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A MARKOV STOPPING PROBLEM FOR WHICH NO ENTRY TIME IS ε-OPTIMAL[1]

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1. Introduction and summary. Let $(X(t): t \geqq 0)$ be a Markov process with lifetime ζ, let g be a bounded continuous nonnegative function on the state space of the process, and let P_x (respectively, E_x) be the probability measure (respectively, expectation operator) associated with paths starting at x. For any (extended real-valued) Markov time T let $f(x, T) = \int_{T<\zeta} g(X(T))\,dP_x$ and let $f(x) = \sup_T f(x, T)$. For any nonnegative ε, a Markov time T^* is called (i) *ε-optimal at x* if $f(x, T^*) \geqq f(x)-\varepsilon$; (ii) *optimal at x* if $f(x, T^*) \geqq f(x)$; (iii) *ε-optimal* if $f(x, T^*) \geqq f(x)-\varepsilon$ for all x; and (iv) *optimal* if $f(x, T^*) \geqq f(x)$ for all x.

We interpret $g(X(T))$ as a reward associated with stopping at time T in state $X(T)$ and we are searching for stopping rules or Markov times T^* which maximize or nearly maximize the expected value of this reward. Since the process is Markov and the reward depends only on the state in which one stops and not on the time nor the previous process history, one would anticipate that an optimal stopping time (provided one exists) would be of the form "Continue as long as $f(X(t)) > g(X(t))$ and stop when first $f(X(t)) \leqq g(X(t))$." That is, at time t in state $X(t) = x$, one continues if the optimal reward from continuing $f(x)$ exceeds the reward from stopping $g(x)$; otherwise, one stops. This reasoning leads one to hope that the search for optimal rules can be restricted without loss to rules specified by a partition of the state space into two sets, one of continuation states and one of stopping states. Thus we ask under what conditions

$$f(x) = \sup f(x, T(\Gamma)) \tag{1}$$

where the supremum is over all appropriately measurable sets Γ and $T(\Gamma)$ is the process entry time to Γ:

$$\begin{aligned} T(\Gamma) &= \infty \qquad &&\text{if} \quad X(t) \notin \Gamma \quad \text{for all} \quad t \geqq 0; \\ &= \inf\{t: t \geqq 0 \text{ and } X(t) \in \Gamma\}, &&\text{otherwise.} \end{aligned} \tag{2}$$

Dynkin (1963) shows that if X is a standard process then for $\varepsilon \geqq 0$, $\Gamma_\varepsilon = \{x: f(x)-\varepsilon \leqq g(x)\}$ is closed in the fine or intrinsic topology and for $\varepsilon > 0$, $T(\Gamma_\varepsilon)$ is ε-optimal, so that in particular (1) holds. When g is unbounded, Dynkin (1968) interprets a result of Chow and Robbins (1967) to state that for $\varepsilon \geqq 0$, if $P_x[T(\Gamma_\varepsilon) < \infty] = 1$ for all x, then $T(\Gamma_\varepsilon)$ is ε-optimal. Taylor (1968) shows that if in addition the transition semigroup of the process is Feller (leaves invariant the space

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of bounded continuous functions) then Γ_0 is closed, and if there exists an optimal Markov time, then $T(\Gamma_0)$ is optimal at all points x for which $f(x) < \infty$.

In this paper we present a Markov process on continuous path space and a bounded continuous nonnegative g such that for some positive ε and some point x, no entry time is ε-optimal at x. It follows that no entry time can satisfy any of the other, more stringent, criteria for optimality. Our process is Markov but not strong Markov (hence, not Feller) and was suggested to me by a related example I learned from David Freedman (1969). Our construction is based on a transformation by a singular continuous strictly increasing function. A local time transformation serves the same purpose in Freedman's example.

The idea behind the example is quite simple, but the details are many and tedious and tend to obscure the basic picture, which is this: Let $y(t)$ be a Brownian motion process starting at the origin. Our process $x(t)$ is a distorted version of $y(t)$. In particular, begin with $x(t) = y(t)$ and continue until first $y(t)$ hits $+1$. Beginning then, distort the process by setting $x(t) = G^{-1}[y(t)]$, where G is strictly increasing, continuous and $G(1) = -G(-1) = 1$. Continue until $y(t)$ next reaches -1, at which time, revert to $x(t) = y(t)$. Repeat, again waiting until $y(t)$ hits $+1$, then switching to $x(t) = G^{-1}[y(t)]$, and so on.

For the $x(t)$ process it is clear that a good decision on whether or not to stop should be based in part on whether currently $x(t) = y(t)$ or $x(t) = G^{-1}[y(t)]$, or equivalently, whether the process is on a -1 to $+1$ section, or on a $+1$ to -1 section. In particular, entry times cannot be good because they do not include this information. Granted, to nail down an example, a sufficiently simple stopping problem must be chosen so that a variety of calculations can be made, but it is intuitively clear why entry times cannot be good and how to improve them. This part of our example is presented in Section 3.

It is not clear, however, that we can distort the Brownian motion as we have described and yet maintain the Markov property. Here is where we require G to be singular, so that G^{-1} carries a set of full Lebesgue measure into a set, call it Λ, which is Lebesgue null. At a fixed time t, $y(t)$ is not in Λ with probability one, since Λ is Lebesgue null. Thus, if we observe $x(t)$ in Λ we may infer, with probability one, that $x(t) = G^{-1}[y(t)]$. Similarly, still for fixed t, if we observe $x(t)$ not in Λ, we infer, with probability one, that $x(t) = y(t)$. This feature preserves the Markov property. A careful development of this idea is given in Section 2.

2. The process. Let G be a singular continuous strictly increasing function of the real line onto itself for which $G(-1) = -1$, $G(0) = 0$ and $G(1) = 1$. Let μ be the measure on the Borel real line that is induced by G and let m be Lebesgue measure. Then for any linear Borel set E, $\mu(E) = m[G(E)]$ where $G(E)$ is the forward image of E under G. Since m and μ are mutually singular, there exists a linear Borel set Λ^*, with complement denoted by Λ^{*c}, for which

$$m(\Lambda^*) = 0 \quad \text{and} \tag{3}$$

$$\mu(\Lambda^{*c}) = m[G(\Lambda^{*c})] = 0. \tag{4}$$

Define

$$\Lambda = [1, \infty) \cup \{(-1, 1) \cap \Lambda^*\} \tag{5}$$

and note that

$$G(\Lambda) = [1, \infty) \cup \{(-1, 1) \cap G(\Lambda^*)\}. \tag{6}$$

Let Ω be the set of real-valued continuous functions on $[0, \infty)$ and for $\omega \in \Omega$ and $t \in [0, \infty)$ let $X(t, \omega) = \omega(t)$. For $t \geqq 0$ let $\mathscr{B}(t)$ be the σ-algebra in Ω generated by $\{X(s): 0 \leqq s \leqq t\}$ and let $\mathscr{B}$ be the σ-algebra generated by $\{X(s): 0 \leqq s < \infty\}$.

For $i = 1, 2$ and $n = 0, 1, \cdots$ define the extended real-valued random variables $S_i(n)$ and $T_i(n)$ by

$$\begin{aligned} &S_1(0) = S_2(0) = 0, \\ &T_1(0) = \inf\{t: t \geqq 0 \text{ and } X(t) \geqq 1\}, \\ &T_2(0) = 0, \quad \text{and for} \quad i = 1, 2 \quad \text{and} \quad n = 1, 2, \cdots, \\ &S_i(n) = \inf\{t: t \geqq T_i(n-1) \text{ and } X(t) \leqq -1\}, \quad \text{and} \\ &T_i(n) = \inf\{t: t \geqq S_i(n) \text{ and } X(t) \geqq +1\}. \end{aligned} \tag{7}$$

As usual in these definitions, the infimum of an empty set is $+\infty$. Then for $i = 1, 2$, $n = 0, 1, \cdots$ and $t \geqq 0$,

$$\{T_i(n) \leqq t\} \in \mathscr{B}(t) \quad \text{and} \quad \{S_i(n) \leqq t\} \in \mathscr{B}(t). \tag{8}$$

Let B_x be Brownian motion with drift $-\frac{1}{2}$ and starting at x. By this we mean B_x is a probability on $\mathscr{B}$, $B_x[X(0) = x] = 1$, and $\{X(t): t \geqq 0\}$ is a Gaussian process with stationary independent increments, $\int X(t)\, dB_x = x - t/2$ and $\operatorname{Var}(X(t)) = t$.

Let $\Omega_0 = \{X(t) \to -\infty \text{ as } t \to \infty\}$. Then $\Omega_0 \in \mathscr{B}$ and $B_x[\Omega_0] = 1$ for all x. Let $\mathscr{B}_0(t) = \mathscr{B}(t) \cap \Omega_0 = \{A \cap \Omega_0 : A \in \mathscr{B}(t)\}$ and let $\mathscr{B}_0 = \mathscr{B} \cap \Omega_0$. On Ω_0, for $i = 1, 2$ define

$$I_i = \min\{n: T_i(n) = \infty\}. \tag{9}$$

For $i = 1, 2$ define the mapping π_i of Ω_0 into Ω_0 by

$$\begin{aligned} \pi_i \omega(t) &= G^{-1}[\omega(t)] \quad \text{for} \quad T_i(n-1) \leqq t < S_i(n), n = 1, \cdots, I_i, \\ &= \omega(t) \qquad\quad\;\; \text{for} \quad S_i(n) \leqq t < T_i(n), n = 0, \cdots, I_i. \end{aligned} \tag{10}$$

Briefly, if the last hit was at $+1$, then π_i distorts the path; if the last hit was at -1, then π_i leaves the path unchanged.

Using (8) one can verify that for $i = 1, 2$ and $t \geqq 0$

$$A \in \mathscr{B}_0(t) \quad \text{implies} \quad \pi_i^{-1}(A) \in \mathscr{B}_0(t). \tag{11}$$

For any real y let P_y be the measure which specifies for $B \in \mathscr{B}_0$ the value

$$\begin{aligned} P_y[B] &= B_{G(y)} \pi_2^{-1}[B] \qquad \text{if} \quad y \in \Lambda; \\ &= B_y \pi_1^{-1}[B] \qquad\quad\; \text{if} \quad y \notin \Lambda. \end{aligned} \tag{12}$$

It is routine to verify that $P_y[X(0) = y] = 1$ and that $P_x[B]$ is Borel measurable in x for $B \in \mathscr{B}_0$.

Let $t > 0$ be fixed and define

$$E = \Omega_0 \cap \{X(t) \in \Lambda\}, \qquad \text{and for} \quad i = 1, 2 \tag{13}$$

$$E_i = \pi_i^{-1} E = \{\omega \in \Omega_0 : \pi_i \omega(t) \in \Lambda\}, \qquad \text{and} \tag{14}$$

$$F_i = \Omega_0 \cap \{T_i(n-1) \leqq t < S_i(n) \text{ for some } n = 1, \cdots, I_i\}. \tag{15}$$

Roughly speaking, E_i is the set of paths for which the transformed point $x(t) = \pi_i[\omega(t)]$ belongs to Λ, and F_i is the set of paths whose last hit was at $+1$.

Let

$$\Omega_1 = \Omega_0 \cap \{\omega\text{: For } i = 1, 2, \omega \in E_i \text{ if and only if } \omega \in F_i\}. \tag{16}$$

Finally, let S_t be the shift so that $S_t \omega = \omega(t + \cdot)$.

PROPOSITION. (i) *E, E_i, F_i and Ω_i are in $\mathscr{B}_0$ and*

$$B_x[\Omega_1] = 1 \tag{17}$$

for all x; that is, with probability one, the transformed point $x(t) = \pi_i[\omega(t)]$ belongs to Λ if and only if the last hit was at $+1$.

(ii) *For $i = 1, 2$ and $B \subset \Omega_1$,*

$$\begin{aligned} F_i \cap \pi_i^{-1} S_t^{-1}(B) &= F_i \cap S_t^{-1} \pi_2^{-1}(B), \\ F_i^c \cap \pi_i^{-1} S_t^{-1}(B) &= F_i^c \cap S_t^{-1} \pi_1^{-1}(B). \end{aligned} \tag{18}$$

PROOF. Since t is fixed, the measurability easily follows from (8) and (11). We first show the symmetric difference $E_i \triangle F_i$ is B_x null for all x. For $\omega \in F_i$, $\omega(t) \in [-1, \infty)$ by (7) and $\pi_i \omega(t) = G^{-1}[\omega(t)] \in [-1, \infty)$ by (10) and (15). Thus using (14) and the complement of (6),

$$\begin{aligned} F_i \backslash E_i &\subset \{\omega \in \Omega_0 : \omega(t) \in [-1, \infty) \backslash G(\Lambda)\} \\ &\subset \{\omega \in \Omega_0 : \omega(t) \in G(\Lambda^{*c})\} \end{aligned}$$

which is B_x null since $G(\Lambda^{*c})$ is Lebesgue null by (4). Similarly, for $\omega \in \Omega_0 \backslash F_i$, $\omega(t) \in (-\infty, 1]$ by (7) and $\pi_i \omega(t) = \omega(t) \in (-\infty, 1]$ by (10) and (15). Again, using (14) and (6),

$$\begin{aligned} E_i \backslash F_i &\subset \{\omega \in \Omega_0 : \omega(t) \in (-\infty, 1] \cap \Lambda\} \\ &\subset \{\omega \in \Omega_0 : \omega(t) \in \Lambda^*\} \end{aligned}$$

which is B_x null since Λ^* is Lebesgue null by (3). Thus, $B_x[E_i \triangle F_i] = 0$ for all x and we conclude $B_x[\Omega_1] = 1$.

We only show the first part of (18), the second being similar. Take $\omega \in F_i$ and $\tau > 0$. It is enough to show

$$S_t[\pi_i(\omega)](\tau) = \pi_2[S_t(\omega)](\tau) \tag{19}$$

and we may suppose

$$\omega(t+\tau) \neq G^{-1}[\omega(t+\tau)] \tag{20}$$

since otherwise, (19) is trivially satisfied. By considering all possible cases, one may verify that $\omega \in F_i$ implies

$$T_i(n-1,\omega) \leqq t+\tau < S_i(n,\omega) \quad \text{for some } n \text{ if and only if} \tag{21}$$

$$T_2(m-1, S_t\omega) \leqq \tau < S_2(m, S_t\omega) \qquad \text{for some } m. \tag{22}$$

From (10) and (20) we have that (21) holds if and only if

$$\pi_i\,\omega(t+\tau) = G^{-1}[\omega(t+\tau)] \tag{23}$$

and (22) holds if and only if

$$\pi_2[S_t(\omega)](\tau) = G^{-1}[\omega(t+\tau)]. \tag{24}$$

Then (23) and (24) yield (19) for $\omega \in F_i$ as desired. □

Roughly speaking, P_y is Markov because, given $x(t)$, we can decide, with probability one, whether the last hit was at $+1$ or -1, by noting whether $x(t) \in \Lambda$ or $x(t) \in \Lambda^c$.

THEOREM. *$\{P_y\}$ is Markov.*

PROOF. Let $t > 0$, $A \in \mathscr{B}_0(t)$ and $B \in \mathscr{B}_0$ be given. We wish to show

$$P_y[A \cap S_t^{-1}B] = \int_A P_{X(t,\omega)}[B]P_y[d\omega].$$

Suppose $y \in \Lambda$, and let $A' = \pi_2^{-1}A$. Then

$$\begin{aligned}
P_y[A \cap S_t^{-1}B] &= B_{G(y)}[A' \cap \pi_2^{-1}S_t^{-1}B] && \text{by (12)}\\
&= B_{G(y)}[A' \cap E_2 \cap F_2 \cap \pi_2^{-1}S_t^{-1}B]\\
&\quad + B_{G(y)}[A' \cap E_2^c \cap F_2^c \cap \pi_2^{-1}S_t^{-1}B] && \text{by (17)}\\
&= B_{G(y)}[A' \cap E_2 \cap F_2 \cap S_t^{-1}\pi_1^{-1}B]\\
&\quad + B_{G(y)}[A' \cap E_2^c \cap F_2^c \cap S_t^{-1}\pi_1^{-1}B] && \text{by (18)}\\
&= \int_{A' \cap E_2 \cap F_2} B_{X(t,\omega')}[\pi_2^{-1}B]B_{G(y)}[d\omega']\\
&\quad + \int_{A' \cap E_2^c \cap F_2^c} B_{X(t,\omega')}[\pi_2^{-1}B]B_{G(y)}[d\omega']
\end{aligned}$$

by using the Markov property for $\{B_x\}$, (11), (17) and that E_i and F_i are in $\mathscr{B}_0(t)$. Changing variables with $\omega' = \pi_2\omega$ and using (10), (12), (14) and (15) we continue with

$$\begin{aligned}
&= \int_{A \cap \{X(t) \in \Lambda\}} P_{X(t,\omega)}[B]P_y[d\omega] + \int_{A \cap \{X(t) \notin \Lambda\}} P_{X(t,\omega)}[B]P_y[d\omega]\\
&= \int_A P_{X(t,\omega)}[B]P_y[d\omega],
\end{aligned}$$

as desired. The case $y \notin \Lambda$ is entirely similar and the proof is omitted. □

3. The stopping problem. Let $g(x) = \min\{1, x^+\}$ for $x \in (-\infty, +\infty)$. For any linear Borel set Γ let $T(\Gamma)$ be the entry time to Γ as defined in (2). Let E_x be the expectation operator associated with P_x as given in (12). For any Markov time T let $f(x, T) = \int_{T<\infty} g[X(T)]\, dP_x$. In this section we specify an x and a positive ε such that for any linear Borel set Γ there exists a Markov time $T_\Gamma{}^*$ for which

$$f(x, T_\Gamma{}^*) \geqq f(x, T(\Gamma)) + \varepsilon. \tag{25}$$

In order to perform the calculations we found it necessary to specify a particular G. Let $Z, Z_1, Z_2, \cdots$ be independent identically distributed random variables on a common probability space with $\text{Prob}\{Z = 0\} = 1 - \text{Prob}\{Z = 1\} = p$, where $0 < p < 1$. Let $V = \sum_1^\infty 2^{-n} Z_n$; that is, $.Z_1 Z_2 Z_3 \cdots$ is a binary expansion for V. For $x \in [0, 1]$ let $H(x) = \text{Prob}\{V \leqq x\}$. Then for $p \neq \frac{1}{2}$, H is singular continuous and strictly increasing on $[0, 1]$. Page 85 of (Dubins and Savage, 1965) contains further remarks and references. Let

$$\begin{aligned} G(x) &= H(x) && \text{for } x \in [0, 1] \\ &= -H(-x) && \text{for } x \in [-1, 0] \end{aligned} \tag{26}$$

and extend to $(-\infty, +\infty)$ by

$$\begin{aligned} G(j + x) &= j + G(x) && \text{for } x \in [0, 1], j = 1, 2, \cdots, \\ &= -j + G(x) && \text{for } x \in [-1, 0], j = -1, -2, \cdots. \end{aligned}$$

Then G satisfies the requirements of Section 2 and in addition

$$G(\tfrac{1}{2}) = -G(-\tfrac{1}{2}) = p. \tag{27}$$

Since G is strictly increasing, Λ^* is dense in $(-\infty, +\infty)$ and $[-\frac{1}{2}, 0) \cap \Lambda$ is not empty. Throughout this section fix $x^* \in [-\frac{1}{2}, 0) \cap \Lambda$ and set $y^* = G(x^*)$.

For $b \leqq a$ let T_{ab} be the entry time to $(-\infty, b) \cup [a, \infty)$ as defined in (2) and let T_a be the entry time to $[a, \infty)$. For $a \in [0, 1]$ and $y \in [-1, 0]$ let

$$v(y, a, p) = a\left\{\frac{e^y - e^{-1}}{e^{G(a)} - e^{-1}} + \left[\frac{e^{G(a)} - e^y}{e^{G(a)} - e^{-1}}\right] e^{-(a+1)}\right\}. \tag{28}$$

PROPOSITION. *For* $a \in [0, 1]$,

$$f(x^*, T_a) = v(y^*, a, p). \tag{29}$$

PROOF.

$$\begin{aligned} f(x^*, T_a) &= \textstyle\int_{T_a<\infty} g[X(T_a)]\, dP_{x^*} \\ &= aP_{x^*}[T_a < \infty] \\ &= a\{P_{x^*}[T_a < T_{-1}] + P_{x^*}[T_{-1} < T_a < \infty]\} \\ &= a\{B_{y^*}[T_{G(a)} < T_{-1}] + B_{y^*}[T_{-1} < T_{G(a)} \text{ and } T_a \circ S_{T_{-1}} < \infty]\} \\ &= v(y^*, a, p), \end{aligned}$$

where we have used (10) and (12), the continuity of the paths, that for $b \leqq x \leqq a$, $B_x[X(T_{ab}) = a] = (e^x - e^b)/(e^a - e^b)$, and the strong Markov property for Brownian motion. □

Let ε_i for $i = 1, 2$ be strictly positive with $0 < \varepsilon_1 + \varepsilon_2 < \frac{1}{2} - 1/e$. For $x^* \in [-\frac{1}{2}, 0]$, we have $y^* \geqq -p = G(-\frac{1}{2})$, and

$$\begin{aligned} v(y^*, \tfrac{1}{2}, p) &= \tfrac{1}{2}\left\{\frac{e^{y^*} - e^{-1}}{e^p - e^{-1}} + \left[\frac{e^p - e^{y^*}}{e^p - e^{-1}}\right] e^{-(\frac{1}{2})}\right\} \\ &\geqq \tfrac{1}{2}\left\{\frac{e^{-p} - e^{-1}}{e^p - e^{-1}} + \left[\frac{e^p - e^{-p}}{e^p - e^{-1}}\right] e^{-\frac{1}{2}}\right\} \end{aligned}$$

and

$$\liminf_{p \downarrow 0} v(y^*, \tfrac{1}{2}, p) \geqq \tfrac{1}{2}.$$

Fix $p^* > 0$ such that $v(y^*, \frac{1}{2}, p^*) \geqq \frac{1}{2} - \varepsilon_1$.

Next, note that

$$\begin{aligned} v(y^*, a, p^*) &\leqq v(0, a, p^*) \\ &= a\left\{\frac{1 - e^{-1}}{e^{G(a)} - e^{-1}} + \left[\frac{e^{G(a)} - e^{y}}{e^{G(a)} - e^{-1}}\right] e^{-(a+1)}\right\}, \end{aligned}$$

and

$$\limsup_{a \uparrow 1} v(y^*, a, p^*) \leqq e^{-1}.$$

Fix $\delta \in (0, \frac{1}{2})$ such that $a \in [1-\delta, 1]$ implies $v(y^*, a, p^*) \leqq e^{-1} + \varepsilon_2$. Then

$$\begin{aligned} v(y^*, \tfrac{1}{2}, p^*) &\geqq \tfrac{1}{2} - \varepsilon_1 \\ &\geqq 1/e + \varepsilon_2 \qquad (30) \\ &\geqq v(y^*, a, p^*) \qquad \text{for} \quad 1 - \delta \leqq a \leqq 1. \end{aligned}$$

Thus $a = \frac{1}{2}$ dominates any $a \in [1-\delta, 1]$. Consider $a \in [0, 1-\delta]$. Let T_a^* be the Markov time

$$\begin{aligned} T_a^* &= T_a \quad \text{on} \quad T_a < T_{-1} \\ &= T_1 \quad \text{on} \quad T_{-1} \leqq T_a. \end{aligned}$$

Note that T_a^* is not an entry time.

PROPOSITION. *For* $a \in [0, 1-\delta]$,

$$f(x^*, T_a^*) = a\left\{\frac{e^{y^*} - e^{-1}}{e^{G(a)} - e^{-1}}\right\} + \left\{\frac{e^{G(a)} - e^{y^*}}{e^{G(a)} - e^{-1}}\right\} e^{-2}. \tag{31}$$

PROOF. Similar to (29).

Thus

$$f(x^*, T_a^*) - v(y^*, a, p^*) = \frac{e^{G(a)} - e^{y^*}}{e^{G(a)} - e^{-1}} \{e^{-2} - ae^{-(a+1)}\}.$$

Let

$$\varepsilon = \inf_{a \in [0, 1-\delta]} \{f(x^*, T_a^*) - v(y^*, a, p^*)\} > 0.$$

Then $f(x^*, T_a^*) \geqq f(x^*, T_a) + \varepsilon$ for $a \in [0, 1-\delta]$ and

$$\begin{aligned} f(x^*, T_{\frac{1}{2}}^*) &\geqq f(x^*, T_{\frac{1}{2}}) + \varepsilon \\ &\geqq f(x^*, T_a) + \varepsilon \quad \text{for} \quad a \in [1-\delta, 1]. \end{aligned}$$

Let Γ be a linear Borel set and $T(\Gamma)$ the entry time. Then $T(\Gamma)$ is a Markov time relative to the appropriate completion of the σ-algebras $\mathscr{B}_0(t)$. To complete the example we need only show that a Markov time of the form T_a for $a \in [0, 1]$ dominates $T(\Gamma)$.

PROPOSITION. *Let* Γ *be a linear Borel set,* $\alpha = \inf\{y \in \Gamma : y \geqq x^*\}$ $(= \infty$ *if* $\Gamma \cap [x^*, \infty) = \varnothing)$, $\beta = \sup\{y \in \Gamma : y \leqq x^*\}$ $(= -\infty$ *if* $\Gamma \cap (-\infty, x^*] = \varnothing)$. *Then* $P_{x^*}[T(\Gamma) = T_{\alpha\beta}] = 1$. *Let* $a = \min\{1, \alpha^+\}$. *Then* $f(x^*, T_a) \geqq f(x^*, T(\Gamma))$.

PROOF. The first assertion follows from the continuity of paths and the corresponding fact for Brownian motion. Then

$$\begin{aligned} f(x^*, T(\Gamma)) &= f(x^*, T_{\alpha\beta}) \\ &= g(\alpha)\textstyle\int_{T_\alpha < T_\beta} dP_{x^*} + g(\beta)\int_{T_\beta \leqq T_\alpha} dP_{x^*} \\ &= g(\alpha)P_{x^*}[T_\alpha < T_\beta] \\ &\leqq g(\alpha)P_{x^*}[T_\alpha < \infty] \\ &\leqq g(a)P_{x^*}[T_a < \infty] \\ &= f(x^*, T_a). \quad \square \end{aligned}$$

This completes the example. For a fixed x^*, I do not know if a Markov time of the form T_a^* is optimal at x^*, nor do I know if an optimal Markov time even exists. I am not even sure how to attack the problem in the absence of the strong Markov property.

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