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ON THE PROBABILITY THAT A SAMPLE DISTRIBUTION FUNCTION LIES BELOW A LINE SEGMENT[1,2]

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0. Summary and introduction. The probability is determined that the sample distribution function (df) of a random sample of any size n, drawn from the uniform distribution, lies below a given line segment of any slope over some $(\alpha, \beta) \subset [0, 1]$ (Section 1, Theorem 1.1 ff.). Probabilities of related events, also under conditioning, are derived. It is well known that results of this type are equivalent to similar ones for random samples from any continuous df. A catalogue of equivalent formulae is given, the various versions being advantageous on certain ranges of the parameters. These results rest upon and generalize a formula (Theorem 2.1 below) of Dempster (1959), Dwass (1959), and Pyke (1959) (his Lemma 1), which gives an explicit expression for any n that the sample df lies below some (straight) line extended over the entire unit interval. Otherwise the proof uses familiar properties of order statistics, the whole argument being essentially a combinatorial one. The present result also generalizes, in particular, results by Wald and Wolfowitz (1939), by Birnbaum and Tingey (1951) (in both papers sample df below a line segment of slope 1 over $[0, 1]$; especially the latter paper, which improves the first, is at the root of the approach of the present article as well as of the papers by Dempster, Dwass, Pyke), by Smirnov (1944 and 1961) (sample df below a line segment with slope 1 over $(0, 1)$ or $(\alpha, 1)$), by Chang (1955) (line segment of any nonnegative slope joining the origin with some point in the open unit square $I_2{}^0$), Csörgö (1965) (line segment of any slope may also end in the point $(1, 1)$), Birnbaum and Lientz (1969) (line segment of any nonnegative slope through the origin over any subinterval of the unit interval). Apparently the first author who determined explicitly the probability that the sample df lies below a line segment over an arbitrary subinterval $(\alpha, \beta) \subset [0, 1]$ was Takacs (1964) (Theorem 3; see also Takacs (1967), pages 176–178). (The author is indebted to Professor J. Kiefer for reminding him of this reference.) However, he had the (not very crucial) restriction that the slope γ of the line be $\geqq 1$. Moreover, his formula contains a double sum, whereas some of ours contain a single one. This fact proves to be of great advantage in the applications we have made so far. Theorem 1.2 below gives certain conditional probabilities that the sample df lies below a line segment over some subinterval of $(0, 1)$. These probability expressions (as any of those described above), for a suitable sequence of line segments depending on n tend to the probability that the Brownian bridge (or conditioned Wiener) process lies below a line segment, as $n \to \infty$ (see [18], in

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particular pages 182–183). This follows from the Doob–Donsker theorem on weak convergence of probability measures (see also [13]). These asymptotic results provide a starting point different from and possibly simpler than that of Section 1 to compute approximate probabilities that the sample df lies below some curved line (for an application and some of the asymptotic formulae compare [7] and [17]). It is the objective of a later paper to determine the error of the approximation.

Theorem 1.3 gives similar conditional probability expressions that the sample probability (a generalization of the sample df, defined below) for intervals of variable lengths lies below a line segment. It is these formulae that are needed in a new version of the proof of the Bahadur–Kiefer representation theorem for sample quantiles [1].

The formulae derived here are explicit (though involved) in contrast to recursive ones first considered in [12] and extended, more recently, e.g., in [9] and [25]. They have been used, e.g., to compute explicitly the probability that the sample df lies below a polygon (compare a forthcoming paper of the author).

Another application is the derivation of asymptotic formulae paralleling and generalizing, e.g., those of Rényi and Csörgö ([4] and [10]). Implications for stochastic processes (such as those studied in [19] and for the theory of goodness of fit tests (compare [1]) are not considered. In [1] also the statistic $F_n(x)-F(x)$, divided by its standard deviation, has been proposed. Certain functionals of this statistic may be studied by the method of the present paper. On the other hand, the method is not immediately applicable to the study of probabilities that the sample df lies, e.g., between two lines (for a recent paper on this compare [16]).

The vast existing literature on Kolmogorov–Smirnov type statistics (for recent surveys compare [20] and the appendix of [12a]) is not being surveyed for possible applications other than the few above-mentioned representative examples.

1. Conditional and unconditional probability of a sample df lying below a line segment.

Notation and abbreviations. R = real line, $R^+ = (0, \infty)$, $[x]$ = largest integer $\leqq x (\in R)$, rv = random variable, i.i.d. = independent identically distributed, df = distribution function; r.[l.]h.s. = right-(left)-hand side, w.l.o.g. = without loss of generality, $I(B)$ = indicator function of the set $B \subset \Omega$. Equations between events in the following sometimes hold only up to sets of probability zero. The binomial df is denoted by $B(\cdot\,; n, p)$. The notation

$$b(k; n, p) := \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \cdots, n,$$

is used also when p is *any real number*. The uniform df over α, β, $-\infty < \alpha < \beta < \infty$ is denoted by $U(\alpha, \beta)$. Throughout the paper, $x^0 = 1$ for all real x. For reals x, y, $x \wedge y := \min(x, y)$. A sum $\sum_k^m$ is to be replaced by zero if $k > m$.

THEOREM 1.1. *Let $\alpha, \beta, \gamma, \delta$ be constants satisfying* $0 \leqq \alpha < \beta \leqq 1$, $0 < \gamma < \infty$,

$$0 \leqq \delta + \gamma\alpha < 1. \tag{1.1}$$

Let the rv's $X_1, \cdots, X_n$ *be* i.i.d. $\sim U(0,1)$, *and let*

$$F_n(x) = n^{-1}\sum_{j=1}^{n} I(X_j < x) \tag{1.2}$$

be the sample df *of* $X_1, \cdots, X_n$, $n = 1, 2, \cdots$. *Denoting*

$$A = (F_n(x) \leqq \delta + \gamma x \quad \textit{for all} \quad x \in (\alpha, \beta)) = \left(\sup_{\alpha < x < \beta} \frac{F_n(x) - x - \delta}{x} \leqq \gamma - 1\right) \tag{1.3}$$

then

$$PA = B(\kappa; n, \beta) + \sum_{h=\kappa+1}^{\lambda^*} \binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda' - h) \cdot \sum_{s=0}^{\kappa} \binom{h}{s}(s - n\delta)^s(\lambda' - s)^{h-s-1} \tag{1.4}$$

where

$$\kappa = [n(\delta + \gamma\alpha)], \qquad \lambda' = n(\delta + \gamma\beta), \tag{1.5)/(1.6}$$

$$\begin{aligned} \lambda^* &= n && \textit{if} \quad \lambda' > n, \\ &= \lambda = [\lambda'] && \textit{if} \quad \lambda' < n \quad \textit{and} \quad \lambda' \quad \textit{not an integer}, \\ &= \lambda' - 1 && \textit{if} \quad \lambda' \leqq n \quad \textit{and} \quad \lambda' \quad \textit{an integer}. \end{aligned} \tag{1.7}$$

Since $\lambda' > \lambda^* \geqq h$, in (1.4) always

$$\lambda' - h > 0 \quad \text{and} \quad \lambda' - s > 0. \tag{1.8}$$

Of course (1.4) remains unchanged if λ^* is replaced by $\lambda \wedge n$. We use λ^* because of its convenience, in view of later transformations. The proof of Theorem 1.1 is given in Section 2.

A disadvantage of formula (1.4) is that in the sum over s the sign of $s - n\delta$ in general oscillates, which makes approximate evaluations difficult. Corollary 1.1 gives equivalent expressions for (1.4) which do not show this defect.

One notes that for $\beta = 1$, $\delta + \gamma \leqq 1$, the r.h.s. of (1.4) gives the correct probability $PA = 0$. If $\kappa \geqq n$ and $\beta < 1$, the r.h.s. of (1.4) gives still the correct probability $PA = 1$.

REMARK 1. The result of the above theorem, (1.4), may be written in various other forms. Special attention is drawn to (1.13 b–d) which are particularly simple. They only have single summations. To begin with, starting from (1.4),

$$PA = \sum_{h=0}^{\lambda^*} \binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda' - h)\sum_{s=0}^{\kappa} \binom{h}{s}(s - n\delta)^s(\lambda' - s)^{h-s-1}, \tag{1.9}$$

where the second sum for $h \leqq \kappa$ reduces to $\sum_{s=0}^{h}$; to obtain (1.9) the formula

$$\sum_{s=0}^{h} \binom{h}{s}(a+s)^s(b-s)^{h-s-1} = (b-h)^{-1}(a+b)^h \tag{1.10}$$

is used ($a, b \in R$; $b \neq h$) (see e.g. Birnbaum and Pyke (1958)).

By (1.8), always $b > h$. The partial sum $\sum_{h=0}^{\kappa}$ in (1.9) is just $B(\kappa; n, \beta)$. (To see this note $s > h \Rightarrow \binom{h}{s} = 0$; hence if $h \leqq \kappa$, then the second sum in (1.9) satisfies: $\sum_{s=0}^{\kappa} = \sum_{s=0}^{h} = n\gamma\beta/(\lambda' - h)$.) Next, by (1.9)

$$(1.11)\quad PA = \sum_{s=0}^{\kappa} \binom{n}{s}(s-n\delta)^s(\lambda'-s)^{-s-1} \sum_{h=s}^{\lambda^*} \binom{n-s}{h-s}(1-\beta)^{n-h}(\lambda'-s)^h(\lambda'-h)(n\gamma)^{-h}$$
$$= \sum_{s=0}^{\kappa} \binom{n}{s}(s-n\delta)^s(\lambda'-s)^{-s-1}(1-\beta)^n a^s \sum_{t=0}^{\lambda^*-s} \binom{n-s}{t} a^t(\lambda'-s-t),$$

$a = (\lambda'-s)/((1-\beta)n\gamma)$. One then obtains with $1+a = (n(\gamma+\delta)-s)\{(1-\beta)n\gamma\}^{-1}$ and

$$(1.12)\qquad p(s) := (\lambda'-s)/(n(\delta+\gamma)-s) = a/(1+a)$$

$$(1.13a)\quad PA = (n\gamma)^{-n} \sum_{s=0}^{\kappa} \binom{n}{s}(s-n\delta)^s(p(s))^s(n(\gamma+\delta)-s)^n(\lambda'-s)^{-s-1}$$
$$\cdot \{(\lambda'-s)B(\lambda^*-s; n-s, p(s)) - (n-s)p(s)\,B(\lambda^*-s-1; n-s-1, p(s))\}$$

$$(1.13b)\quad = \sum_{s=0}^{\kappa} b\left(s; n, \frac{s-n\delta}{n\gamma}\right)\left\{B(\lambda^*-s; n-s, p(s)) - \frac{n-s}{n(\gamma+\delta)-s}\right.$$
$$\left.\cdot B(\lambda^*-s-1; n-s-1, p(s))\right\}$$

$$(1.13c)\quad = \sum_{s=0}^{\kappa} b\left(s; n, \frac{s-n\delta}{n\gamma}\right)\left\{\frac{n(\delta+\gamma-1)}{n(\delta+\gamma)-s} B(\lambda^*-s-1; n-s-1, p(s))\right.$$
$$\left. + \frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s} b(\lambda^*-s; n-s-1, p(s))\right\}$$

$$(1.13d)\quad = n(\delta+\gamma-1) \sum_{s=0}^{\kappa} b\left(s; n, \frac{s-n\delta}{n\gamma}\right)(n(\gamma+\delta)-s)^{-1}$$
$$\times B(\lambda^*-s-1; n-s-1, p(s))$$
$$+ (n-\lambda^*)b(\lambda^*; n, \beta) \sum_{s=0}^{\kappa} (n-s)^{-1} b\left(s; \lambda^*, \frac{s-n\delta}{n\gamma\beta}\right)$$

where (1.13c) has been obtained using the identity

$$(1.13e)\qquad B(k+1; n+1, p) = B(k; n, p) + qb(k+1; n, p),$$

(Feller (1957), page 163). One notes that here λ^* could have been replaced by $\lambda \wedge n$. In all of the preceding expressions the sign of $(s-n\delta)^s$ alternates if $\delta > 0$. Expressions avoiding this are given below in Corollary 1.1.

REMARK 2. If $\lambda' > n$, β may obviously be replaced by any $\tilde{\beta} \geqq \gamma^{-1}(1-\delta)$ without changing the r.h.s. of (1.4). (1.4) then equals

$$P(F_n(x) \leqq \delta + \gamma x \text{ for } x \in (\alpha, \tilde{\beta}))$$
$$(1.14)\quad = B(\kappa; n, \tilde{\beta}) + \sum_{h=\kappa+1}^{\tilde{\lambda}^*} \binom{n}{h}(1-\tilde{\beta})^{n-h}(n\gamma)^{-h}(n-h)$$
$$\times \sum_{s=0}^{\kappa} \binom{h}{s}(s-n\delta)^s(n-s)^{h-s-1},$$

$$\tilde{\lambda}^* = n-1 \quad \text{if } \tilde{\lambda} = n,$$
$$= n \quad \text{if } \tilde{\lambda} > n, \qquad \tilde{\lambda} = n(\delta + \gamma\tilde{\beta}).$$

The independence of (1.14) of $\tilde{\beta}$ can be checked if one writes PA in the form (1.17c). Assuming $\tilde{\lambda} > n$,

$$(1.15)\qquad PA = 1-(n\gamma)^{-n}\textstyle\sum_{s=\kappa+1}^{n}\binom{n}{s}(s-n\delta)^s(n(\delta+\gamma)-s)^{n-s-1}n(\delta+\gamma-1)$$

which does not involve $\tilde{\beta}$; it is the same probability as (1.24) below, since $\tilde{\lambda} > n$ is the Dempster–Dwass case. If $\tilde{\lambda} = n$, we refer to (1.13b) and the statement at the end of Remark 1; thus we put $\tilde{\lambda}^* = n$. Then

$$(1.16)\qquad PA = (n\gamma)^{-n}n(\gamma+\delta-1)\textstyle\sum_{s=0}^{\kappa}\binom{n}{s}(s-n\delta)^s(n(\gamma+\delta)-s)^{n-s-1}$$

which again is (1.15) if (1.10) is applied.

REMARK 3. The familiar transformation of rv's $Y = F(X)$ yields a formula corresponding to (1.4) for i.i.d. rv's $X_j \sim F$ where F is continuous. Then under the assumptions (1.1) the probability

$$(1.16)\qquad P(F_n(x) \leqq \delta+\gamma F(x) \text{ for } F(x)\in(\alpha,\beta))$$

is given by (1.4) and its equivalent expressions.

Formula (1.4) can be written in various other forms (*particular attention is drawn to* (1.17c–d')) which are advantageous on different ranges of the parameters.

COROLLARY 1.1. *Under the assumptions of Theorem* 1.1 (*and using the function* $b(k;n,p)$ *also for values* $p < 0$ *and* $p > 1$)

$$(1.17a)\qquad PA = B(\lambda^*;n,\beta)-\textstyle\sum_{h=\kappa+1}^{\lambda^*}\binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda'-h) \cdot\textstyle\sum_{s=\kappa+1}^{h}\binom{h}{s}(s-n\delta)^s(\lambda'-s)^{h-s-1},$$

$$(1.17b)\qquad = B(\lambda^*;n,\beta)-\textstyle\sum_{s=\kappa+1}^{\lambda^*}\binom{n}{s}(s-n\delta)^s(\lambda'-s)^{-s-1} \cdot\textstyle\sum_{h=s}^{\lambda^*}\binom{n-s}{h-s}(1-\beta)^{n-h}(\lambda'-s)^h(\lambda'-h)(n\gamma)^{-h},$$

$$(1.17c)\qquad = B(\lambda^*;n,\beta)-\sum_{s=\kappa+1}^{\lambda^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right)\left\{B(\lambda^*-s;n-s,p(s)) - \frac{n-s}{n(\gamma+\delta)-s}B(\lambda^*-s-1;n-s-1,p(s))\right\},$$

$$(1.17d)\qquad = B(\lambda^*;n,\beta)-\sum_{s=\kappa+1}^{\lambda^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right) \times\left\{\frac{n(\delta+\gamma-1)}{n(\delta+\gamma)-s}B(\lambda^*-s-1;n-s-1,p(s)) + \frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s}b(\lambda^*-s;n-s-1,p(s))\right\},$$

$$(1.17d')\qquad = B(\lambda^*;n,\beta)-n(\delta+\gamma-1)\sum_{s=\kappa+1}^{\lambda^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right)(n(\gamma+\delta)-s)^{-1} \times B(\lambda^*-s-1;n-s-1,p(s))-(n-\lambda^*)b(\lambda^*;n,\beta) \cdot\sum_{s=\kappa+1}^{\lambda^*}(n-s)^{-1}b\left(s;\lambda^*,\frac{s-n\delta}{n\gamma\beta}\right)$$

with

$$p(s) = 1-q(s) = (\lambda'-s)/(n(\delta+\gamma)-s). \tag{1.17d''}$$

The proof is given below.

One notes that all the summands in any of the sums above are nonnegative while the terms $s-n\delta$ in (1.4) can be negative. This makes it easier to obtain bounds.

Still other, sometimes more convenient, expressions can be obtained by writing for the bracket $\{\cdots\}$ in (1.17c)

$$\sum_{t=0}^{\lambda^*-s-1} b(t;n-s,p(s))\left(1-\frac{n-s-t}{n\gamma(1-\beta)}\right)+b(\lambda^*-s;n-s,p(s)) \tag{1.17e}$$

where the term $1-(n-s-t)(n\gamma(1-\beta))^{-1}$ often becomes relatively small since it lies between $(\gamma-1+\delta(\beta-\alpha))(\gamma(1-\beta))^{-1}$ and $(\gamma-1+\delta)(\gamma(1-\beta))^{-1}$. The terms in (1.17c), when (1.17e) has been applied, can be further simplified using the identity

$$\sum_{s=\kappa+1}^{\lambda^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right)b(\lambda^*-s;n-s,p(s)) = b(\lambda^*;n,\beta)\sum_{s=\kappa+1}^{\lambda^*} b\left(s;\lambda^*,\frac{s-n\delta}{n\gamma\beta}\right). \tag{1.17f}$$

The proof of (1.17a–d′) follows from (1.4) via (1.10), putting $a=-n\delta$, $b=\lambda'$. Then $b\neq h$, since always $\lambda'>\lambda^*\geqq h$ (this guarantees that zero never occurs in the denominator). The sum over s in (1.4) becomes

$$(n\gamma\beta)^h/(\lambda'-h)-\textstyle\sum_{s=\kappa+1}^{h}\binom{h}{s}(s-n\delta)^s(\lambda'-s)^{h-s-1} \tag{1.17g}$$

which yields (1.17a). (1.17b) is obtained in the same manner as (1.13b). (1.17d′) is obtained from (1.17d) by observing

$$\begin{aligned} &b\left(s;n,\frac{s-n\delta}{n\gamma}\right)\frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s}\,b(\lambda^*-s;n-s-1,p(s)) \\ &= \frac{n!}{s!(\lambda^*-s)!(n-\lambda^*)!}(n-s-1)(n-\lambda^*)\frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s}\left(\frac{s-n\delta}{n\gamma}\right)^s \\ &\qquad\cdot\left(\frac{n(\gamma+\delta)-s}{n\gamma}\right)^{n-s} p(s)^{\lambda^*-s}q(s)^{n-\lambda^*-1} \\ &= (n-\lambda^*)\binom{n}{\lambda^*}\binom{\lambda^*}{s}(n-s)^{-1}n\gamma(1-\beta)(n\gamma)^{-n}(s-n\delta)^s(\lambda'-s)^{\lambda^*-s} \\ &\qquad\cdot(n\gamma(1-\beta))^{n-\lambda^*-1} \\ &= (n-\lambda^*)b(\lambda^*;n,\beta)\binom{\lambda^*}{s}(n-s)^{-1}(n\gamma\beta)^{-\lambda^*}(s-n\delta)^s(n\gamma\beta-(s-n\delta))^{\lambda^*-s} \\ &= (n-\lambda^*)b(\lambda^*;n,\beta)b\left(s;\lambda^*,\frac{s-n\delta}{n\gamma\beta}\right)(n-s)^{-1}. \end{aligned} \tag{1.17h}$$

Special case 1. The case $\alpha=\delta=0$ has been considered by Chang (1955) (Theorem 4, page 26), and the identity of his results with ours follows after some algebraic computations. Since $\kappa = 0$, one obtains respectively from (1.4), (1.13c), or (1.17b), with $\lambda = [n\gamma\beta] \wedge n$, $\lambda' = n\gamma\beta$,

$$PA = \sum_{h=0}^{\lambda}\left(1-\frac{h}{n\gamma\beta}\right)b(h;n,\beta) = B(\lambda;n,\beta)-\gamma^{-1}B(\lambda-1;n-1,\beta)$$

$$(1.18)\qquad = B(\lambda;n,\beta)-\textstyle\sum_{s=1}^{\lambda}\binom{n}{s}s^s\sum_{h=s}^{\lambda}\binom{n-s}{h-s}(1-\beta)^{n-h}(\lambda'-s)^{h-s-1}(\lambda'-h)(n\gamma)^{-h}$$

$$= (1-\gamma^{-1})B(\lambda-1;n-1,\beta)+(1-\beta)b(\lambda;n-1,\beta).$$

In our notation Chang's formula reads as follows for $0 < \gamma\beta \leqq 1$:

$$(1.19)\qquad PA = B(\lambda;n,\beta)-\textstyle\sum_{s=1}^{\lambda}\binom{n}{s}s^{s-1}\sum_{h=s}^{\lambda}\binom{n-s}{n-s}(1-\beta)^{n-h}(\lambda'-s)^{h-s}(n\gamma)^{-h}$$

$$(1.20)\qquad = B(\lambda;n,\beta)-(n\gamma)^{-n}\textstyle\sum_{s=1}^{\lambda}\binom{n}{s}s^{s-1}(n\gamma-s)^{n-s}B(\lambda-s;n-s,p(s))$$

with $p(s) = (n\gamma\beta-s)(n\gamma-s)^{-1}$ [same as (1.12) for $\delta = 0$]. To show the identity with (1.18a–e) we need the algebraic identity (see Dwass (1959), page 1027, Lemma d)

$$(1.21)\qquad \textstyle\sum_{t=0}^{n-1}\binom{n}{t+1}(A+t)^t(B-t)^{n-t-1} = ((A+B)^n-(B+1)^n)(A-1)^{-1}, \qquad A \neq 1$$

$$= n(1+B)^{n-1}, \qquad A = 1$$

where the lower expression follows from the upper by de l'Hospitâl's rule. Let us write (1.19) in the form

$$PA = B(\lambda;n,\beta)-\textstyle\sum_{h=1}^{\lambda}\binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}\sum_{s=1}^{h}\binom{h}{s}s^{s-1}(\lambda'-s)^{h-s}.$$

The sum over s by (1.21) is equal to

$$(1.22)\qquad \textstyle\sum_{t=0}^{h-1}\binom{h}{t+1}(1+t)^t(\lambda'-1-t)^{h-t-1} = h(\lambda')^{h-1}.$$

Hence

$$(1.23)\qquad PA = B(\lambda;n,\beta)-\gamma^{-1}\textstyle\sum_{h=1}^{\lambda}\binom{n-1}{h-1}(1-\beta)^{n-1-(h-1)}\beta^{h-1}$$

$$= B(\lambda;n,\beta)-\gamma^{-1}B(\lambda-1;n-1,\beta),$$

the same as (1.18 tres) (which again yields immediately $PA = 1-\gamma^{-1}$ for $\lambda = n$, any $\beta \geqq \gamma-1$; $\gamma \leqq 1$).

Special case 2. $\alpha = 0$, $\beta = 1$, $0 \leqq \delta < 1$, $\delta+\gamma > 1$ (the Dempster–Dwass–Pyke case; Theorem 2.1 below). Starting from (1.9) of (1.17a), one has with $\kappa = [n\delta]$, $\lambda^* = n$,

$$(1.24)\qquad PA = \frac{\delta+\gamma-1}{\gamma}\sum_{s=0}^{\kappa}\binom{n}{s}\left(\frac{s}{n\gamma}-\frac{\delta}{\gamma}\right)^s\left(1-\left(\frac{s}{n\gamma}-\frac{\delta}{\gamma}\right)\right)^{n-s-1}$$

which is, of course, (2.20) below (as seen upon substitution $t = n-s$ and observing (2.10) below). PA remains unchanged (Remark 2) if $\beta = 1$ is replaced by any $\tilde{\beta} \geqq \gamma^{-1}(1-\delta)$, implying $\tilde{\lambda} = n(\delta+\gamma\tilde{\beta}) \geqq n$.

Special case 3. If $\delta = 0$, $\gamma = 1$, any $0 \leqq \alpha < \beta \leqq 1$,

$$(1.25a)\quad PA = (n-\lambda)b(\lambda^*; n, \beta)\sum_{s=0}^{\kappa}(n-s)^{-1}b(s; \lambda^*, s/(n\beta))$$

$$(1.25b)\quad = B(\lambda^*; n, \beta)-(n-\lambda)b(\lambda^*; n, \beta)\sum_{s=\kappa+1}^{\lambda^*}(n-s)^{-1}b(s; \lambda^*, s/(n\beta)),$$

by (1.13c) and (1.17d); here $\kappa = [n\alpha]$, λ^* defined by (1.6).

The following simple lemma and corollary are often useful:

LEMMA 1.1. *Given constants* α, β *satisfying* $0 \leqq \alpha \leqq \beta \leqq 1$ *and a non-decreasing function* $h: [\alpha, \beta] \to [0, 1]$. *Then*

$$(1.26)\quad P(\textstyle\bigcap_{\alpha \leqq x \leqq \beta}(F_n(x) \geqq h(x))) = P(\bigcap_{1-\beta \leqq x \leqq 1-\alpha}(F_n(x) \leqq 1-h(1-x))).$$

PROOF. One substitutes $x = 1-x'$ and utilizes the symmetry of the new problem with the original one: $G_n(1-x) = 1-F_n(x)$, $0 \leqq x \leqq 1$, is a (now right-continuous) sample df and one has

$$\textstyle\bigcap_{\alpha \leqq x \leqq \beta}(F_n(x) \geqq h(x)) = \bigcap_{\alpha \leqq x \leqq \beta}(G_n(1-x) \leqq 1-h(x)).$$

Due to continuity of $U(0, 1)$ and since the probability laws of X_j and X_j' are identical, the probability of the event on the right can be written in the form of the right-hand side of (1.26).

COROLLARY 1.2. *Let* $\alpha, \beta, \gamma, \delta$ *be constants satisfying* $0 \leqq \alpha < \beta \leqq 1$, $\gamma \in R^+$, $0 \leqq 1-\delta-\gamma\beta < 1$. *Then*

$$(1.26a)\quad P(F_n(x) \geqq \delta+\gamma x \textit{ for all } x \in (\alpha, \beta))$$

$$= P(F_n(x) \leqq 1-\delta-\gamma+\gamma x \textit{ for all } x \in (1-\beta, 1-\alpha))$$

$$(1.26b)\quad = B(\tilde{\kappa}; n, 1-\alpha)+\sum_{h=\kappa+1}^{\tilde{\lambda}}\binom{n}{h}\alpha^{n-h}(n\gamma)^{-h}(n(1-\delta-\gamma\alpha)-h)$$

$$\times \sum_{s=0}^{\tilde{\kappa}}\binom{h}{s}(s-n(1-\delta-\gamma))^s(n(1-\delta-\gamma\alpha)-s)^{h-s-1},$$

$$(1.27)\quad \tilde{\kappa} = [n(1-\delta-\gamma\beta)], \qquad \tilde{\lambda} = \min(n, [n(1-\delta-\gamma\alpha)]).$$

Next we deal with the event that the sample df lies below a line segment L in the unit square for all $x \in (\alpha, \beta)$ (or above L for some $x \in (\alpha, \beta)$) and at the same time lies below given real numbers at the two endpoints α and β. The probabilities of these events are often useful (compare the introduction).

THEOREM 1.2. *Let the assumptions and notations of Theorem* 1.1 *be valid. Moreover, let (without loss of generality)* $\delta+\gamma\beta \leqq 1$ *and let* $0 \leqq a \leqq b \leqq n$ *be given integers. Then the probability*

$$(1.28)\quad P(A, F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n)$$

$$= P(F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n)$$

$$-P(\textstyle\bigcup_{\alpha < x < \beta}(F_n(x) > \delta+\gamma x), F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n)$$

equals in case $b \leqq \kappa' := n(\delta+\gamma\alpha)$

(1.28a) $P(F_n(\alpha) \leqq n^{-1}a, F_n(\beta) \leqq n^{-1}b) = B(a;n,\beta)+\sum_{k=a+1}^{b} b(k;n,\beta)B(a;k,\alpha/\beta).$

(Alternative expressions are given in Remark 4 below.) *If* $b>\kappa'$, *we obtain with*

$$a' := \min(a,\kappa), \qquad b' := \min(b,\lambda), \qquad b^* := \min(b,\lambda^*),$$

$$\begin{aligned} &P(A, F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n) \\ (1.29) \quad &= B(b^*;n,\beta)-\textstyle\sum_{h=a'+1}^{\kappa} b(h;n,\beta)(1-B(a';h,\alpha/\beta)) \\ &\quad -\textstyle\sum_{h=\kappa+1}^{b^*} b(h;n,\beta)\{B(\kappa;h,\alpha/\beta)-B(a;h,\alpha/\beta)\} \\ &\quad -\sum_{s=\kappa+1}^{b^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right)\Psi(s,b^*-s)\left\{1-\left[B\left(\kappa;s,\frac{n\gamma\alpha}{s-n\delta}\right)\right.\right. \\ &\qquad\qquad \left.\left. -B\left(a';s,\frac{n\gamma\alpha}{s-n\delta}\right)\right]\right\} \end{aligned}$$

$$\begin{aligned} (1.29a) \quad &= B(a';n,\beta)+\textstyle\sum_{h=a'+1}^{\kappa} b(h;n,\beta)B(a';h,\alpha/\beta) \\ &\quad +\textstyle\sum_{h=\kappa+1}^{b^*} b(h;n,\beta)\{1-[B(\kappa;h,\alpha/\beta)-B(a';h,\alpha/\beta)]\} \\ &\quad -\sum_{s=\kappa+1}^{b^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right)\Psi(s,b^*-s)\left\{1-\left[B\left(\kappa;s,\frac{n\gamma\alpha}{s-n\delta}\right)\right.\right. \\ &\qquad\qquad \left.\left. -B\left(a';s,\frac{n\gamma\alpha}{s-n\delta}\right)\right]\right\}; \end{aligned}$$

here $\Psi(s,\chi)$ *with* $p(s) = (\lambda'-s)/(n(\delta+\gamma)-s)$ *is given by*

$$(1.30) \quad \Psi(s,\chi) := \frac{n(\delta+\gamma-1)}{n(\delta+\gamma)-s}B(\chi-1;n-s-1,p(s))+\frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s}\,b(\chi;n-s-1,p(s)).$$

The sum over k in (1.30) *could have been replaced by the expression* (2.29d). *One further step of simplification can be done by the transformation that led from* (1.17d) *to* (1.17d'). (For other expressions equivalent to (1.29) see Remark 5 below.)

For $a' = \kappa$, $b' = \lambda$, (1.30) gives the former (1.17d).

The proof of Theorem 1.2 is given in Section 2.

For ease of reference we rephrase Theorem 1.2 as follows:

COROLLARY 1.3. *With the assumption and notations of Theorem* 1.2, *for* $\kappa < b$,

$$\begin{aligned} (1.31) \quad &P(\textstyle\bigcup_{\alpha<x<\beta}(F_n(x) > \delta+\gamma x), F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n) \\ &= B(a;n,\beta)-B(a';n,\beta)+\textstyle\sum_{h=a+1}^{b} b(h;n,\beta)B(a;h,\alpha/\beta) \\ &\quad -\textstyle\sum_{h=a'+1}^{\kappa} b(h;n,\beta)B(a';h,\alpha/\beta) \\ &\quad -\textstyle\sum_{h=\kappa+1}^{b^*} b(h;n,\beta)\{1-[B(\kappa;h,\alpha/\beta)-B(a';h,\alpha/\beta)]\} \\ &\quad +\sum_{s=\kappa+1}^{b^*} b\left(s;n,\frac{s-n\delta}{n\gamma}\right)\Psi(s,b^*-s)\left\{1-\left[B\left(\kappa;s,\frac{n\gamma\alpha}{s-n\delta}\right)\right.\right. \\ &\qquad\qquad \left.\left. -B\left(a';s,\frac{n\gamma\alpha}{s-n\delta}\right)\right]\right\}. \end{aligned}$$

The sum $\sum_s$ may be rewritten as in (1.17d′) in a form that sometimes circumvents the bounding of the b-function in Ψ. It will then be replaced by (using (1.17h))

$$(1.32)\quad \sum_{s=\kappa+1}^{b^*} \frac{n(\delta+\gamma-1)}{n(\delta+\gamma)-s} b\left(s; n, \frac{s-n\delta}{n\gamma}\right) B(b^*-s-1; n-s-1, p(s))$$
$$+ b(b^*; n, \beta) \sum_{s=\kappa+1}^{b^*} \frac{n-b^*}{n-s} b\left(s; b^*, \frac{s-n\delta}{n\gamma\beta}\right)$$
$$- \sum_{s=\kappa+1}^{b^*} b\left(s; n, \frac{s-n\delta}{n\gamma}\right) \Psi(s, b^*-s) \left\{ B\left(\kappa; s, \frac{n\gamma\alpha}{s-n\delta}\right) - B\left(a'; s, \frac{n\gamma\alpha}{s-n\delta}\right) \right\}.$$

Special case. The case $a \leqq \kappa < b \leqq \lambda - 1 \leqq n$ is of particular interest. Then $a' = a$, $b^* = b' = b$, and the probability (1.31) simplifies to

$$(1.33)\quad -\textstyle\sum_{h=\kappa+1}^{b} b(h; n, \beta)(1 - B(\kappa; h, \alpha/\beta))$$
$$+ \sum_{s=\kappa+1}^{b} b\left(s; n, \frac{s-n\delta}{n\gamma}\right) \Psi(s, b-s) \left\{ 1 - \left[B\left(\kappa; s, \frac{n\gamma\alpha}{s-n\delta}\right) - B\left(a'; s, \frac{n\gamma\alpha}{s-n\delta}\right) \right] \right\}.$$

PROOF OF COROLLARY 1.3. We use the notations and formulae of the proof of Theorem 1.2. The probability on the left of (1.31) is by (1.28)

$$(1.34)\quad P(A^c\bar{C}\bar{D}) = P(\bar{C}\bar{D}) - P(ACD),$$

utilizing the remark following (2.24). Here $P(ACD)$ may be expressed by (1.29a), and $P(\bar{C}\bar{D})$ by (1.28a).

Remark 4. Equivalent expressions for the trinomial probability (1.28a) are

$$P(F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n)$$
$$(1.35)\quad = \sum_{k=0}^{a} b(k; n, \alpha) B\left(b-k; n-k, \frac{\beta-\alpha}{1-\alpha}\right)$$
$$= \sum_{k=0}^{b-a} b(k; n, \beta-\alpha) B\left(a; n-k, \frac{\alpha}{1-\beta+\alpha}\right)$$
$$+ \sum_{k=b-a+1}^{b} b(k; n, \beta-\alpha) B\left(b-k; n-k, \frac{\alpha}{1-\beta+\alpha}\right),$$

as will be shown in the proof of Theorem 1.2.

Remark 5. Equivalent expressions for the probability (1.28) in the case $b > \kappa'$ are given by (2.26) and (2.29) in the proof of Theorem 1.2. A sometimes more convenient expression is (note $\lambda' > s$)

$$P(A, F_n(\alpha) \leqq a/n, F_n(\beta) \leqq b/n)$$
$$(1.36)\quad = B(b^*; n, \beta) - P(F_n(\alpha) > a'/n, F_n(\beta) \leqq \kappa/n)$$
$$-\textstyle\sum_{h=\kappa+1}^{b^*} \binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda'-h)\{\sum_{s=\kappa+1}^{h}\binom{h}{s}(s-n\delta)^s(\lambda'-s)^{h-s-1}$$
$$+\textstyle\sum_{s=a'+1}^{\kappa}\binom{h}{s}(\lambda'-s)^{h-s-1}\sum_{k=a'+1}^{s}\binom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k}\}.$$

In the sum over s and k the signs of the terms still oscillate. A version which avoids this is given in (2.29).

A property of sample probabilities. Let $X_1, \cdots, X_n$ be i.i.d. rv's from the df F defined on $(a,b) \subset R$. We call

$$F_n(B) := n^{-1}\textstyle\sum_{j=1}^n I(X_j \in B) \quad \text{for} \quad B \in (a,b)\mathfrak{B},$$

with $(a,b)\mathfrak{B}$ the Borel field on (a,b) the *sample probability* of the sample $X_1, \cdots, X_n$ since $F_n(\cdot)$ is a transition probability (or Markov kernel). If e.g. $[c,d) \subset (a,b)$ we write $F_n([c,d)) = F_n[c,d)$. The preceding results generalize to sample probabilities in various ways, e.g.:

THEOREM 1.3. *Under the assumptions of Theorem* 1.1, *with a constant* $p = 1 - q \in [0, \alpha]$, *and with the notations* (1.5–1.7), (1.12) *and*

$$A := (F_n[p,x) < \delta + \gamma x \ \forall\, x \in (\alpha, \beta)), H_k := (F_n(p) = k/n), \quad k = 0, 1, \cdots n,$$

$$\begin{aligned}(1.37)\qquad \lambda_k^* &:= n-k && \textit{if } \tilde{\lambda}' > n-k [\Leftrightarrow \lambda' > n-k],\\ &= \lambda := [\lambda'] && \textit{if } \lambda' < n-k \textit{ and } \lambda' \textit{ is not an integer},\\ &= \lambda' - 1 && \textit{if } \lambda' \leqq n-k \textit{ and } \lambda' \textit{ is an integer},\end{aligned}$$

there holds for $k = 0, 1, \cdots, n - \kappa - 1$

$$(1.38)\quad P(AH_k) = b(k;n,p)\left[B\left(\lambda_k^*; n-k, \frac{\beta-p}{q}\right) - \sum_{s=\kappa+1}^{\lambda_k^*} b\left(s; n-k, \frac{s-n(\delta+\gamma p)}{n\gamma q}\right)\right.$$
$$\times \left\{\frac{n(\delta+\gamma-1)+k}{n(\delta+\gamma)-s} B(\lambda_k^* - s - 1; n-k-s-1, p(s))\right.$$
$$\left.\left. + \frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s} b(\lambda_k^* - s; n-k-s-1, p(s))\right\}\right],$$

$$P(AH_k) = PH_k = b(k;n,p), \qquad k \geqq n-\kappa.$$

For subsets $\mathscr{M} \subset \{0/n, 1/n, \cdots, n/n\}$, $P(A(F_n(p) \in \mathscr{M}))$ = *sum of corresponding terms* (1.38). *In particular,*

$$PA = 1 - B(n-\kappa; n, p)$$
$$+ \sum_{k=0}^{n-\kappa} b(k;n,p)\left[B\left(\lambda_k^*; n-k, \frac{\beta-p}{q}\right) - \sum_{s=\kappa+1}^{\lambda_k^*} b\left(s; n-k, \frac{s-n(\delta+\gamma p)}{n\gamma q}\right)\right.$$
$$\times \left\{\frac{n(\delta+\gamma-1)+k}{n(\delta+\gamma)-s} B(\lambda_k^* - s - 1; n-k-s-1, p(s))\right.$$
$$\left.\left. + \frac{n\gamma(1-\beta)}{n(\delta+\gamma)-s} b(\lambda_k^* - s; n-k-s-1, p(s))\right\}\right].$$

For the proof, some ramifications and an application see [17].

2. Proofs.

PROOF OF THEOREM 1.1. The assertions for $\kappa < 0$, and $\kappa \geqq n$ are obvious. For $n > \kappa \geqq 0$ put

(2.1) $$B = (F_n(\beta) \leqq \delta + \gamma\alpha) = (X_j < \beta \text{ at most } \kappa \text{ times}),$$

$PB = B(\kappa; n, \beta)$. Given B the graph of F_n lies below $\delta + \gamma x$ for all $x \in (\alpha, \beta)$, hence $B \subset A$. Obviously

(2.2) $$(F_n(\alpha) > \delta + \gamma\alpha)A = \emptyset, \qquad (F_n(\beta) \geqq \delta + \gamma\beta)A = \emptyset \quad \text{a.s.}$$

Hence given B^c only the following situation needs to be considered:

(2.3) $$\begin{aligned} &X_j < \alpha \quad \text{at most} \quad \kappa \quad \text{times and simultaneously,} \\ &X_j < \beta \quad \text{at least} \quad \kappa+1 \quad \text{times and at most} \quad [\lambda'] = \lambda \quad \text{times,} \end{aligned}$$

and $n > \kappa \geqq 0$. If $[\lambda'] > n$ it obviously suffices to replace $[\lambda']$ by n and β by $\gamma^{-1}(1-\delta)$ (Remark 2). Let $Y_1, \cdots, Y_n$ be the order statistics of $X_1, \cdots, X_n$. For convenience put $Y_0 \equiv -1$, $Y_{n+1} \equiv 2$. The events

(2.4) $$A_{k,h} = (Y_k < \alpha < Y_{k+1}; Y_h < \beta < Y_{h+1}), \qquad k = 0, 1, \cdots, h; h = 0, 1, \cdots, n,$$

are disjoint and their union is a.s. the entire sample space Ω. Due to (2.3) only index pairs k, h satisfying

(2.5) $$0 \leqq k \leqq \kappa < h < \lambda' \leqq n$$

need to be considered in partitioning B^c by the A_{kh}, and

(2.6) $$PA = B(\kappa; n, \beta) + \sum_{k=0}^{\kappa} \sum_{h=\kappa+1}^{\lambda} P(AA_{k,h}).$$

Now on $AA_{k,h}$ by $Y_k < \alpha, \beta < Y_{h+1}$ and by

(2.7) $$F_n(x) = n^{-1}(k + \sum_{j=k+1}^{h} I(Y_j < x)), \qquad x \in (\alpha, \beta),$$

$$AA_{k,h} = A_{k,h}((h-k)^{-1} \sum_{j=k+1}^{h} I(Y_j < x) \leqq (h-k)^{-1}(n(\delta + \gamma x) - k), x \in (\alpha, \beta)).$$

Let the second event on the right be denoted by $B_{k,h}$. The probability of this event may be determined by using the density function of the vector $(Y_k, \cdots, Y_{h+1})$. For $k > 0$, $h < n$, it is given by

(2.8) $$\frac{n!}{(k-1)!(n-h-1)!} y_k^{\,k-1}(1-y_{h+1})^{n-h-1}$$

on $0 < y_k < y_{k+1} < \cdots < y_{h+1} < 1$, zero otherwise. (Compare Rényi (1953).) Hence for $k > 0$, $h < n$

(2.9a) $P(AA_{k,h})$

$$= \frac{n!}{(k-1)!(n-h-1)!} \int_0^\alpha y_k^{\,k-1}\, dy_k$$

$$\times \left[\int_\alpha^\beta dy_{k+1} \int_{y_{k+1}}^\beta dy_{k+2} \cdots \int_{y_{h-1}}^\beta dy_h I(B_{k,h})\right] \int_\beta^1 dy_{h+1}(1-y_{h+1})^{n-h-1}$$

(2.9b) $$= \frac{n!}{k!(n-h)!(h-k)!} \alpha^k (1-\beta)^{n-h} (\beta-\alpha)^{h-k} \{(h-k)!(\beta-\alpha)^{-(h-k)}[\ldots]\}$$

where [...] is the same expression as in the preceding member of the equation and where of course in $I(B_{k,h})$ the Y_j are replaced by y_j. It can be easily seen that (2.9b) gives the correct probability also for $k = 0$ and/or $h = 0$. Since $(h-k)!\,(\beta-\alpha)^{-(h-k)}$ is the value of the joint density, where it is not zero, of the order statistics of a size $h-k$ random sample from the df $\tilde{F} = U(\alpha,\beta)$, the expression $\{\dots\}$ in (2.9b) is identical with

$$P(\tilde{F}_{h-k}(x) \leqq \tilde{\delta}+\tilde{\gamma}\tilde{F}(x) \text{ for } x\in(\alpha,\beta)), \tag{2.10}$$

$$\tilde{\delta}: = (h-k)^{-1}(n(\delta+\gamma\alpha)-k), \qquad \tilde{\gamma}: = (h-k)^{-1}n\gamma(\beta-\alpha), \tag{2.11}$$

and where $\tilde{F}_{h-k}$ is the sample df of the new sample. By (2.5),

$$0 \leqq \tilde{\delta} < 1, \qquad \tilde{\gamma} > 0, \qquad \tilde{\delta}+\tilde{\gamma} = (h-k)^{-1}(n(\delta+\gamma\beta)-k) \geqq 1. \tag{2.12}$$

(In order to simplify the notation the dependence of the numbers $\tilde{\delta}, \tilde{\gamma}$ on h and k is not indicated.) Theorem 2.1 below now yields for (2.10)

$$\textstyle\sum_{j=k_{\tilde{\delta}}+1}^{h-k} C_j(\tilde{\delta},\tilde{\gamma},h-k), \tag{2.13}$$

$k_{\tilde{\delta}}$ = largest integer $< (1-\tilde{\delta})(h-k) = h-n(\delta+\gamma\alpha)$, equivalently $k_{\tilde{\delta}}$ = largest integer $< h-\kappa$, i.e.,

$$k_{\tilde{\delta}}+1 = h-\kappa. \tag{2.14}$$

(By (2.5), $1 \leqq h-\kappa \leqq h-k$, thus the sum (2.13) has always at least one summand.) The double sum in (2.6) now becomes, taking into account (2.9b) and (2.13),

$$\sum_{k=0}^{\kappa}\sum_{h=\kappa+1}^{\lambda} \frac{n!}{k!(n-h)!(h-k)!}\alpha^k(1-\beta)^{n-h}(\beta-\alpha)^{h-k} \times \sum_{j=h-\kappa}^{h-k} \tilde{\varepsilon}\binom{h-k}{j}\left(\tilde{\varepsilon}+\frac{j}{\tilde{\gamma}(h-k)}\right)^{j-1}\left(1-\left(\tilde{\varepsilon}+\frac{j}{\tilde{\gamma}(h-k)}\right)\right)^{h-k-j}, \tag{2.15}$$

$$\tilde{\varepsilon}: = \tilde{\gamma}^{-1}(\tilde{\gamma}+\tilde{\delta}-1) = (n\gamma(\beta-\alpha))^{-1}(n(\delta+\gamma\beta)-h)\ [\in[0,1)]. \tag{2.16}$$

Since $\tilde{\gamma}(h-k) = n\gamma(\beta-\alpha)$, (2.15) can be rewritten with $\kappa': = n(\delta+\gamma\alpha)$ as

$$\textstyle\sum_{h=\kappa+1}^{\lambda}\binom{n}{h}(1-\beta)^{n-h}(\lambda'-h)\sum_{k=0}^{\kappa}\binom{h}{k}\alpha^k(n\gamma)^{k-h} \times\sum_{j=h-\kappa}^{h-k}\binom{h-k}{j}(\lambda'-h+j)^{j-1}(h-j-\kappa')^{h-k-j}. \tag{2.17}$$

Substituting $h-j = s$ [$s = k,\cdots,\kappa$] and permuting summations over k and s:

$$\textstyle\sum_{k=0}^{\kappa}\sum_{j=h-\kappa}^{h-k} = \sum_{k=0}^{\kappa}\sum_{s=k}^{\kappa} = \sum_{s=0}^{\kappa}\sum_{k=0}^{s}, \tag{2.17a}$$

one obtains for (2.15) or (2.17)

$$\textstyle\sum_{h=\kappa+1}^{\lambda}\binom{n}{h}(1-\beta)^{n-h}(\lambda'-h)(n\gamma)^{-h}\sum_{s=0}^{\kappa}\binom{h}{s}(\lambda'-s)^{h-s-1} \times\sum_{k=0}^{s}\binom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k}. \tag{2.18}$$

The sum over k equals $(s-n\delta)^s$, thus yielding for (2.15)

$$\textstyle\sum_{h=\kappa+1}^{\lambda}\binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda'-h)\sum_{s=0}^{\kappa}\binom{h}{s}(s-n\delta)^s(\lambda'-s)^{h-s-1}. \tag{2.19}$$

Inserting this into (2.6) yields (1.4). In the above proof the following result has been used.

THEOREM 2.1. (Dempster (1959), Dwass (1959), Pyke (1959)). *Let* $X_1, \cdots, X_n$ i.i.d. $\sim F$ *(continuous on R),* δ, γ *constants with* $\gamma > 0$, $\delta + \gamma \geqq 1$, $0 \leqq \delta < \infty$. *Then*

(2.20) $$P(F_n(z) \leqq \delta + \gamma F(z) \text{ for } z \in R) = 1 - \sum_{j=0}^{k_\delta} C_j = \sum_{j=k_\delta+1}^{n} C_j,$$

k_δ = *largest integer* $< (1-\delta)n$,

(2.21) $$C_j \equiv C_j(\delta, \gamma, n) := \varepsilon\binom{n}{j}(\varepsilon + j/(\gamma n))^{j-1}(1-(\varepsilon + j/(\gamma n)))^{n-j}, \quad \varepsilon := \gamma^{-1}(\gamma + \delta - 1).$$

Even if some assumptions about the parameters are not satisfied, (2.20) still holds.

PROOF OF THEOREM 1.2. (1) To prove (1.28a), put

$$\bar{C} := (F_n(\alpha) \leqq a/n), \qquad \bar{D} := (F_n(\beta) \leqq b/n), \qquad \Delta := F_n(\beta) - F_n(\alpha).$$

Then

$$P(\bar{C}\bar{D}) = \sum_{k=0}^{b} P(F_n(\beta) = k/n, \Delta \geqq (k-a)/n)$$
$$= B(a; n, \beta) + \sum_{k=a+1}^{b} \sum_{m=k-a}^{k} P(1 - F_n(\beta) = (n-k)/n, \Delta = m/n)$$

(2.21a) $$= B(a; n, \beta) + \sum_{k=a+1}^{b} \sum_{m=k-a}^{k} \frac{n!}{(n-k)!m!(k-m)!}(1-\beta)^{n-k}(\beta-\alpha)^m \alpha^{k-m}$$

$$= B(a; n, \beta) + \sum_{k=a+1}^{b} \sum_{m=k-a}^{k} b(k; n, \beta) b(k-m; k, \alpha/\beta)$$
$$= B(a; n, \beta) + \sum_{k=a+1}^{b} b(k; n, \beta) B(a; k, \alpha/\beta).$$

The equivalent formulae of Remark 4 are proved by similar decompositions:

(2.21b) $$P(\bar{C}\bar{D}) = \sum_{k=0}^{a} P((F_n(\alpha) = k/n)(\Delta \leqq (b-k)/n))$$
$$= \sum_{k=0}^{a} b(k; n, \alpha) B(b-k; n-k, (\beta-\alpha)/(1-\alpha)).$$

Decomposing according to the values of Δ yields

$$\bar{D} = \sum_{k=0}^{b} (\Delta = k/n) \sum_{m=0}^{b-k} (F_n(\alpha) = m/n) F,$$

$$P(\bar{C}\bar{D}) = \sum_{k=0}^{b} \sum_{m=0}^{(b-k)\wedge a} \binom{n}{k}\binom{n-k}{m} \alpha^m (\beta-\alpha)^k (1-\beta)^{n-k-m}$$

(2.21c) $$= \sum_{k=0}^{b-a} \sum_{m=0}^{a} \cdots + \sum_{k=b-a+1}^{b} \sum_{m=0}^{b-k} \cdots$$

$$= \sum_{k=0}^{b-a} b(k; n, \beta-\alpha) B\left(a; n-k, \frac{\alpha}{1-\beta+\alpha}\right)$$
$$+ \sum_{k=b-a+1}^{b} b(k; n, \beta-\alpha) B\left(b-k; n-k, \frac{1}{1-\beta+\alpha}\right).$$

(2) In order to prove (1.36) of Remark 5 and subsequently (1.29), put

(2.22) $$C := (F_n(\alpha) \leqq a'/n), \qquad D := (F_n(\beta) \leqq b'/n)$$

(2.23) $$a' := \min(a, \kappa), \qquad b' := \min(b, \lambda).$$

Obviously,

$$P(A, F_n(\alpha) \leqq a/n,\ F_n(\beta) \leqq b/n) \tag{2.24}$$

is not changed by replacing a by a' and b by b'. The proof seems quite similar to that of Theorem 1.1, if one restricts the range of pairs (k, h) in (2.4) to $0 \leqq k \leqq a' \leqq \kappa < h \leqq b'$ if $\kappa < b$ instead of the range (2.5). Correspondingly, the summation ranges in (2.6), (2.15) and (2.17) are altered. If $b \leqq \kappa$, then $B^c D = \varnothing$ and (2.24) equals $P(CD)$. Instead of (2.17a) one has, always for $\kappa < b$ now,

$$\sum_{k=0}^{a'} \sum_{s=k}^{\kappa} = \sum_{k=0}^{a'} (\sum_{s=k}^{a'} + \sum_{s=a'+1}^{\kappa}) = \sum_{s=0}^{a'} \sum_{k=0}^{s} + \sum_{s=a'+1}^{\kappa} \sum_{k=0}^{a'} .$$

Analogously to (2.18) one obtains, denoting $B := (nF_n(\beta) \leqq \kappa)$,

$$\begin{aligned} P(ACD) = P(BCD) &+ \sum_{h=\kappa+1}^{b'} \tbinom{n}{h}(1-\beta)^{n-h}(\lambda'-h)(n\gamma)^{-h} \\ &\times \{\sum_{s=0}^{a'} \tbinom{h}{s}(\lambda'-s)^{h-s-1} \sum_{k=0}^{s} \tbinom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k} \\ &+ \sum_{s=a'+1}^{\kappa} \tbinom{h}{s}(\lambda'-s)^{h-s-1} \sum_{k=0}^{a'} \tbinom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k}\}. \end{aligned} \tag{2.25}$$

The first sum over k equals $(s-n\delta)^s$.

The second sum over k cannot be expressed in terms of a cumulative binomial since $s-\kappa'$ is mostly negative. But since in interesting cases a' is near κ, the number of summands is then reduced by replacing the sum by

$$(s-n\delta)^s - \sum_{k=a'+1}^{s} \tbinom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k}.$$

Hence (2.25) equals, observing $BD = B$ because of $\kappa < b$,

$$\begin{aligned} P(ACD) = P(BC) &+ \sum_{h=\kappa+1}^{b'} \tbinom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda'-h) \\ &\times \{\sum_{s=0}^{\kappa} \tbinom{h}{s}(s-n\delta)^s(\lambda'-s)^{h-s-1} \\ &- \sum_{s=a'+1}^{\kappa} \tbinom{h}{s}(\lambda'-s)^{h-s-1} \sum_{k=a'+1}^{s} \tbinom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k}\}. \end{aligned} \tag{2.26}$$

We now derive equivalent expressions for this, always under $\kappa < b$. In order to apply (1.17h), the case $\lambda' = h$ has to be ruled out. This can be done, as is easily checked, by replacing b' in (2.26) by

$$b^* := \min(b, \lambda^*), \qquad \lambda^* \quad \text{given by (1.7).}$$

Application of (1.10) to the first sum over s in (2.26) (since always $\lambda' \neq h$ after replacing b' by b^*) results in an additive term

$$B(b^*; n, \beta) - B(\kappa; n, \beta). \tag{2.27}$$

By $P(BC) = PB - P(BC^c) = B(\kappa; n, \beta) - P(BC^c)$, (1.36) of Remark 5 follows.

In order to get rid of the oscillation of signs in the sums over k and s in (1.36) we apply again (1.17h), which is allowed since $\lambda' - k > h - k$. First we introduce $t := s-k$ $[\geqq 0]$ as a new summation variable. Then

$$\begin{aligned} &\sum_{s=a'+1}^{\kappa} \tbinom{h}{s}(\lambda'-s)^{h-s-1} \sum_{k=a'+1}^{s} \tbinom{s}{k}(n\gamma\alpha)^k(s-\kappa')^{s-k} \\ &= \sum_{t=0}^{\kappa-a'-1} \sum_{k=a'+1}^{\kappa-t} \tbinom{h}{k}(n\gamma\alpha)^k \tbinom{h-k}{t}(-(\kappa'-k)+t)^t(\lambda'-k-t)^{(h-k)-t-1} \\ &= \sum_{k=a'+1}^{\kappa} \tbinom{h}{k}(n\gamma\alpha)^k\{[(\lambda'-\kappa')^{h-k}/(\lambda'-h)] \\ &\qquad - \sum_{t=\kappa-k+1}^{h-k} \tbinom{h-k}{t}(-(\kappa'-k)+t)^t(\lambda'-k-t)^{(h-k)-t-1}\} \end{aligned} \tag{2.28}$$

where now in the sum over t all terms are positive. (2.28) equals, summing in the double sum over k and over the new variable

$$\tau := t + k,$$

$$\frac{(n\gamma\beta)^h}{\lambda' - h}(B(\kappa; h, \alpha/\beta) - B(a'; h, \alpha/\beta))$$
$$- \sum_{\tau=\kappa+1}^{h} \binom{h}{\tau}(\lambda' - \tau)^{h-\tau-1}(\tau - n\delta)^{\tau}\left\{B\left(\kappa; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right) - B\left(a'; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right)\right\}$$
$$= \frac{(n\gamma\beta)^h}{\lambda' - h}(B(\kappa; h, \alpha/\beta) - B(a'; h, \alpha/\beta))$$
$$-(n\gamma\beta)^h \sum_{\tau=\kappa+1}^{h} \frac{1}{\lambda' - \tau} b\left(\tau; h, \frac{\tau - n\delta}{n\gamma\beta}\right)\left\{B\left(\kappa; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right) - B\left(a'; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right)\right\}.$$

To summarize, we have by now for (1.36)

(2.29) $P(ACD)$

$$= B(b^*; n, \beta) - P(BC^c)$$
$$- \textstyle\sum_{s=\kappa+1}^{b^*} \sum_{h=s}^{b^*} \binom{n}{h}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda' - h)\binom{h}{s}(s - n\delta)^s(\lambda' - s)^{h-s-1}$$
$$- \textstyle\sum_{h=\kappa+1}^{b^*} b(h; n, \beta)\Big\{B(\kappa; h, \alpha/\beta) - B(a'; h, \alpha/\beta)$$
$$- \sum_{\tau=\kappa+1}^{h} \frac{\lambda' - h}{\lambda' - \tau} b\left(\tau; h, \frac{\tau - n\delta}{n\gamma\beta}\right)\left[B\left(\kappa; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right) - B\left(a'; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right)\right]\Big\}.$$

Using Lemma 2.1 below, the first summation over h may be performed, since $s < \lambda$ resulting in

(2.29a) $P(ACD) = B(b^*; n, \beta) - P(BC^c)$

$$- \sum_{s=\kappa+1}^{b^*} b\left(s; n, \frac{s - n\delta}{n\gamma}\right)\Psi(s, b^* - s)$$
$$- \textstyle\sum_{h=\kappa+1}^{b^*} b(h; n, \beta)\{B(\kappa; h, \alpha/\beta) - B(a'; h, \alpha/\beta)\}$$
$$+ \sum_{\tau=\kappa+1}^{b^*} \left[B\left(\kappa; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right) - B\left(a'; \tau, \frac{n\gamma\alpha}{\tau - n\delta}\right)\right]$$
$$\times \left(\frac{\tau - n\delta}{\lambda' - \tau}\right)^{\tau} \frac{1}{\lambda' - \tau} \sum_{h=\tau}^{b^*} \binom{n}{h}\binom{h}{\tau}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda' - \tau)^h(\lambda' - h).$$

The expression in the last row equals $b(\tau; n, (\tau - n\delta/(n\gamma)))\Psi(\tau, b^* - \tau)$ again by Lemma 2.1. This yields (1.29), if the probability of BC^c is determined analogously to (2.21a):

$$
\begin{aligned}
P(BC^c) &= \textstyle\sum_{k=a'+1}^{K} P(F_n(\beta) = k/n, (k-a')/n > \Delta) \\
\text{(2.29b)} \qquad &= \textstyle\sum_{k=a'+1}^{K} \sum_{m=0}^{k-a'-1} P(F_n(\beta) = k/n, \Delta = m/n) \\
&= \textstyle\sum_{k=a'+1}^{K} \sum_{m=0}^{k-a'-1} b(k; n, \beta) b(k-m; k, \alpha/\beta) \\
&= \textstyle\sum_{k=a'+1}^{K} b(k; n, \beta)(1 - B(a'; k, \alpha/\beta)).
\end{aligned}
$$

This proves Theorem 1.2.

LEMMA 2.1. *For any integer* $K \in [\kappa+1, \lambda]$, *and* $s \in [n\delta, K]$, with $s < \lambda'$, $K \leqq n$, *and with* $p(s)$ *given by* (1.12),

$$
\begin{aligned}
&\left(\frac{s-n\delta}{\lambda'-s}\right)^s \frac{1}{\lambda'-s} \sum_{h=s}^{K} \binom{n}{h}\binom{h}{s}(1-\beta)^{n-h}(n\gamma)^{-h}(\lambda'-h)(\lambda'-s)^h \\
\text{(2.30)} \qquad &= \binom{n}{s}\left(\frac{s-n\delta}{n\gamma}\right)^s \frac{1}{\lambda'-s} \sum_{h=s}^{K} \binom{n-s}{h-s}(1-\beta)^{n-s-(h-s)}\left(\frac{\lambda'-s}{n\gamma}\right)^{h-s} (\lambda'-s-(h-s)) \\
&= \binom{n}{s}\left(\frac{s-n\delta}{n\gamma}\right)^s \left(\frac{n(\gamma+\delta)-s}{n\gamma}\right)^{n-s} \sum_{h'=0}^{K-s} b(h'; n-s, p(s))\left(1 - \frac{h'}{\lambda'-s}\right) \\
&= b\left(s; n, \frac{s-n\delta}{n\gamma}\right)\Psi(s, K-s)
\end{aligned}
$$

where for $\chi = 0, 1, \cdots, n-s$ $\Psi(s, \chi) := \sum_{h=0}^{\chi} b(h; n-s, p(s))(1 - h/(\lambda'-s))$. *Using* (1.13e), Ψ *can be brought into the form* (1.30).

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