The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 2130–2131

AN EXTENSION OF THE HEWITT–SAVAGE ZERO-ONE LAW[1]

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Let $(\Omega^\infty, \mathfrak{A}^\infty)$ be the direct product of countably many copies of the measurable space $(\Omega, \mathfrak{A})$ and let $\mu = \prod \mu_i$ be a product probability measure on $(\Omega^\infty, \mathfrak{A}^\infty)$. The Hewitt–Savage Zero-One Law says that if all μ_i are equal then the sets of $\mathfrak{A}^\infty$ which are invariant under all permutations of finitely many coordinates have μ-measure either zero or one. We derive an extension of this theorem to a case where the μ_i are not all identical.

A product probability measure $\mu = \prod \mu_i$ is said to be *recurring* if for each $i = 1, 2, \cdots$ there is some $j > i$ such that $\mu_j = \mu_i$, i.e., each factor of μ occurs infinitely often.

THEOREM. *If $\mu = \prod \mu_i$ is recurring then $\mu(S)$ is zero or one for every set $S \in \mathfrak{A}^\infty$ which is invariant under all permutations of finitely many coordinates.*

PROOF. Let S be such a permutation invariant set. Let $\mathscr{F}_n$ be the σ-field of cylinder sets generated by the first n factors of $(\Omega^\infty, \mathfrak{A}^\infty)$. Then there exist sets $A_n \in \mathscr{F}_n$ such that $\mu(S \Delta A_n) \to 0$ as $n \to \infty$. Since μ is recurring, for each n there are n distinct indices $j_i > n$ such that $\mu_{j_i} = \mu_i$, $1 \leqq i \leqq n$. Hence for each n there exists a measure preserving transformation φ_n induced by a permutation π_n of a finite number of indices such that A_n is independent of $\varphi_n(A_n)$. Thus as $n \to \infty$

$$\mu^2(S) \leftarrow \mu(A_n)\,\mu(\varphi_n(A_n)) = \mu(A_n \cap \varphi_n(A_n)) \to \mu(S)$$

which implies $\mu(S)$ is either zero or one.

LEMMA. *Let $\mathscr{F} \supset \mathscr{S}_1 \supset \mathscr{S}_2 \supset \cdots$ be a decreasing sequence of σ-fields and let λ and ν be two probability measures on $\mathscr{F}$ whose restrictions to $\bigcap_{n=1}^\infty \mathscr{S}_n$ are identical. Then*

$$\sup_{S \in \mathscr{S}_n} |\lambda(S) - \nu(S)| \to 0 \qquad \textit{as} \quad n \to \infty.$$

PROOF. Let $\alpha \equiv \frac{1}{2}(\lambda + \nu)$ and let α_n, λ_n, ν_n be the restrictions of α, λ, ν to $\mathscr{S}_n$. Let $f_n \equiv d\lambda_n/d\alpha_n$ and $g_n \equiv d\nu_n/d\alpha_n$. Then f_n and g_n are reversed martingales which converge in $L_1(\alpha)$ to a common limit (Doob (1953)). Hence $f_n - g_n \to_{L_1(\alpha)} 0$, which implies the desired result since for $S \in \mathscr{S}_n$,

$$|\lambda(S) - \nu(S)| \leqq \|f_n - g_n\|_{L_1(\alpha)}.$$

COROLLARY. *Let μ be a recurring product probability measure on $(\Omega^\infty, \mathfrak{A}^\infty)$. Let $\mathscr{S}_n$ be the σ-field of sets in $\mathfrak{A}^\infty$ that are invariant under all permutations of the first n coordinates. If probability measures λ and ν are absolutely continuous with respect to μ, then $\sup_{S \in \mathscr{S}_n} |\lambda(S) - \nu(S)| \to 0$ as $n \to \infty$.*

Received June 9, 1969; revised April 27, 1970.
[1] Research supported by USPHS-GM 14554-03.

PROOF. According to the theorem, μ assigns measure zero or one to every set invariant under all permutations of finitely many coordinates. The same then holds for λ and ν. The lemma now applies to complete the proof.

Special cases of this Corollary have been obtained before: Hannan (1953) and Hannan and Robbins (1955) treated the case of two different μ_i-measures; Horn (1968) obtained a similar result for a finite number of different factor measures. Our present method generalizes those results in various directions and considerably shortens the proof.

Acknowledgment. The authors wish to thank the referee for his helpful suggestions.

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