The Annals of Mathematical Statistics
1970, Vol. 41, No. 6, 2126–2129

PROBABILISTIC TECHNIQUES LEADING TO A VALIRON-TYPE THEOREM IN SEVERAL COMPLEX VARIABLES

BY J. GOPALA KRISHNA

Andhra University

0. Introduction. Rosenbloom [5] initiated a probabilistic approach for proving Wiman-type theorems for entire power series in one complex variable, which is extended to the case of several complex variables and studied extensively by Schumitzky [6]. Their technique consists of studying the relations among certain functions such as the cumulants, the modal mass and the modes of a stochastic process (of which the classical Poisson Process is a particular case) associated with a non-constant entire power series. The purpose of this work is to discuss some properties of such a process, which lead to a well-known theorem of Valiron in one complex variable and its generalization to the case of several complex variables (see [3]). It incidentally turns out that the theorem may be proved bypassing the considerations of the central index.

1. Notation. We follow throughout the standard or suggestive notation (see [2]). Throughout k denotes a positive integer and $\mathscr{C}^k$ the Cartesian product of k copies of the complex plane. We denote $(r_1, \cdots, r_k)$, $(n_1, \cdots, n_k)$, $(z_1, \cdots, z_k)$, $(|z_1|, \cdots, |z_k|)$ etc. $\in \mathscr{C}^k$ by their respective unsuffixed symbols r, n, z, $|z|$ etc. and denote $(0, \cdots, 0) \in \mathscr{C}^k$ by $\mathbf{0}$. We say, in the case of $r, s \in \mathscr{C}^k$, that $r \leqq s$ or $s \geqq r$, iff (if and only if) r_j, s_j are real and $r_j \leqq s_j$ for $1 \leqq j \leqq k$, and that $r \ll s$ or $s \gg r$, if and only if $r \leqq s$ and $r_j < s_j$ for $1 \leqq j \leqq k$. We write $|\mathscr{C}^k| = [r : r \in \mathscr{C}^k, r \geqq \mathbf{0}]$, $\mathscr{C}^{k+} = [r : r \in \mathscr{C}^k, r \gg \mathbf{0}]$ and $I = I^k = [n : n \in |\mathscr{C}^k|$, where each n_j is an integer].

Throughout f denotes a non-constant entire power series in $\mathscr{C}^k$ defined by $f(z) = \sum_{n \in I} a_n z^n$, for $z \in \mathscr{C}^k$ $(z^n = z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k})$ and F denotes the function on $|\mathscr{C}^k|$ defined by $F(r) = \sum_{n \in I} |a_n| r^n$, for $r \in |\mathscr{C}^k|$. We define the functions: the maximum term μ and the maximum modulus $\mathscr{M}$ of f by

$$\mu(r) = \max_{n \in I}(|a_n| r^n), \qquad \mathscr{M}(r) = \max_{|z| = r} |f(z)|, \quad \text{for} \quad r \in |\mathscr{C}^k|.$$

We say that a real valued function H on $\mathscr{C}^{k+}$ is of finite order if and only if there exist $A \in |\mathscr{C}^1|$, $\alpha \in \mathscr{C}^{k+}$ such that $H^+(r) \leqq A r^\alpha$ asymptotically as $r \to +\infty = (+\infty, \cdots, +\infty)$ in $\mathscr{C}^{k+}$. We say that f is of finite order (in $\mathscr{C}^k$) if and only if $\log \mathscr{M}$ is of finite order.

2. A distribution valued function associated with f and its properties. Throughout this section $\mathscr{F}$ denotes the function over $\mathscr{C}^{k+}$ with values as probability measure distributions (see for definitions etc. [1], [4]) such that the distribution $\mathscr{F}_r$ associated with $r \in \mathscr{C}^{k+}$ is the discrete distribution over $|\mathscr{C}^k|$ having mass $|a_n| r^n / F(r)$ at n, for $n \in I$. We write χ_n, for $n \in I$, to denote the nth cumulant of $\mathscr{F}$ (so that $\chi_n(r)$ is the nth cumulant of $\mathscr{F}_r$). We first prove

Received December 4, 1968; revised April 25, 1970.

THEOREM 2.1. *The following three statements are equivalent*:

(a) $\log F$ *is of finite order*;

(b) *each cumulant of* $\mathscr{F}$ *is of finite order*;

(c) *there exists a positive integer* p *such that each cumulant of degree* p *(i.e.,* χ_n *with* $n_1 + \cdots + n_k = p$*—usually referred to as "$\cdots$ of order p") is of finite order.*

We require three lemmas.

LEMMA 2.2. *For* $r \in \mathscr{C}^{k+}$, $n \in I$,

$$\chi_n(r) = (\partial_1^{n_1}\partial_2^{n_2}\cdots\partial_k^{n_k})\log F(r) = \partial^n \log F(r),$$

where ∂_j *is the operator* $r_j(\partial/\partial r_j)$ *for* $1 \leqq j \leqq k$.

PROOF OF LEMMA 2.2. Let $r \in \mathscr{C}^{k+}$ and be $= (e^{s_1}, \cdots, e^{s_k})$, denoted by e^s. Now the moment generating function $\mathscr{M}_r$ of $\mathscr{F}_r$ is easily seen to be analytic over the entire $\mathscr{C}^k$ and is given by

$$\mathscr{M}_r(t) = g(e^{s+t})/F(r), \quad \text{for} \quad t \in \mathscr{C}^k,$$

where $g(z) = \sum_{n \in I} |a_n| z^n$ for $z \in \mathscr{C}^k$. Hence the cumulant generating function of $\mathscr{F}_r$ is analytic in a neighborhood of $\mathbf{0} \in \mathscr{C}^k$ and

$$\chi_n(r) = \frac{\partial^{n_1+\cdots+n_k}\log g(e^{s+t})}{\partial t_1^{n_1}\cdots\partial t_k^{n_k}} \quad \text{at} \quad t = \mathbf{0} \quad \text{in} \quad \mathscr{C}^k$$

$$= \frac{\partial^{n_1+\cdots+n_k}\log F(e^s)}{\partial s_1^{n_1}\cdots\partial s_k^{n_k}} \quad \text{in} \quad \mathscr{C}^{k+},$$

which implies the lemma.

LEMMA 2.3. *Let* ϕ *and* $\phi_j (1 \leqq j \leqq k)$ *be nonnegative real valued and locally bounded functions on* $\mathscr{C}^{k+}$ *such that the jth section of* ϕ_j *is increasing (i.e.* $\phi_j(r) \leqq \phi_j(s)$ *if* $r, s, \in \mathscr{C}^{k+}$, $r \leqq s$, $r_t = s_t$ *for* $t \neq j$, $1 \leqq t \leqq k$), *for* $1 \leqq j \leqq k$. *Let further the line integral*

$$\int_r^s [\textstyle\sum_{j=1}^k x_j^{-1}\phi_j(x)\,dx_j]$$

taken over any polygonal path from r *to* s *with sides parallel to the axes, exist and be* $= \phi(s) - \phi(r)$. *Then* (i) *the following two statements are equivalent*:

(a) ϕ *is of finite order*;

(b) *each* $\phi_j (1 \leqq j \leqq k)$ *is of finite order.*

(ii) (b) *implies* (a) *even if the jth section of* ϕ_j *is not increasing for any* $1 \leqq j \leqq k$.

PROOF OF LEMMA 2.3. For any $r \in \mathscr{C}^{k+}$, $1 \leqq j \leqq k$ we have

$$\phi_j(r) \leqq \int_{r_j}^{er_j} \phi_j(r_1, \cdots, r_{j-1}, x_j, r_{j+1}, \cdots, r_k) x_j^{-1}\,dx_j$$

$$\leqq \phi(r_1, \cdots, r_{j-1}, er_j, r_{j+1}, \cdots, r_k),$$

which shows that (a) implies (b).

We easily verify that for any d, $r \in \mathscr{C}^{k+}$, $d \leqq r$,

$$\phi(r) \leqq \phi(d) + \sum_{j=1}^{k} \sup_{d \leqq t \leqq r} \phi_j(t) \log (r_j/d_j),$$

which (with the components of d "fixed but chosen sufficiently large") shows that (b) implies (a). Hence the lemma.

LEMMA 2.4. *If f is of finite order, then any partial derivative of f is of finite order (all in $\mathscr{C}^k$).*

PROOF OF LEMMA 2.4. The proof may be carried out as in the case of one variable using Cauchy's Integral Formula (see [2]).

PROOF OF THEOREM 2.1. We first prove that (a) implies (b). Let (a) hold. By Lemma 2.2, $\partial_j^2 \log F(r) \geqq 0$ and hence the jth section of $\partial_j \log F(r)$ is increasing with r_j, for $1 \leqq j \leqq k$. It is now easy to verify that the hypothesis of Lemma 2.3 holds with $\phi = \log F$ and $\phi_j = \partial_j \log F$ and hence by Lemma 2.2, $\partial_j \log F = (\partial_j F)/F$ is of finite order for $1 \leqq j \leqq k$. By virtue of Lemma 2.4 we might repeat the above argument with $\partial^n F$, for any particular $n \in I$ in the place of F to conclude that $\partial_j \partial^n F/\partial^n F$ is of finite order, and (b) now follows from the fact that each cumulant of $\mathscr{F}$ is a polynomial in the functions $(\partial_j \partial^n F)/\partial^n F$ $(n \in \mathscr{I}, 1 \leqq j \leqq k)$.

That (b) implies (c) is trivial. Let (c) hold. Since any cumulant α of $\mathscr{F}$ of order $p-1$ ($\log F$ being regarded as the cumulant of order 0, for the moment) is expressible by a line integral of the kind mentioned in Lemma 2.3 in terms of $\partial_j \alpha (1 \leqq j \leqq k)$, it follows by Lemma 2.3 (ii) that α is of finite order and (a) now follows by induction. □

REMARK 2.5. Following the ideas of Ronkin and Fuks (cf. Section 26.2, Ch. V of [2]) one might define the more precise concept of the hypersurface of systems of conjugate orders in the case of a function of finite order in $\mathscr{C}^{k+}$ and observe through our discussion of Theorem 2.1 that if $\log F$ is of finite order, then itself and the function $F_1 = \max [\chi_n : n \in I, n_1 + \cdots + n_k = 1]$ have the same hypersurface of systems of conjugate orders. It may however be realised that our growth indicators are based on the asymptotic considerations "as $r \to +\infty$" and that the theorems of Section 3 would be false, if "$r \to +\infty$" in their statements is replaced by "$\sum r_j \to +\infty$", even if "$\cdots$ of finite order" is interpreted in the sense of Fuks or in that of Gol'dberg [2] (consider the example suggested in Remark 5.6 of [3] and ignore the rest of the remark).

THEOREM 2.6. *Let $\log F$ be of finite order. Then the reciprocal of the modal mass of $\mathscr{F}$ viz, F/μ is of finite order.*

PROOF OF THEOREM 2.6. The theorem follows from Theorem 2.1, Lemma 2.2 and the following lemma (which we mention separately mainly because of its elegance):

LEMMA 2.7. *There exists a positive real number $A = A(k)$ such that*

$$F(r) \leqq \mu(r) A \prod_{j=1}^{k} [1 + \partial_j^2 \log F(r)]^{\frac{1}{2}}, \quad \text{for} \quad r \in \mathscr{C}^{k+}.$$

PROOF OF LEMMA 2.7. Using a multi-dimensional version of Chebyschev's Lemma, Schumitzky [6] proved the existence of a positive real $A = A(k)$ such that for any $r \in \mathscr{C}^{k+}$, $F(r) \leqq \mu(r)A[\det(\Lambda_r + U)]^{\frac{1}{2}}$, where Λ_r is the moment matrix of $\mathscr{F}_r$ and U is the k/k unit matrix. The lemma now follows from the fact that Λ_r is a non-negative definite matrix.

3. A Valiron-type theorem. We define the power series g in $\mathscr{C}^k$ by $g(z) = \sum_{n \in I} |a_n| z^n$ and as in the case of f we say that g is of finite order (in $\mathscr{C}^k$) if and only if its maximum modulus F is such that $\log F$ is of finite order. We first prove

THEOREM 3.1. *Let g be of finite order. Then*

$$\log \mu(r) \sim \log F(r), \quad \text{as} \quad r \to +\infty.$$

PROOF OF THEOREM 3.1. The theorem readily follows from Theorem 2.6 in case F is "purely trancendental" i.e. if there exists no $m \in I$ with the property that $a_n = 0$ for all $n \geqq m$. The theorem follows in particular when $k = 1$. The rest of the proof may be carried out using induction on k, the number of variables (see for details the proof of Theorem 5.2 of [3]).

We finally deduce

THEOREM 3.2. *Let f be of finite order. Then*

$$\log \mu(r) \sim \log \mathscr{M}(r), \quad \text{as} \quad r \to +\infty.$$

PROOF OF THEOREM 3.2. It easily follows from Cauchy's inequality that $\mu(r) \leqq \mathscr{M}(r)$, while it is obvious that $\mathscr{M}(r) \leqq F(r)$, for $r \in \mathscr{C}^{k+}$. Thus the theorem follows from Theorem 3.1 because of the fact that the finite orderedness of f is equivalent to a statement involving only the absolute values of its coefficients a_n, $n \in I$ (cf. Theorems 26.1, 26.2, Chapter V of [2]), which implies that g is of finite order.

REFERENCES

[1] DOOB, J. L. (1965). *Stochastic Processes.* Wiley, New York.

[2] FUKS, B. A. (1965). Introduction to the theory of functions of several complex variables. *Amer. Math. Soc. Transl.*

[3] KRISHNA, GOPALA J. (1969). Maximum term of a power series in one and several complex variables. *Pacific J. Math.* **29** 609–622.

[4] RAO, RADHAKRISHNA C. (1967). *Linear Statistical Inference and its Applications.* Wiley, New York.

[5] ROSENBLOOM, P. C. (1962). Probability and entire functions. *Studies in Mathematical Analysis and Related Parts.* Stanford Univ. Press.

[6] SCHUMITZKY, A. (1966). Probabilistic approach to Wiman–Valiron theory. Notes, Summer Institute on entire functions, LaJolla. Pp. III R 1–9. (To appear in *J. Math. Anal. Appl.*)