[\liminf'_{n\to\infty}(\min_{i \in \{1,\cdots,s\}} P\{M_i{}^n\})]]^{-1}$ where by lim inf′ we understand that lim inf is taken over all subsequences of the type constructed in the proof of the Theorem, eliminating the events corresponding to those states whose probabilities are going to 0. In the case when $\sup[\liminf'_{n\to\infty}(\min_{1 \leqq i \leqq s} P\{M_i{}^n\})] > \frac{1}{2}$, $\{X_n : n = 1, 2, \cdots\}$ has a trivial tail and therefore follows the $0-1$ law.

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