The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 2155–2157

REVERSE SUBMARTINGALE AND SOME FUNCTIONS OF ORDER STATISTICS

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1. Introduction. Let $\{X_n, n \geqq 1\}$ be an exchangeable sequence of random variables and $Y_{1,n} \leqq Y_{2,n} \leqq \cdots \leqq Y_{n,n}$ be the order statistics based on $X_1, X_2, \cdots, X_n$. The object of this note is to show that $(Y_{n,n} - Y_{1,n})/\binom{n}{2}$ forms a reverse submartingale sequence of random variables, if $E(X_i) < \infty$. Moreover, if the X_i's are nonnegative random variables then $Y_{n,n}/n$ also forms a reverse submartingale sequence. Some moment properties of these statistics follow from these observations. We have also shown that an upper bound of $E(Y_{n,n} - Y_{1,n})$ is the expected value of range of n observations from a sequence of independent and identically distributed random variables having the same marginal distribution as that of X_i.

2. Some inequalities. Consider a set of $n+m$ finite real numbers $x_1, x_2, \cdots, x_{n+m}$ with $y_1 \leqq y_2 \leqq \cdots \leqq y_{n+m}$ as the corresponding ordered set. Let $(x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)})$, $i = 1, 2, \cdots, \binom{n+m}{n}$, be the all possible subsets of n tuples that can be formed from the $n+m$ x's and $\{y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}\}$ denote the corresponding ordered set. The range of $(x_1, x_2, \cdots, x_{n+m})$ is $R_{n+m} = y_{n+m} - y_1 = \max_{1 \leqq h \neq k \leqq n+m} |x_h - x_k|$. The range of the ith subset of n x's is indicated by $R_n^{(i)}$.

LEMMA 1.

$$\binom{n+m}{n}\binom{n}{2}R_{m+n} \leqq \binom{n+m}{2}\sum_{i=1}^{\binom{n+m}{n}} R_n^{(i)}. \tag{2.1}$$

PROOF. Notice that there are $\binom{m+n-2}{n-2}$ subsets in which $R_n^{(i)}$ is greater or equal to $|x_h - x_k|$. Hence

$$\binom{n+m}{2}\sum_{i=1}^{\binom{n+m}{n}} R_n^{(i)} \geqq \binom{n+m}{2}\binom{m+n-2}{n-2}|x_h - x_k| = \binom{n+m}{n}\binom{n}{2}|x_h - x_k|. \tag{2.2}$$

The lemma follows by taking the maximum on both sides of (2.2).

LEMMA 2. *If the x_i's are nonnegative, then*

$$\binom{m+n}{n} n y_{n+m} \leqq (n+m)\sum_{i=1}^{\binom{n+m}{n}} y_n^{(i)}. \tag{2.3}$$

PROOF. We know that $y_n^{(i)} \geqq y_j$ for all $j \leqq n$. Hence

$$(n+m)\sum_{i=1}^{\binom{n+m}{n}} y_n^{(i)} \geqq \binom{n+m}{n} n y_j \qquad \text{for} \quad j \leqq n \quad \text{and} \quad m \geqq 0. \tag{2.4}$$

For $j > n$, there are $\binom{n+m}{n} - \binom{j-1}{n}$ subsets where $y_n^{(i)} \geqq y_j$. Hence

$$\begin{aligned}(n+m)\sum_{i=1}^{\binom{n+m}{n}} y_n^{(i)} &\geqq (n+m)\left[\binom{n+m}{n} - \binom{j-1}{n}\right]y_j \\ &\geqq (n+m)\left[\binom{n+m}{n} - \binom{n+m-1}{n}\right]y_j \\ &= \binom{n+m}{n} n y_j \qquad \text{for} \quad j > n.\end{aligned} \tag{2.5}$$

Received March 2, 1970; revised May 26, 1970.

By combining (2.4) and (2.5) and taking maximum on both sides of the inequality, we obtain Lemma 2.

THEOREM 1. *Let* $\{X_n, n \geqq 1\}$ *be an exchangeable sequence of random variables with* $E(X_i) < \infty$. *Then* $\{Y_{n,n} - Y_{1,n}\}/\binom{n}{2}$ *forms a reverse submartingale sequence.*

PROOF. Let $R_n = Y_{n,n} - Y_{1,n}$. Since $\{X_n\}$ is an exchangeable sequence of random variables, therefore the random variables $\{X_1, X_2, \cdots, X_{n+m}\}$ given R_{n+m}, $R_{n+m+1}, \cdots, R_{n+m+k}$ for any finite $k \geqq 0$ are also exchangeable. Multiplying both sides of (2.1) by the conditional distribution of $(X_1, X_2, \cdots, X_{n+m})$ given R_{n+m}, $R_{n+m+1}, \cdots, R_{n+m+k}$ we get

$$R_{n+m}/\binom{n+m}{2} \leqq E[\{R_n/\binom{n}{2}\}/\{R_{n+m}/\binom{n+m}{2}\}, \cdots, \{R_{n+m+k}/\binom{n+m+k}{2}\}]. \tag{2.6}$$

Moreover $E(R_n) < \infty$ since $E(X_i) < \infty$. The theorem follows by letting k go to infinity. The validity of the limiting process follows from Doob ([1] page 332).

THEOREM 2. *Let* $\{X_n, n \geqq 1\}$ *be an exchangeable sequence of nonnegative random variables with finite expectation. Then* $\{Y_{n,n}/n\}$ *forms a reverse submartingale sequence.*

PROOF. This theorem follows from Lemma 2 exactly in the same way as Theorem 1 from Lemma 1.

THEOREM 3. *If the random variables* $\{X_n, n \geqq 1\}$ *form an exchangeable sequence with* kth *moment finite then*

$$\text{(i) } E\{R_{n+m}\}^u \leqq \frac{(n+m)^u(n+m-1)^u}{n^u(n-1)^u} E\{R_n{}^u\} \tag{2.7}$$

$$\text{(ii) } E\{\max_{1 \leqq i \leqq n+m} |x_i|^u\} \leqq (1+m/n)^u E\{\max_{1 \leqq i \leqq n} |x_i|^u\} \tag{2.8}$$

for $1 \leqq u \leqq k$.

PROOF. The theorem follows from Theorem 1 and Theorem 2 and by using the fact that if a reverse submartingale sequence belongs to L_r, then the sequence of rth moments form a monotone decreasing sequence (Loève [2] page 397).

COROLLARY. *If the random variables* $\{X_n, n \geqq 1\}$ *form an* i.i.d. *sequence with finite first moment then*

$$\frac{n}{n-1} E(R_{n-1}) \leqq E(R_n) \leqq \frac{n}{n-2} E(R_{n-1}). \tag{2.9}$$

PROOF. The rhs of the inequality follows directly from Theorem 3 and the lhs of the inequality is well known.

3. Upper bound of $E(R_n)$. Let the random variables $\{X_n, n \geqq 1\}$ form an exchangeable sequence having marginal distribution function F and let $\{U_n, n \geqq 1\}$ be an i.i.d. sequence of random variables having the same marginal distribution function F. Let

$$V_{nn} = \max_{1 \leqq i \leqq n}\{U_i\} \quad \text{and} \quad V_{1n} = \min_{1 \leqq i \leqq n}\{U_i\}. \tag{3.1}$$

THEOREM 4. *For all t*

$$\Pr\{Y_{n,n} \leqq t\} \geqq \Pr\{V_{n,n} \leqq t\} \quad \textit{and} \quad \Pr\{Y_{1n} \leqq t\} \leqq \Pr\{V_{1n} \leqq t\}. \tag{3.2}$$

PROOF. Since $\{X_n, n \geqq 1\}$ is an exchangeable sequence, by deFinnetti–Dynkin theorem, there exists a random variable W such that

$$\Pr\{X_1 \leqq t_1, X_2 \leqq t_2, X_n \leqq t_n\} = E\{\textstyle\prod_{i=1}^n \Pr X_i \leqq t_i \mid W\} \tag{3.3}$$

where the conditional distributions are identical. Therefore

$$\begin{aligned} \Pr\{Y_{n,n} \leqq t\} &= E\Pr\{X_i \leqq t \mid W\}^n \\ &\geqq [E\Pr\{X_i \leqq t \mid W\}]^n = F^n(t) \\ &= \Pr\{V_{nn} \leqq t\}. \end{aligned} \tag{3.4}$$

By following the same reasoning, we can establish $\Pr\{Y_{1n} \leqq t\} \leqq \Pr\{V_{1n} \leqq t\}$.

COROLLARY.

$$E(R_n) \leqq \textstyle\int_{-\infty}^{\infty} \{1-(1-F)^n - F^n\}\, dx. \tag{3.5}$$

PROOF. From Theorem 4,

$$\begin{aligned} E(R_n) &= E\{Y_{nn} - Y_{1n}\} \leqq E\{V_{nn} - V_{1n}\} \\ &= \textstyle\int_{\infty}^{\infty} \{1-(1-F)^n - F^n\}\, dx. \end{aligned}$$

4. Comments. It can easily be shown that if X_i's are i.i.d. random variables then $\sup_{1 \leqq i \leqq n} |X_i|/n$ converges to zero if $E(|X_i|) < \infty$. If the X_i's are exchangeable, it can be shown by elementary calculations that $\sup_{1 \leqq i \leqq n} |X_i|/n$ converges almost surely to zero if $E(|X_i|) < \infty$. But this may be viewed as a consequence of Theorem 2 since a reverse submartingale converges to a random variable a.s. if it is uniformly integrable (see Loève [2] page 397). That it is uniformly integrable follows from the observation that it is less than $\sum_{i=1}^n |X_i|/n$ and the latter statistic converges almost surely to a random variable as $E|X_i| < \infty$ {Loève ([2] page 400)}. Now by using Theorem 4 we get $\sup_{1 \leqq i \leqq n} |X_i|/n \to_p 0$ and hence by equivalence theorem $\sup_{1 \leqq i \leqq n} |X_i|/n \to_{\text{a.s.}} 0$.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Process.* Wiley, New York.
[2] LOÈVE, M. (1960). *Probability Theory.* Van Nostrand, Princeton.