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ON THE RELATION BETWEEN BAHADUR EFFICIENCY AND POWER

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1. Definitions and introduction. For a set of probability measures $\{P_\theta\}$, $\theta \in \Omega$, defined on an abstract sample space S, let H be the hypothesis: $\theta \in \Omega_0 \subset \Omega$, and let $\{T_n\}$ be a sequence of test statistics defined on S. Following Bahadur [1], $\{T_n\}$ is called a *standard sequence* if (I) for each $\theta \in \Omega_0$ there exists a continuous probability distribution function $F(x)$ such that $\lim_{n\to\infty} P_\theta\{T_n < x\} = F(x)$ for each x, (II) there is a positive constant a such that $\log[1-F(x)] = -\frac{1}{2}ax^2[1+o(1)]$ as $x \to \infty$, and (III) there is a function $b(\theta)$, $0 < b(\theta) < \infty$, on $\Omega - \Omega_0$ such that $\lim_{n\to\infty} P_\theta\{|T_n/n^{\frac{1}{2}} - b(\theta)| > \varepsilon\} = 0$ for each $\varepsilon > 0$ and for each $\theta \in \Omega - \Omega_0$. The function $c(\theta)$ on Ω defined by $c(\theta) = 0$ for $\theta \in \Omega_0$ and $c(\theta) = ab^2(\theta)$ for $\theta \in \Omega - \Omega_0$ is called the *slope* of $\{T_n\}$ when θ obtains. If $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ are two standard sequences with slopes $c_1(\theta)$ and $c_2(\theta)$, respectively, then the ratio $\phi_{12}(\theta) = c_1(\theta)/c_2(\theta)$ is called the asymptotic efficiency of $\{T_n^{(1)}\}$ relative to $\{T_n^{(2)}\}$. For $0 < \alpha < 1$ and $\theta \in \Omega$, let $\beta_n^{(i)}(\alpha \mid \theta) = P_\theta\{F^{(i)}(T_n^{(i)}) < 1-\alpha\}$, $i = 1, 2$, and let $\delta_n(1, 2 \mid \theta) = \sup_\alpha [\beta_n^{(2)}(\alpha \mid \theta) - \beta_n^{(1)}(\alpha \mid \theta)]$. Then $\{T_n^{(2)}\}$ is said to *dominate* $\{T_n^{(1)}\}$ at θ if $\lim_{n\to\infty} \delta_n(1, 2 \mid \theta) = 0$.

The function ϕ_{12} has many important and interesting properties (cf. [1], [2], [3]) some of which concern the power of the test statistics. Among other things, R. R. Bahadur [1] shows (i) if $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ at θ, then $\phi_{12}(\theta) \leqq 1$ and (ii) if $\phi_{12}(\theta) < 1$, then $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ at θ. The assertion is then made that from these results it follows that $\phi_{12} < 1$ if and only if $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ but $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$. However, this is not the case. If $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ but $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$, one can conclude from (i) and (ii) only that $\phi_{12} \leqq 1$. To illustrate that $\phi_{12} \leqq 1$ is the correct conclusion rather than $\phi_{12} < 1$, the authors give an example of two standard sequences $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ for which $\{T_n^{(2)}\}$ dominates $\{T_n^{(1)}\}$ but $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$, and for which $\phi_{12} = 1$.

2. Example. Let $s = (x_1, x_2, \cdots)$ where the x_i are independent and normally distributed on the real line with $E(x_i) = \theta$, $\operatorname{Var}(x_i) = 1$. Let Ω consist of two points 0 and μ, where $\mu > 0$, and let H be the hypothesis that $\theta = 0$. For each $n = 1, 2, \cdots$ let $T_n^{(2)} = \sum_{i=1}^n x_i/n^{\frac{1}{2}}$. It is well known and readily verified that $\{T_n^{(2)}\}$ is a standard sequence and its slope $c_2(\mu) = \mu^2$. Let $k_1 \leqq k_2 \leqq \cdots$ be a sequence of positive integers such that

$$k_n \leqq n \qquad \text{for each } n, \tag{1}$$

and

$$k_n/n \to 1 \qquad \text{as } n \to \infty, \tag{2}$$

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and let

$$T_n^{(1)} = T_{k_n}^{(2)} \qquad (n = 1, 2, \cdots). \tag{3}$$

It follows from (1) that (3) is a valid definition, and from (2) and (3) that $\{T_n^{(1)}\}$ is a standard sequence and that its slope $c_1(\mu)$ also equals μ^2. Thus $\phi_{12}(\mu) = 1$.

We proceed to show that the k_n can be chosen so that $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$. Since $T_n^{(2)}$ is normally distributed with mean $n^{\frac{1}{2}}\theta$ and variance 1,

$$\beta_n^{(2)}(\alpha) = \Phi(z_\alpha - n^{\frac{1}{2}}\mu) \tag{4}$$

for $0 < \alpha < 1$, where Φ is the standard normal distribution function and $1 - \Phi(z_\alpha) = \alpha$. Choose and fix constants l_1 and l_2, $-\infty < l_1 < l_2 < \infty$. For each n let α_n be defined by

$$z_{\alpha_n} - n^{\frac{1}{2}}\mu = l_1 \tag{5}$$

Then $z_{\alpha_n} \to \infty$ (so $\alpha_n \to 0$) as $n \to \infty$. Hence there exists n_0 such that $z_{\alpha_n} > l_2$ for all $n > n_0$. For $n \leqq n_0$ let $k_n = n$ and for $n > n_0$ let k_n be the positive integer such that

$$z_{\alpha_n} - (k_n + 1)^{\frac{1}{2}}\mu < l_2 \leqq z_{\alpha_n} - k_n^{\frac{1}{2}}\mu. \tag{6}$$

It follows from (5) and (6) that (1) holds, and that

$$l_1 + n^{\frac{1}{2}}\mu - (k_n + 1)^{\frac{1}{2}}\mu < l_2 \leqq l_1 + n^{\frac{1}{2}}\mu - k_n^{\frac{1}{2}}\mu \tag{7}$$

for $n > n_0$. It follows from (7) and $\mu > 0$ first that $k_n \to \infty$ and next that (2) holds.

Now, $\beta_n^{(1)}(\alpha) = \beta_{k_n}^{(2)}(\alpha) = \Phi(z_\alpha - k_n^{\frac{1}{2}}\mu)$ by (3) and (4). Hence, for $n > n_0$,

$$\begin{aligned} \beta_n^{(1)}(\alpha_n) - \beta_n^{(2)}(\alpha_n) &= \Phi(z_{\alpha_n} - k_n^{\frac{1}{2}}\mu) - \Phi(z_{\alpha_n} - n^{\frac{1}{2}}\mu) \\ &\geqq \Phi(l_2) - \Phi(l_1) \end{aligned} \tag{8}$$

by (5) and (6). Hence $\sup\{\beta_n^{(1)}(\alpha) - \beta_n^{(2)}(\alpha) \colon 0 < \alpha < 1\} \geqq \Phi(l_2) - \Phi(l_1)$ for all $n > n_0$. Since $l_2 > l_1$, $\{T_n^{(1)}\}$ does not dominate $\{T_n^{(2)}\}$.

3. Remarks. The example reveals that the efficiency ϕ_{12} does not distinguish between tests quite as sharply as previously thought. That is, there is a class of pairs of tests which are ordered with respect to the domination criterion but are not ordered by the efficiency function.

Leon J. Gleser [3] gives results analogous to (i) and (ii) and makes the same assertion under generalized rates of convergence. The example is applicable in this setting also. Further, it is worth noting that our example treats in fact the exact slopes and exact sizes of the tests since the null distribution of the statistics is the same for each n. For a discussion of the difference between exact slopes and approximate slopes, cf. Bahadur [2].

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