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DESIGNS FOR REGRESSION PROBLEMS WITH CORRELATED ERRORS III

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1. Introduction. Consider the linear regression model in which one may observe a stochastic process Y having the form

$$Y(t) = \sum_{j=1}^{J} \beta_j f_j(t) + Z(t), \qquad t \in [0, 1]. \tag{1.1}$$

Here the β_j are taken as unknown constants, the f_j as known functions and Z is assumed to have mean function zero and known covariance kernel R. Let T be a subset of $[0, 1]$ and let $\hat{\beta}_T$ denote the best linear estimate (if it exists) of $\beta = (\beta_1, \cdots, \beta_J)'$ based on observing $\{Y(t), t \in T\}$. When the covariance matrix of $\hat{\beta}_T$ is nonsingular it will be denoted by A_T^{-1}; when $T = [0, 1]$ we will use the notation A^{-1}.

In an earlier paper [1], we treated the special case $J = 1$ of (1.1). The problem posed was that of finding a member T_n in the class $\mathscr{D}_n = \{T \mid T = \{t_0, t_1, \cdots, t_n\}, 0 = t_0 < t_1 < \cdots < t_n = 1\}$ of all $n+1$ point "designs" for which $A_{T_n}^{-1} = \inf_{T \in \mathscr{D}_n} A_T^{-1}$. We assumed there that $f_1 = f$ had the form

$$f(t) = \int_0^1 R(s, t)\varphi(s)\, ds \tag{1.2}$$

for some continuous function φ and that R satisfied assumptions slightly weaker than those labelled A, B and C in Section 2 below (see also the Remark at the end of Section 2). It was then shown that

$$\inf_{T \in \mathscr{D}_n} A_T^{-1} - A^{-1} = \frac{c^3(\varphi)}{12n^2A^2} + o(1) \tag{1.3}$$

$$A_{T_n^*}^{-1} - A^{-1} = \frac{c^3(\varphi)}{12n^2A^2} + o(1) \tag{1.4}$$

where T_n^* is a set of n-tiles of the probability distribution function with density $c^{-1}(\varphi)\varphi^{\frac{2}{3}}$. Thus our approximate solution to the design problem in $\mathscr{D}_n$ is T_n^*. We say when (1.3) and (1.4) are satisfied that sampling according to $\varphi^{\frac{2}{3}}$ is asymptotically optimum.

In a second paper [2], the full model (1.1) was discussed. There, for a variety of criteria ψ which would measure the size of A_T^{-1} (e.g. the generalized variance), we sought T_n in $\mathscr{D}_n$ for which $\psi(A_{T_n}^{-1}) = \inf_{T \in \mathscr{D}_n} \psi(A_T^{-1})$. It was assumed that each f_j had the form (1.2) with associated φ_j and that R was subject to the same restrictions as in [1]. Our results then had the following character: given a criterion

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ψ, sampling according to $(\sum_{i,j}\varphi_i\varphi_j M_{ij}(\psi))^{\frac{1}{3}}$ is asymptotically optimum where $M(\psi)$ is a suitable nonnegative definite matrix.

It is our intention here to extend results of this type to a different though related class of covariance kernels. The main feature of our assumptions on R in [1] and [2] was that R should be non-smooth at diagonal points in the unit square as exemplified by kernels like $R(s,t) = \min(s,t)$ and $R(s,t) = \exp(-|s-t|)$. This guaranteed that the process Z had no quadratic mean derivatives. In the present paper we permit Z to have such derivatives. In fact, using the notation $a_+{}^n = a^n$ if $a > 0$, $= 0$ if $a \leqq 0$, we assume that, for some integer $k \geqq 0$,

$$Z(t) = \int_0^1 X(u)\frac{(t-u)_+{}^{k-1}}{(k-1)!}\,du, \qquad t \in [0,1], \tag{1.5}$$

where the process X has mean function zero and covariance kernel K satisfying Assumptions A, B, C of Section 2. It follows that $Z^{(k)} = X$ (in quadratic mean) and that

$$R(s,t) = \int_0^1\int_0^1 \frac{(s-u)_+{}^{k-1}(t-v)_+{}^{k-1}}{(k-1)!^2}K(u,v)\,du\,dv. \tag{1.6}$$

Processes Z which satisfy (1.5) for $k > 0$ cannot, of course, be stationary and this restricts the class of Z's for which our results are applicable. However, we will note (see the remark following Theorem 1) that the right side of (2.27) is an *upper* bound if Z is *any* process with $Z^{(k)} = X$ where X satisfies the above assumptions. For some practical purposes this may be enough. In a recent unpublished manuscript, G. Wahba [4] has obtained results like ours for a class of processes not covered by (1.5) and including some stationary processes.

In Section 2 it will be shown that results analogous to those in [1] and [2] obtain provided $A_T{}^{-1}$ is replaced by $A_{k,T}^{-1}$, the covariance matrix of the best linear estimate of β based on observing $\{Y(t), Y'(t), \cdots, Y^{(k)}(t), t \in T\}$. With this modification and $J = 1$, the convergence rate of A_{k,T_n}^{-1} to A^{-1} is $O(n^{-2k-2})$ and sampling according to $\varphi^{2/2k+3}$ is asymptotically optimum. Similarly, for $J > 1$ we can demonstrate the asymptotic optimality of sampling according to $(\sum_{i,j}\varphi_i\varphi_j M_{ij}(\psi))^{1/2k+3}$ for various criteria ψ. While the observation sets $\{Y(t), Y'(t), \cdots, Y^{(k)}(t), t \in T\}$ are not the natural ones, these results do provide us with lower bounds on the convergence rates of $A_{T_n}^{-1}$ to A^{-1}. Further, we can obtain upper bounds on these rates which are of the same order of magnitude.

The case $k = 1$ is considered in greater detail in Section 3. The result there is that use of $\{Y'(t), t \in T\}$ does not improve the asymptotic convergence rates, i.e., when $k = 1$ we can demonstrate the asymptotic optimality of our sampling schemes for the more natural observation sets $\{Y(t), t \in T\}$.

As was the case in [1] and [2], the behavior of linear combinations of the functions φ_j around their zeros presents the main obstacle to simple statements of theorems and short-winded proofs. To counter this, we adopt a somewhat different approach than the one taken in [2]. We place here a condition on the φ_j's which allows reasonable proofs of all our results. The uniformity this engenders means that most of our theorems are true under less restrictive assumptions and, indeed, we have no example of a function φ_1 for which the basic Theorem 1 fails. The specific assumption we make on the φ_j's is stated just prior to Theorem 1 in Section 2.

The proof of Theorem 1 is due to the referee and is an improvement on our original proof which was less elegant and required a superfluous condition on K. We thank the referee for his remarks.

What we do in the present paper is related to certain integral prediction problems and to the theory of spline approximation. These details will appear in [3].

2. Main results. C_r is the space of functions r times continuously differentiable on [0, 1]. D will denote the derivative operator. If G is a function of two variables on the unit square, we will denote

$$\frac{\partial^{p+q}}{\partial u^p\, \partial v^q} G(u, v)\,\Big|_{u=s, v=t}$$

by $G^{p,q}(s, t)$ and we let, for example, $G^{p,q}_{+-}(s_0, t_0) = \lim_{s\downarrow s_0, t\uparrow t_0} G^{p,q}(s, t)$.

Suppose $\{Z(t), t\in[0, 1]\}$ is a stochastic process as defined through the process $\{X(t), t\in[0, 1]\}$ in (1.5). The covariance kernel R of Z is then given in terms of the covariance kernel K of X by (1.6). Let $\mathcal{M}$ denote the L_2-space generated by the random variables $\{Z(t), t\in[0, 1]\}$. Since $D^kZ(t) = X(t)$ in quadratic mean, $\mathcal{M}$ is also the L_2-space generated by $\{X(t), t\in[0, 1]\}$. Let $H(R)$ and $H(K)$ denote the reproducing kernel spaces associated with the kernels R and K. $\mathcal{M}$ and $H(R)$ are isomorphic under the mapping $U\to EUZ(\cdot)$. Let $\mathcal{M}_0$ be a closed subspace of $\mathcal{M}$. If f, H_0 are the isomorphic images of U and $\mathcal{M}_0$ then $\|f-P_{H_0}f\|^2_{H(R)} = E(U-P_{\mathcal{M}_0}U)^2$ where P_{H_0} is projection of $H(R)\to H_0$ and $P_{\mathcal{M}_0}$ is projection from $\mathcal{M}\to\mathcal{M}_0$. We also note that $D^k: H(R)\to H(K)$ is an isomorphism since $E[UD^kZ(\cdot)] = D^kE[UZ(\cdot)]$.

An alternative way to view the projections is to assume that X is a Gaussian process (there is no loss in doing so), to let $\mathcal{S}$ be the σ-algebra generated by the set of random variables which generates $\mathcal{M}_0$, and then notice that $E[U-P_{\mathcal{M}_0}U]^2 = \text{Var}(U\,|\,\mathcal{S}) = E([U-E(U\,|\,\mathcal{S})]^2\,|\,\mathcal{S})$.

The assumptions we make on K are like those used in [1] and [2]. Assumptions A and C below are identical with A and C in [1] while the present B is a restricted version of B in [1] which nevertheless covers the examples of most interest. A discussion of them can be found in [1].

ASSUMPTION A. Let p and q be nonnegative integers with $p+q \leqq 2$. If $s\neq t$' $K^{p,q}(s, t)$ exists and is continuous and $K^{p,q}_{++}(t, t)$, $K^{p,q}_{+-}(t, t)$, etc., all exist.

ASSUMPTION B. $K^{1,0}_{--}(t, t) - K^{1,0}_{++}(t, t) = 1$ for all $t\in(0, 1)$. (1 can be replaced by any positive constant.)

ASSUMPTION C. $K^{2,0}_{++}(s, \cdot)\in H(K)$ for each $s\in[0, 1]$ and

$$\sup_{0\leqq s\leqq 1} \|K^{2,0}_{++}(s, \cdot)\|_{H(K)} < \infty.$$

An example of particular importance (Lemma 1 below) is obtained by taking $K(s, t) = \min(s, t)$—the Brownian motion covariance kernel. In this case we get from (1.6)

$$R(s,t) = \int_0^1 \frac{(s-u)_+{}^k(t-u)_+{}^k}{k!^2}\,du. \tag{2.1}$$

Let Z be an associated Gaussian process, i.e. the kth integral of a Brownian motion process and define

$$W(t) = Z(t) - E[Z(t)\,|\,D^rZ(1),\ r = 0, 1, \cdots, k]. \tag{2.2}$$

Then

$$B_k(s,t) = EW(s)W(t) = EZ(s)Z(t) - \gamma(s)'B^{-1}\gamma(t) \tag{2.3}$$

where B is the covariance matrix of $\{D^0Z(1), DZ(1), \cdots, D^kZ(1)\}$ and $\gamma(s)$ is the vector with rth coordinate $E[Z(s)D^{r-1}Z(1)] = R^{0,r-1}(s, 1)$. (2.3) is easy to obtain because $E[Z(s)\,|\,D^rZ(1),\ r = 0, 1, \cdots, k]$ is a linear combination of the $D^rZ(1)$'s and the vector of coefficients of this linear combination can be written as $B^{-1}\gamma(s)$. The point of introducing the kernel B_k is the following

LEMMA 1. *If $h \in C_{2k+2}$ and $D^rh(0) = D^rh(1) = 0$ for $0 \leqq r \leqq k$, then*

$$\textstyle\int_0^1 D^{2k+2}h(s)B_k(s,t)\,ds = (-1)^{k+1}h(t). \tag{2.4}$$

PROOF. It is not hard to obtain from (2.3) that $B_k^{r,0}(0, t) = B_k^{r,0}(1, t) = 0$ for $0 \leqq r \leqq k$. This and the assumptions on h yield, after successive integrations by parts,

$$\begin{aligned}\textstyle\int_0^1 D^{2k+2}hB_k(\cdot,t) &= \textstyle(-1)^{k+1}\int_0^1 D^{k+1}hB_k^{k+1,0}(\cdot,t)\\ &= \textstyle(-1)^{2k+1}\int_0^1 DhB_k^{2k+1,0}(\cdot,t).\end{aligned} \tag{2.5}$$

On the set where $s \neq t$ we have from (2.1)

$$D^{2k+1}EZ(\cdot)Z(t) = D^{2k+1}R(\cdot,t) = (-1)^kI_{[0,t]}(\cdot) \tag{2.6}$$

where $I_{[0,t]}(\cdot)$ is the indicator function of $[0, t]$. Since $R^{2k+1,p}(s, t) = 0$ if $s \neq t$ and $p > 0$ and $R^{2k+1,0}(\cdot, 1) = (-1)^k$ on $[0, 1)$ we have $D^{2k+1}\gamma(\cdot)$ is constant on $[0, 1)$ so that $D^{2k+1}\gamma(\cdot)'B^{-1}\gamma(t)$ is constant on $[0, 1)$. This last fact together with (2.6) gives $B_k^{2k+1,0}(s, t) = (-1)^kI_{[0,t]} + c$ if $s \neq t$, $s \neq 1$. This and the assumption that $h(0) = h(1) = 0$ allows the evaluation of the right-hand expression in (2.5) giving (2.4) and proving Lemma 1.

For later use we want the following version of Lemma 1.

COROLLARY. *If $0 \leqq a < b \leqq 1$, $h \in C_{2k+2}([a, b])$ and $D^rh(a) = D^rh(b) = 0$ for $0 \leqq r \leqq k$ then, putting*

$$Q_{a,b}(x,y) = (b-a)^{2k+1}B_k\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right), \tag{2.7}$$

we have

$$\int_a^b D^{2k+2}h(x)Q_{a,b}(x,y)\,dx = (-1)^{k+1}h(y). \tag{2.8}$$

Note that by choosing $h(x) = (x^{k+1}(1-x)^{k+1}(-1)^{k+1})/(2k+2)!$ we can apply Lemma 1 and its corollary to obtain

$$\begin{aligned}\int_a^b\int_a^b Q_{a,b}(x,y)\,dx\,dy &= (b-a)^{2k+3}\int_0^1\int_0^1 B_k(x,y)\,dx\,dy \\ &= (b-a)^{2k+3}\frac{(k+1)!^2}{(2k+2)!(2k+3)!}.\end{aligned} \tag{2.9}$$

We now turn to the general setup and suppose that K satisfies Assumptions A, B, C and that R is given by (1.6). Let $\varphi \in C_0$ and let $f \in H(R)$ be of the special form

$$f(s) = \int_0^1 R(s,t)\varphi(t)\,dt. \tag{2.10}$$

In case $K(s,t) = \min(s,t)$ we have

$$\varphi = (-1)^{k+1}D^{2k+2}f. \tag{2.11}$$

Let $T = \{t_0, t_1, \cdots, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n = 1$. Let $L(k, T)$ be the subspace in $H(R)$ spanned by $\{R(\cdot, t), R^{0,1}(\cdot, t), \cdots, R^{0,k}(\cdot, t), t \in T\}$. $P_{k,T}$ will denote the projection operator from $H(R)$ onto $L(k, T)$. For a given $f \in H(R)$, $P_{k,T}f$ is characterized by the fact that

$$D^r(f - P_{k,T}f)(t) = 0 \qquad \text{if } t \in T \text{ and } 0 \leqq r \leqq k. \tag{2.12}$$

When $K(s,t) = \min(s,t)$ and f is of the form (2.10) we get (see (2.11) also)

$$D^{2k+2}(f - P_{k,T}f) = D^{2k+2}f = (-1)^{k+1}\varphi \quad \text{on} \quad [0,1] - T. \tag{2.13}$$

LEMMA 2. *Let $K(s,t) = \min(s,t)$ and let f be of the form* (2.10). *Then*

$$\|f - P_{k,T}f\|_{H(R)}^2 = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\varphi(x)\varphi(y)Q_{t_i,t_{i+1}}(x,y)\,dx\,dy = \bar{Q}_k(\varphi) \ (say). \tag{2.14}$$

If $k = 0$, K satisfies Assumptions A, B, C, *and f is of the form* (2.10) *then, setting* $\delta = \sup_{0 \leqq j \leqq n-1}(t_{j+1} - t_j)$,

$$1 - a_0\delta \leqq \|f - P_T f\|_{H(K)}^2/\bar{Q}_0(\varphi) \leqq 1 + a_0\delta \tag{2.15}$$

where a_0 is a constant independent of φ.

PROOF. Abbreviate $Q_{t_i,t_{i+1}}$ by Q_i. Since $P_{k,T}f$ is orthogonal to $f - P_{k,T}f$ we can write, using (2.10),

$$\begin{aligned}\|f - P_{k,T}f\|^2 = \langle f - P_{k,T}f, f\rangle &= \int_0^1 \varphi(y)(f - P_{k,T}f)(y)\,dy \\ &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\varphi(y)(f - P_{k,T}f)(y)\,dy.\end{aligned}$$

Now use (2.8), (2.12), and (2.13) in the last expression and obtain (2.14).

To obtain (2.15) we appeal to Lemma 3.2 of [2] which is applicable directly when $\varphi \geqq 0$. For arbitrary φ the proof of Lemma 3.2 of [2] is easily adapted to give (2.15) when $\alpha(t) = K_{--}^{1,0}(t,t) - K_{++}^{1,0}(t,t) = 1$.

For h a continuous density on [0, 1], let $T_n = \{t_{0n}, t_{1n}, \cdots, t_{nn}\}$ be defined by

$$\int_0^{t_{in}} h(x)\,dx = i/n, \qquad i = 0, 1, 2, \cdots, n, \tag{2.16}$$

with the convention (in case of ambiguity) that $t_{0n} = 0$ and $t_{nn} = 1$. In other cases of ambiguity (which will arise if h is zero on some intervals) take any t_{in} which satisfies (2.16) e.g., the smallest such. The sequence $\{T_n; n \geqq 1\}$ so defined is called a *Regular Sequence generated by h* (RS(h)). By parroting the proofs of Lemma 3.3 and Lemma 3.4 in [2] and using (2.9) we can prove

LEMMA 3. *If* $K(s, t) = \min(s, t)$ *and* $\{T_n\}$ *is* RS(h) *then*

$$\liminf_{n\to\infty} n^{2k+2}\,||f - P_{k,T_n}f||^2 \geqq \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\int_0^1 \varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx.$$

(If the right-hand side of this inequality is infinite, so is the left-hand side. Also, the integrand is 0 at any x for which $0 = h(x) = \varphi(x)$).

We would like to demonstrate that

$$\lim_{n\to\infty} n^{2k+2}\,||f - P_{k,T_n}f||^2 = \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\int_0^1 \varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx \tag{2.17}$$

for $\{T_n\}$ a RS(h) but have been unable to do so without restrictions on φ and/or h. We choose therefore to restrict φ in such a way that (2.17) holds for all h (in contrast to placing a variety of assumptions on h as was ultimately done in [2]). This enables us to prove all consequent results with little further ado. Specifically, we will assume φ satisfies the condition

(*) φ has at most finitely many zeros (perhaps none) and if $\varphi(z) = 0$ then in some neighborhood of z there are numbers $0 < m = m(z) < M = M(z)$ and $p = p(z) > 0$ so that for any x in this neighborhood, $m|z-x|^p \leqq |\varphi(x)| \leqq M|z-x|^p$.

It may be observed below that we use only the consequence (2.21) of (*). Amongst other possible assumptions on φ, (2.21) holds (with a slight modification) if φ has finitely many intervals of zeros and around these zeros φ behaves as required in (*). We will not spell this out.

To prove Lemma 5 we first establish

LEMMA 4. *Suppose* φ *satisfies* (*). *There exists a* $\gamma > 0$ *and a* $B > 0$ *so that for any interval* (a, b) *with* $b - a \leqq \gamma$,

$$\int_a^b\int_a^b \varphi(x)\varphi(y)Q_{a,b}(x, y)\,dx\,dy \leqq B\left(\int_a^b \varphi^{2/2k+3}(x)\,dx\right)^{2k+3}. \tag{2.18}$$

PROOF. Let the zeros of φ be $z_1, \cdots, z_r$ and cover the z_i's with disjoint intervals I_i on which $m_i|x-z_i|^{p_i} \leqq |\varphi(x)| \leqq M_i|x-z_i|^{p_i}$ for choices of $0 < m_i < M_i$ and $p_i > 0$ as provided by (*). We may suppose these intervals are symmetrically placed around the z_i's which are in (0, 1) and that the shortest I_i has length 4γ.

Suppose η is such that $|\varphi(x)| \geqq \eta > 0$ for $x \notin \bigcup_i (z_i - \gamma, z_i + \gamma)$. Using (2.1), (2.3) and (2.7)

$$\int_a^b \int_a^b \varphi(x)\varphi(y) Q_{a,b}(x, y)\, dx\, dy$$

$$= \int_a^b \int_a^b \varphi(x)\varphi(y)(b-a)^{2k+1} B_k\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right) dx\, dy$$

$$\leqq \int_a^b \int_a^b \varphi(x)\varphi(y)(b-a)^{2k+1}\left[\int_0^1 \frac{\left(\frac{x-a}{b-a}-u\right)_+^k \left(\frac{y-a}{b-a}-u\right)_+^k}{k!^2}\, du\right] dx\, dy \tag{2.19}$$

$$= (b-a)^{2k+1} \int_0^1 \left[\int_a^b \varphi(x) \frac{\left(\frac{x-a}{b-a}-u\right)_+^k}{k!}\, dx\right]^2 du$$

$$= (b-a)^{2k+3} \int_0^1 \left[\int_0^1 \varphi((b-a)v+a) \frac{(v-u)_+{}^k}{k!}\, dv\right]^2 du$$

$$\leqq \frac{(b-a)^{2k+3}}{k!^2}\left[\int_0^1 |\varphi((b-a)v+a)|\, dv\right]^2.$$

Since

$$\left(\int_a^b \varphi^{2/2k+3}(x)\, dx\right)^{2k+3} = (b-a)^{2k+3}\left(\int_0^1 \varphi^{2/2k+3}((b-a)v+a)\, dv\right)^{2k+3} \tag{2.20}$$

to obtain (2.18) we need only find an upper bound for the ratio

$$\left[\int_0^1 |\varphi((b-a)v+a)|\, dv\right]^2 \Big/ \left(\int_0^1 \varphi^{2/2k+3}((b-a)v+a)\, dv\right)^{2k+3} \tag{2.21}$$

for all (a, b), $b-a \leqq \gamma$. If (a, b) intersects no $(z_i-\gamma, z_i+\gamma)$, (2.21) is no larger than $\max|\varphi|^2/\eta^2$. If (a, b) intersects $(z_i-\gamma, z_i+\gamma)$ then for all $x \in (a, b)$, $m_i|x-z_i|^{p_i} \leqq |\varphi(x)| \leqq M_i|x-z_i|^{p_i}$. Suppose first that $z_i \notin [a, b]$, say $z_i < a$, and write $\theta = (a-z_i)/(b-a)$. Using the bounds on $|\varphi|$, (2.21) is bounded by

$$\frac{\frac{M_i^2}{(1+p_i)^2}\left[(1+\theta)^{1+p_i} - \theta^{1+p_i}\right]^2}{m_i^2\left(\frac{2k+3}{2p_i+2k+3}\right)^{2k+3}\left[(1+\theta)^{1+(2p_i/2k+3)} - \theta^{1+(2p_i/2k+3)}\right]^{2k+3}} \tag{2.22}$$

and this ratio is bounded in $\theta \geqq 0$. If on the other hand $z_i \in [a, b]$, write $\theta = (z_i-a)/(b-a)$ and bound φ as before to get (2.21) bounded by

$$\frac{\frac{M_i^2}{(1+p_i)^2}\left[\theta^{1+p_i} + (1-\theta)^{1+p_i}\right]^2}{m_i^2\left(\frac{2k+3}{2p_i+2k+3}\right)^{2k+3}\left[\theta^{1+(2p_i/2k+3)} + (1-\theta)^{1+(2p_i/2k+3)}\right]^{2k+3}} \tag{2.23}$$

(2.23) in turn is bounded on $0 \leqq \theta \leqq 1$. The existence of B follows and this finishes the proof of Lemma 4.

LEMMA 5. *Let $K(s, t) = \min(s, t)$ and let f be of the form* (2.10) *with φ satisfying* (*). *Then* (2.17) *holds for any continuous h.*

PROOF. Because of Lemma 3, we need only consider h's for which $\varphi^2[h^{2k+2}]^{-1}$ is integrable and it will suffice to show

$$(2.24)\quad \limsup_{n\to\infty} n^{2k+2}\,\|f-P_{k,T_n}f\|^2 \leqq \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\int_0^1 \varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx\,.$$

If $\varphi^2[h^{2k+2}]^{-1}$ is integrable and $\{T_n\}$ is RS (h) then, from Lemma 2,

$$\begin{aligned}&\limsup_{n\to\infty} n^{2k+2}\,\|f-P_{k,T_n}f\|^2\\ &\quad= \limsup_{n\to\infty} n^{2k+2}\textstyle\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\varphi(x)\varphi(y)Q_{t_i,t_{i+1}}(x,y)\,dx\,dy.\end{aligned}$$

In the sum on the right-hand side let $I_1 = \{i \mid h(x) > \varepsilon \text{ for all } x\in[t_i, t_{i+1}]\}$ and let $I_2 = \{0, 1, \cdots, n-1\} - I_1$. As in (3.27) of [2] we have

$$(2.25)\quad \begin{aligned}&n^{2k+2}\textstyle\sum_{i\in I_1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\varphi(x)\varphi(y)Q_{t_i,t_{i+1}}(x,y)\,dx\,dy\\ &\quad= \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\int_{h>\varepsilon}\varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx + o(1).\end{aligned}$$

The uniform continuity of h and the fact that $\delta_n = \sup_{0\leqq j\leqq n-1}(t_{j+1}-t_j)\to 0$ implies that, for n large enough, $h(x) < 2\varepsilon$ for $x\in[t_i, t_{i+1}]$ and all $i\in I_2$. For $i\in I_2$, Lemma 4 and then a Hölder inequality give, provided n is large enough that $\delta_n \leqq \gamma$,

$$\begin{aligned}&\textstyle\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\varphi(x)\varphi(y)Q_{t_i,t_{i+1}}(x,y)\,dx\,dy\\ &\quad\leqq B(\textstyle\int_{t_i}^{t_{i+1}}\varphi^{2/2k+3}(x)\,dx)^{2k+3}\\ &\quad\leqq B(\textstyle\int_{t_i}^{t_{i+1}} h)^{2k+2}\int_{t_i}^{t_{i+1}}\varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx\\ &\quad= (B/n^{2k+2})\textstyle\int_{t_i}^{t_{i+1}}\varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx.\end{aligned}$$

Thus

$$(2.26)\quad \begin{aligned}&n^{2k+2}\textstyle\sum_{i\in I_2}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\varphi(x)\varphi(y)Q_{t_i,t_{i+1}}(x,y)\,dx\,dy\\ &\quad\leqq B\textstyle\sum_{i\in I_2}\int_{t_i}^{t_{i+1}}\varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx\\ &\quad\leqq B\textstyle\int_{h<2\varepsilon}\varphi^2(x)[h^{2k+2}(x)]^{-1}\,dx.\end{aligned}$$

(2.24) follows from (2.25) and (2.26) if we let $n\to\infty$ and then $\varepsilon\to 0$.

THEOREM 1. *Let Z satisfy* (1.5) *with K satisfying Assumptions* A, B, C. *Let f be of the form* (2.10) *with φ satisfying* (*). *Then, if h is a continuous density and $\{T_n\}$ is* RS (h)

$$(2.27)\quad \lim_{n\to\infty} n^{2k+2}\,\|f-P_{k,T_n}f\|_{H(R)}^2 = \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\int_0^1 \varphi^2(x)[h(x)]^{-2k-2}\,dx.$$

PROOF. Let $K_0(s,t) = \min(s,t)$. Let $\tilde{H} = \{g \in H(K) \mid g(s) = \int_0^1 K(s,t)\psi(t)\,dt,\ \psi$ continuous$\}$ and let $\tilde{H}_0$ be the corresponding subset of $H(K_0)$. Let $\mu: \tilde{H} \to \tilde{H}_0$ by $\int K\psi \to \int K_0 \psi$. According to Lemma 2

$$(2.28)\qquad (1-a_0\delta) \leqq \|g-P_T g\|_{H(K)}^2 / \|\mu(g)-P_T\mu(g)\|_{H(K_0)}^2 \leqq 1+a_0\delta$$

for all $g \in \tilde{H}$. Since $\tilde{H}\,(\tilde{H}_0)$ is dense in $H(K)(H(K_0))$ μ can be uniquely extended to a mapping from $H(K)$ onto $H(K_0)$ and (2.28) will hold for all $g \in H(K)$. (2.28) can be written more conveniently as

$$(2.29)\qquad (1-a_0\delta) \leqq \frac{\inf_{\{c_i\}} E[U-\sum_0^n c_i X(t_i)]^2}{\inf_{\{\theta_i\}} E[\bar{\mu}(U)-\sum_0^n \theta_i X_0(t_i)]^2} \leqq 1+a_0\delta$$

where $U \in \mathscr{M} = L_2$-space of X (see the second paragraph of this section), $\bar{\mu}$ is the mapping of $\mathscr{M}$ onto the L_2-space of X_0 induced by μ (X_0 is, of course, the process with covariance kernel K_0).

There is a $U \in \mathscr{M}$ such that

$$\|f-P_{k,T}f\|_{H(R)}^2 = E[U-\sum_{r=0}^k \sum_{i=0}^n b_{ir}^* Z^{(r)}(t_i)]^2$$

where $U^* = U-\sum_{i,r} b_{ir}^* Z^{(r)}(t_0)$ is orthogonal to $\{X(t_i);\ i=0,\cdots,n\}$ (recall $Z^{(k)} = X$), and lies in $\mathscr{M}$. Hence, by (2.29),

$$\begin{aligned}
\|f-P_{k,T}f\|_{H(R)}^2 &\geqq (1-a_0\delta)\inf_{\{\theta_i\}} E[\bar{\mu}(U^*)-\textstyle\sum \theta_i X_0(t_i)]^2 \\
(2.30)\qquad &= (1-a_0\delta)\inf_{\{\theta_i\}} E[\bar{\mu}(U)-\textstyle\sum_{i,n} b_{ir}^* Z_0{}^{(r)}(t_i)-\sum_i \theta_i X_0(t_i)]^2 \\
&\geqq (1-a_0\delta)\inf_{\{\gamma_{ir}\}} E[\bar{\mu}(U)-\textstyle\sum \gamma_{ir} Z^{(r)}(t_i)]^2 \\
&= (1-a_0\delta)\|\mu(f)-P_{k,T}\mu(f)\|_{H(R_0)}^2 .
\end{aligned}$$

Interchanging the role of K_0, K, etc. we obtain from (2.30)

$$(2.31)\qquad 1-a_0\delta \leqq \|f-P_{k,T}f\|_{H(R)}^2 / \|\mu(f)-P_{k,T}\mu(f)\|_{H(R_0)}^2 \leqq 1+a_0\delta$$

If f is of the form (2.10) then $\mu(f)(s) = \int_0^1 R_0(s,t)\varphi(t)\,dt$ so we can use Lemma 5 and (2.31) to obtain (2.27). The proof of Theorem 1 is complete.

REMARK. Suppose Z is a process with $Z^{(k)} = X$ and the covariance kernel K of X satisfies Assumptions A, B, C. We are not assuming that Z satisfies (1.5). If we let $\bar{Z}(t) = Z(t) - \sum_{j=0}^{k-1} t^j Z^{(j)}(0)$ then $\bar{Z}$ does satisfy (1.5) and $\bar{Z}^{(k)} = Z^{(k)} = X$ so that Theorem 1 is applicable to $\bar{Z}$. Since

$$\begin{aligned}
\|f-P_{k,T}f\|_{H(R)}^2 &= \inf_{\{c_{ir}\}} E[\textstyle\int_0^1 \varphi(t)Z(t)\,dt - \sum_{i=0}^n \sum_{r=0}^k c_{ir} Z^{(r)}(t_i)]^2 \\
&= \inf_{\{a_j\},\{\gamma_{ir}\}} E[\textstyle\int \varphi\bar{Z} - \sum_0^{k-1} a_j Z^{(j)}(0) - \sum_{i,r}\gamma_{ir}\bar{Z}^{(r)}(t_i)]^2 \\
&\leqq \inf_{\{\gamma_{ir}\}} E[\textstyle\int \varphi\bar{Z} - \sum_{i,r}\gamma_{ir}\bar{Z}^{(r)}(t_i)]^2
\end{aligned}$$

we can apply Theorem 1 and obtain the result that the right side of (2.27) is an upper bound for $\limsup_{n\to\infty} n^{2k+2}\|f-P_{k,T_n}f\|_{H(R)}^2$.

THEOREM 2. *Let $f_1, \cdots, f_p$ be p functions of the form* (2.10) *with associated continuous functions $\varphi_1, \cdots, \varphi_p$ each satisfying* (*). *Let $a_1, \cdots, a_p$ be positive numbers. If $\{T_n\}$ is any sequence of designs, then*

$$\liminf_{n\to\infty} n^{2k+2} \sum_{j=1}^{p} a_j \|f_j - P_{k,T_n} f_j\|^2 \geqq \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\left(\int_0^1 \left(\sum_{j=1}^{p} a_j \varphi_j^2\right)^{1/2k+3}\right)^{2k+3}$$

$$= \lambda\,(say).$$

If $h(x) \sim (\sum_{j=1}^{p} a_j \varphi_j^2(x))^{1/2k+3}$ and if $\{T_n^*\}$ is RS(h) *then*

$$\lim_{n\to\infty} n^{2k+2} \textstyle\sum_{j=1}^{p} a_j \|f_j - P_{k,T_n^*} f_j\|^2 = \lambda.$$

We omit the proof of Theorem 2 since it goes along the lines of the proof of Theorem 3.2 of [2] with few changes.

Consider now the regression model defined in (1.1) of the introduction. Assume it is possible to observe $\{Y(t), \cdots, Y^{(k)}(t), t \in T\}$. We write the covariance matrix of the best linear estimates of the regression coefficients as the $J \times J$ matrix $A_{k,T}^{-1}$. For $T = [0, 1]$, the (limit) covariance matrix is denoted by A^{-1}. This regression problem is tied to the results above about projections in the following way: the matrix $A - A_{k,T}$ has as i, jth entry, $\langle f_i - P_{k,T} f_i, f_j - P_{k,T} f_j\rangle$. We are now able to assert the asymptotic optimality of certain sampling schemes for criteria placed on the size of $A_{k,T}^{-1}$. We suppose that the regression functions $f_1, \cdots, f_J$ are of the form (2.10) with associated continuous functions $\varphi_1, \cdots, \varphi_J$. To apply the present Theorem 1 and Theorem 2, we further assume that

(**) Each non-degenerate linear combination of the φ_j's satisfies (*).

With straightforward modifications being made, Theorems 4.1 through 4.9 of [2] and their corollaries remain true in the context of this paper. The proofs in [2] depend on Theorem 3.1 and Theorem 3.2 there while here we rely on the corresponding Theorem 1 and Theorem 2. In fact, assuming (**) holds, the restrictions found in Theorem 4.5 and Theorem 4.7 of [2] may be eliminated. As a sample result, we give the analogue to the Corollary to Theorem 4.2 of [2]: For the problem of minimizing the generalized variance $\det A_{k,T}^{-1}$, the regular sequence $\{T_n^*\}$ generated by the density $h \sim (\sum_{i,j} \varphi_i \varphi_j A^{ij})^{1/2k+3}$ is asymptotically optimum in the sense that

$$\frac{\det A_{k,T_n^*}^{-1} - \det A^{-1}}{\inf_{T\in\mathscr{D}_n}[\det A_{k,T}^{-1} - \det A^{-1}]} \to 1 \qquad \text{as} \quad n \to \infty.$$

(A^{ij} denotes the i, jth entry in the matrix A^{-1}.) As a second example, suppose M is a $J \times J$ nonnegative definite matrix. We obtain, corresponding to Theorem 4.5,

$$\text{(2.32)} \quad \inf_{T\in\mathscr{D}_n} n^{2k+2}[\operatorname{tr}(A_{k,T}^{-1} M) - \operatorname{tr}(A^{-1}M)]$$

$$= \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\left(\int_0^1 (\phi' A^{-1} M A^{-1} \phi)^{1/2k+3}\right)^{2k+3} + o(1)$$

where ϕ is the $J \times 1$ vector of functions φ_j. Furthermore, if $\{T_n\}$ is RS(h) where $h \sim (\phi' A^{-1} M A^{-1} \phi)^{1/2k+3}$ then $\{T_n\}$ is asymptotically optimum. (2.32) can be viewed as a measure of the natural convergence rate of $A_{k,T}^{-1}$ to A^{-1}, viz., n^{-2k-2}. We will show that this is also the convergence rate of $A_T{}^{-1}$ to A^{-1}.

Suppose now that one may observe only $\{Y(t), t \in T\}$ in the regression problem (1.1) so that the covariance matrix of the best linear estimates of the regression coefficients is $A_T{}^{-1}$. Clearly $A_T{}^{-1} - A_{k,T}^{-1}$ is a nonnegative definite matrix so for an arbitrary nonnegative definite M,

$$\text{(2.33)} \quad \inf_{T \in \mathscr{D}_n} n^{2k+2}[\operatorname{tr}(A_T{}^{-1}M) - \operatorname{tr}(A^{-1}M)] \geqq \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\left(\int_0^1 (\phi' A^{-1} M A^{-1} \phi)^{1/2k+3}\right)^{2k+3} + o(1).$$

We can also bound the infimum in (2.33) above and we turn to that now.

For $T = \{t_0, \cdots, t_n\}$, $t_0 = 0 < t_1 < \cdots < t_n = 1$, let $L(T)$ be the subspace of $H(R)$ spanned by $\{R(\cdot, t), t \in T\}$. P_T will denote the projection operator on $H(R)$ to $L(T)$. It is assumed still that R satisfies Assumptions A, B, C and that $f_1, \cdots, f_J$ are functions of the form (2.10) with the associated functions $\varphi_1, \cdots, \varphi_J$ satisfying (**).

THEOREM 3. *For any nonnegative definite matrix M,*

$$\inf_{T \in \mathscr{D}_n} n^{2k+2}[\operatorname{tr} AM - \operatorname{tr} A_T M] \leqq (k+1)^{2k+2}(\textstyle\int_0^1 (\phi' M \phi)^{1/2k+3})^{2k+3} + o(1).$$

PROOF. Let M be written in eigenvector form as $\sum_{j=1}^p \lambda_j \theta_j \theta_j'$ for $\lambda_1, \cdots, \lambda_p$ the positive eigenvalues of M. Let $h \sim (\phi' M \phi)^{1/2k+3}$ and let $\{T_n{}^*\}$ be RS(h). From Theorem 2 one obtains

$$\text{(2.34)} \quad \begin{aligned} &n^{2k+2}[\operatorname{tr}(AM) - \operatorname{tr}(A_{k,T_n{}^*}M)] \\ &\quad = n^{2k+2} \textstyle\sum_{j=1}^p \lambda_j \|\theta_j' f - P_{k,T_n{}^*}(\theta_j' f)\|^2 \\ &\quad = \frac{(k+1)!^2}{(2k+2)!(2k+3)!}\left(\int_0^1 (\phi' M \phi)^{1/2k+3}\right)^{2k+3} + o(1) \end{aligned}$$

where f denotes the vector of functions f_j. For each $t_i{}^* \in T_n{}^*$, take $k+1$ distinct points $s_{i0} < s_{i1} < \cdots < s_{ik}$ with $|s_{ij} - t_i{}^*| < \rho$ and $0 < \rho < \sup_{0 \leqq i \leqq n-1}(t_{i+1}^* - t_i{}^*)/4$. Let $S_k(\rho) = \{s_{ir}, i = 0, \cdots, n, r = 0, \cdots, k\}$ and note that $S_k(\rho) \in \mathscr{D}_{(n+1)(k+1)-1}$. We will use the fact that for any $f \in H(R)$, $\lim_{\rho \to 0} \|P_{S_k(\rho)} f\| = \|P_{k,T_n{}^*} f\|$ (the proof of this is fairly routine and is omitted). Now $\operatorname{tr}(AM) - \operatorname{tr}(A_T M) = \sum_{j=1}^p \lambda_j \|\theta_j' f - P_T(\theta_j' f)\|^2$ so that

$$\text{(2.35)} \quad \begin{aligned} &\inf_{T \in \mathscr{D}_{(n+1)(k+1)-1}} [\operatorname{tr}(AM) - \operatorname{tr}(A_T M)] \\ &\quad \leqq \textstyle\sum_{j=1}^p \lambda_j \|\theta_j' f - P_{S_k(\rho)}(\theta_j' f)\|^2 \\ &\quad \to \textstyle\sum_{j=1}^p \lambda_j \|\theta_j' f - P_{k,T_n{}^*}(\theta_j' f)\|^2 \qquad \text{as } \rho \to 0. \end{aligned}$$

For $(n+1)(k+1)-1 \leqq m < (n+2)(k+1)-1$, (2.35) yields

$$\begin{aligned} &\inf_{T\in\mathscr{D}_m} m^{2k+2}[\operatorname{tr}(AM)-\operatorname{tr}(A_T M)] \\ (2.36)\qquad &\leqq \inf_{T\in\mathscr{D}_{(n+1)(k+1)-1}} (n+2)^{2k+2}(k+1)^{2k+2}[\operatorname{tr}(AM)-\operatorname{tr}(A_T M)] \\ &\leqq \left(\frac{n+2}{n}\right)^{2k+2}(k+1)^{2k+2}n^{2k+2}\sum_{j=1}^{p}\lambda_j\|\theta_j'f-P_{k,T_n*}(\theta_j'f)\|^2 \\ &= (k+1)^{2k+2}n^{2k+2}[\operatorname{tr}(AM)-\operatorname{tr}(A_{k,T_n*}M)]+o(1). \end{aligned}$$

Theorem 3 then follows from (2.34) and (2.36).

To obtain an inequality in the other direction from the one given in (2.33), we use Theorem 3 and the inversion technique employed in Theorem 4.5 of [2] to get, for any nonnegative definite matrix M,

$$\begin{aligned} (2.37)\quad &\inf_{T\in\mathscr{D}_n} n^{2k+2}[\operatorname{tr}(A_T^{-1}M)-\operatorname{tr}(A^{-1}M)] \\ &\leqq (k+1)^{2k+2}\frac{(k+1)!^2}{(2k+2)!(2k+3)!}\left(\int_0^1(\phi'A^{-1}MA^{-1}\phi)^{1/2k+3}\right)^{2k+3}+o(1). \end{aligned}$$

Thus A_T^{-1}, like $A_{k,T}^{-1}$, converges to A^{-1} at the rate of n^{-2k-2}.

REMARK. For regression functions f of the form

$$f(t) = \int R(s,t)\phi(s)\,ds + \textstyle\sum_{r=0}^{k}\sum_{j=1}^{j_r}\alpha_{jr}R^{0,r}(t,t_{jr})$$

for given t_{jr} in [0, 1] and constants α_{jr}, our results hold with suitable modifications. The details parallel those described in Remark 3.3 of [1].

3. The case $k = 1$. In this section with $k = 1$ we eliminate the disparity between the conclusions of Theorem 1 and Theorem 2, as expressed for example in (2.32), and the conclusion of Theorem 3, as expressed in (2.37). Thus if M is a nonnegative definite matrix, (2.32) gives

$$\begin{aligned} (3.1)\quad n^4\inf_{T\in\mathscr{D}_n}[\operatorname{tr}(A_{1,T}^{-1}M)-\operatorname{tr}(A^{-1}M)] &= \frac{1}{720}\left(\int_0^1(\phi'A^{-1}MA^{-1}\phi)^{1/5}\right)^5+o(1) \\ n^4[\operatorname{tr}(A_{1,T_n*}^{-1}M)-\operatorname{tr}(A^{-1}M)] &= \frac{1}{720}\left(\int_0^1(\phi'A^{-1}MA^{-1}\phi)^{1/5}\right)^5+o(1) \end{aligned}$$

where $\{T_n{}^*\}$ is RS(h), $h \sim (\phi'A^{-1}MA^{-1}\phi)^{1/5}$, while (2.37) gives only

$$(3.2)\quad n^4\inf_{T\in\mathscr{D}_n}[\operatorname{tr}(A_T^{-1}M)-\operatorname{tr}(A^{-1}M)] \leqq \frac{1}{45}\left(\int_0^1(\phi'A^{-1}MA^{-1}\phi)^{1/5}\right)^5+o(1).$$

As a consequence of Theorem 4 below, one finds

$$(3.3)\quad n^4[\operatorname{tr}(A_{T_n*}^{-1}M)-\operatorname{tr}(A^{-1}M)] = \frac{1}{720}\left(\int_0^1(\phi'A^{-1}MA^{-1}\phi)^{1/5}\right)^5+o(1)$$

for the same sequence $\{T_n{}^*\}$. Thus the two problems of minimizing $\operatorname{tr} A_T^{-1}M$ and minimizing $\operatorname{tr} A_{1,T}^{-1}M$ have a common convergence rate and, more importantly, sampling according to $h \sim (\phi' A^{-1} M A^{-1} \phi)^{1/5}$ is asymptotically optimum in both instances. Put another way, the information $\{Y(t), t \in T\}$ is asymptotically as effective as the information $\{Y(t), Y'(t), t \in T\}$ with respect to reducing the size (as measured by the criterion $\psi(B) = \operatorname{tr} BM$) of the covariance matrix of best regression coefficient estimates.

To establish (3.3) and the asymptotic optimality of certain sampling schemes for the above and related problems, it will suffice for us to show that the analogue to (2.27) is true, viz., for f given by (2.10) and $\{T_n\}$ an RS(h),

$$\lim_{n\to\infty} n^4 \|f - P_{T_n} f\|^2 = \frac{1}{720}\int_0^1 \frac{\varphi^2}{h^4}. \tag{3.4}$$

Having at our disposal

$$\liminf n^4 \|f - P_{T_n} f\|^2 \geq \lim n^4 \|f - P_{1,T_n} f\|^2 = \frac{1}{720}\int_0^1 \frac{\varphi^2}{h^4} \tag{3.5}$$

and noting that $n^4 \|f - P_{T_n} f\|^2 \leq n^4 \|f - g_{T_n}\|^2$ if g_{Tn} is any function in $L(T_n)$, it is enough to exhibit g_{Tn} in $L(T_n)$ for which

$$n^4 \|f - g_{T_n}\|^2 \leq \frac{1}{720}\int_0^1 \frac{\varphi^2}{h^4} + o(1). \tag{3.6}$$

Now functions g_{T_n} in $L(T_n)$ are of the form $\sum_{t \in T_n} c_t R(\cdot, t)$ so the approach taken below is that of providing coefficients c_t^*, $t \in T_n$ for which (3.6) holds. These coefficients are defined in (3.8), an expression for $\|f - g_{T_n}\|^2_{H(R)} = \|(f - g_{T_n})'\|^2_{H(K)}$ is given at (3.15) and Theorem 4 establishes (3.6) for such a choice. Following Theorem 4, we discuss briefly the improvements made possible in the direction suggested by (3.1) and (3.3).

Let R be given by (1.6) with $k = 1$ and K a covariance kernel satisfying Assumptions A–C and let f be a function of the form (2.10) for a continuous function φ. For $T_n = \{t_0, t_1, \cdots, t_n\}$, $0 = t_0 < t_1 < \cdots < t_n = 1$, d_i as usual denotes $t_{i+1} - t_i$ for $i = 0, 1, \cdots, n-1$ and we set $d_{-1} = d_n = 0$. Let

$$L_i = (d_i + d_{i-1})^{-1} \textstyle\int_{t_i}^{t_{i+1}} \varphi(x)(x - t_i)(t_{i+1} - x)\,dx, \qquad i = 0, \cdots, n-1 \tag{3.7}$$
$$L_n = 0.$$

We define coefficients $c_j^*, j = 1, \cdots, n$, by

$$\textstyle\sum_{j=i+1}^n c_j^* = d_i^{-1}[L_{i+1} - L_i] + \int_{t_{i+1}}^1 \varphi(x)\,dx + d_i^{-1} \int_{t_i}^{t_{i+1}} \varphi(x)(x - t_i)\,dx, \qquad i = 1, \cdots, n-1, \tag{3.8}$$

$$\textstyle\sum_{j=1}^n c_j^* = d_0^{-1} L_1 + \int_{t_1}^1 \varphi(x)\,dx + d_0^{-1} \int_0^{t_1} \varphi(x) x\,dx.$$

For $t \in (t_i, t_{i+1})$ let

$$U_i(t) = \textstyle\int_t^{t_{i+1}} \varphi(x)\,dx - d_i^{-1} \int_{t_i}^{t_{i+1}} \varphi(x)(x - t_i)\,dx, \qquad i = 0, \cdots, n-1. \tag{3.9}$$

For future reference we note

$$(3.10)\qquad \int_{t_i}^{t_{i+1}} U_i(t)\,dt = 0$$

$$\int_{t_i}^{t_{i+1}} U_i(t)(t_{i+1}-t)\,dt = L_i(d_i+d_{i-1})/2 = -\int_{t_i}^{t_{i+1}} U_i(t)(t-t_i)\,dt.$$

Define ψ so that on (t_i, t_{i+1}),

$$(3.11)\qquad \begin{aligned}\psi(t) = \int_t^1 \varphi(x)\,dx - \textstyle\sum_{j=i+1}^{n} c_j{}^* &= U_i(t)+d_i{}^{-1}(L_i-L_{i+1}), && i=1,\cdots,n-1\\ &= U_0(t)-d_0{}^{-1}L_1, && i=0.\end{aligned}$$

We now take $g(\cdot)=g_{T_n}(\cdot)=\sum_{j=1}^{n} c_j{}^*R(\cdot, t_j)$ and give an expression for $\|f-g\|_{H(R)}^2 = \|(f-g)'\|_{H(K)}^2$. For $h\in C_0$ let

$$(3.12)\qquad J_r h(s) = \int_0^1 h(u)\frac{(u-s)_+{}^{r-1}}{(r-1)!}\,du.$$

Then, if f is of the form (2.10), $Df(s) = \int_0^1 J_1\varphi(u)K(s,u)\,du$. Since $R^{1,0}(\cdot, t) = \int_0^t K(\cdot, u)\,du$ we then get

$$(3.13)\qquad (f-g)'(s) = \int_0^1 \psi(t)K(s,t)\,dt$$

and then

$$\|(f-g)'\|_{H(K)}^2 = \int_0^1\int_0^1 \psi(s)\psi(t)K(s,t)\,ds\,dt.$$

Writing $\int_0^1 \psi(s)K(s,t)\,ds$ as $A(t)$ and using Assumptions A–C,

$$(3.14)\qquad \begin{aligned}A'(t) &= \int_0^1 \psi(s)K_+{}^{0,1}(s,t)\,ds\\ A''(t) &= -\psi(t)+\langle (f-g)', K_+{}^{0,2}(\cdot, t)\rangle.\end{aligned}$$

Perform three integrations by parts using (3.14), and use $J_1\psi(1) = J_2\psi(1) = 0$ to obtain

$$(3.15)\qquad \begin{aligned}\|(f-g)'\|_{H(K)}^2 = \int_0^1 \psi(t)A(t)\,dt &= J_1\psi(0)A(0) + \textstyle\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} J_1\psi\cdot A'\\ &= J_1\psi(0)A(0)+J_2\psi(0)A'(0)-J_1\psi(0)J_2\psi(0)\\ &\quad+\int_0^1 (J_1\psi)^2+\int_0^1 J_2\psi(t)\langle (f-g)', K_+{}^{0,2}(\cdot,t)\rangle\,dt.\end{aligned}$$

The right-hand side of (3.15) is the expression for $\|f-g\|_{H(R)}^2$ which is suited to our purpose. We make some preliminary estimates of quantities appearing there prior to stating Theorem 4.

$J_1\psi$ and $J_2\psi$ may be seen from (3.10) to be

$$(3.16)\qquad \begin{aligned}J_1\psi(x) &= \int_x^{t_{i+1}} U_i(t)\,dt + d_i{}^{-1}[L_i(t_{i+1}-x)+L_{i+1}(x-t_i)] && \text{if } i\geqq 1 \text{ and } x\in[t_i, t_{i+1}]\\ &= \int_x^{t_1} U_0(t)\,dt + d_0{}^{-1}L_1 x && \text{if } x\in[0,t_1]\end{aligned}$$

and

$$(3.17)\qquad \begin{aligned}J_2\psi(x) &= \int_x^{t_{i+1}}\int_s^{t_{i+1}} U_i(t)\,dt\,ds + (L_i-L_{i+1})(t_{i+1}-x)^2/2d_i\\ &\quad + L_{i+1}(t_{i+1}-x) - L_{i+1}d_i/2 && \text{if } i\geqq 1 \text{ and } x\in[t_i,t_{i+1}]\\ &= \int_x^{t_1}\int_s^{t_1} U_i(t)\,dt\,ds - L_1(t_1-x)^2/2d_0 + L_1(t_1-x) - L_1 d_0/2 && \text{if } x\in[0,t_1].\end{aligned}$$

From (3.16), (3.17) and (3.10) we find

$$J_1\psi(0) = 0, \qquad J_2\psi(0) = -L_0 d_0/2. \tag{3.18}$$

Our next step is to get an estimate of $A'(0)$. An integration by parts and using $J_1\psi(0) = J_1\psi(1) = 0$ yields

$$A'(0) = \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} J_1\psi(s)K_{++}^{1,1}(s,0)\,ds.$$

$K_{++}^{1,1}(s, 0)$ is bounded by Assumption A so that from (3.16), (3.7), (3.9) and (3.11) there is a constant a_1 for which

$$|J_1\psi(s)| \leqq a_1 d_j \int_{t_j}^{t_{j+1}} |\varphi(x)|\,dx + a_1 d_{j+1}\int_{t_{j+1}}^{t_{j+2}} |\varphi(x)|\,dx \tag{3.19}$$

for $s \in [t_i, t_{i+1}]$. Then for some constant a_2

$$|A'(0)| \leqq a_2 \sum_{j=0}^{n-1} d_j(d_{j-1}+d_j)\int_{t_j}^{t_{j+1}} |\varphi(x)|\,dx. \tag{3.20}$$

Having gathered the above facts, we now state the result.

THEOREM 4. *Let φ satisfy (*) and let $\{T_n\}$ be* RS(h). *If $\{c_j^*\}$ is defined for each T_n by* (3.8) *and if $g(\cdot) = \sum_{j=1}^n c_j^* R(\cdot, t_j)$ (the dependence on n being suppressed)*

$$\lim n^4 \|f-g\|_{H(R)}^2 = \frac{1}{720}\int_0^1 \frac{\varphi^2}{h^4}.$$

PROOF. In view of the discussion at (3.8), we may suppose $\int_0^1 \varphi^2/h^4 < \infty$. Using the condition (*), $\delta_n = \max_{0\leqq j\leqq n-1} d_j \to 0$. The condition (*) also produces at (2.21) numbers B and $\gamma > 0$ so that

$$\begin{aligned}(\textstyle\int_0^1 |\varphi((b-a)v+a)|\,dv)^2/(\int_0^1 \varphi^{2/5}((b-a)v+a)\,dv)^5 \\ = (b-a)^3(\textstyle\int_a^b |\varphi(x)|\,dx)^2/(\int_a^b \varphi^{2/5}(x)\,dx)^5 \leqq B\end{aligned} \tag{3.21}$$

for all (a, b) with $(b-a) \leqq \gamma$. We take n so large that $\delta_n \leqq \gamma$. Let us write $\eta_j = d_j\int_{t_j}^{t_{j+1}} |\varphi(x)|\,dx$ and use again the Hölder inequality used above (2.26) to obtain

$$\begin{aligned}d_j\eta_j^2 = d_j^3(\textstyle\int_{t_j}^{t_{j+1}} |\varphi(x)|\,dx)^2 &\leqq B(\textstyle\int_{t_j}^{t_{j+1}} \varphi^{2/5}(x)\,dx)^5 \\ &\leqq \frac{B}{n^4}\int_{t_j}^{t_{j+1}} \frac{\varphi^2}{h^4}.\end{aligned} \tag{3.22}$$

We repeatedly take advantage below of the following estimates given by (3.22):

$$d_j\eta_j \leqq (d_j\eta_j^2)^{\frac{1}{2}} = o\left(\frac{1}{n^2}\right) \tag{3.23}$$

$$\sum_{j\in I} d_j\eta_j^2 \leqq \frac{B}{n^4}\sum_{j\in I}\int_{t_j}^{t_{j+1}} \frac{\varphi^2}{h^4} = O\left(\frac{1}{n^4}\right)$$

where I denotes some index set contained in $\{0, 1, \cdots, n-1\}$. There are three terms to consider in the right-hand side of equation (3.15). We first show that $|J_2\psi(0)A'(0)| = o(1/n^4)$.

From (3.18), (3.7) and (3.23) note that

$$|J_2\psi(0)| = \frac{d_0}{2}|L_0| \leqq \frac{d_0^{\,2}}{8}\int_0^{t_1}|\varphi(x)|\,dx = \frac{d_0}{8}\eta_0 = o\left(\frac{1}{n^2}\right). \tag{3.24}$$

According to (3.20)

$$\begin{aligned}|A'(0)| &\leqq a_2\textstyle\sum_{j=0}^{n-1} d_j(d_{j-1}+d_j)\int_{t_j}^{t_{j+1}}|\varphi(x)|\,dx\\ &= a_2\textstyle\sum_{j=0}^{n-1} d_j\eta_j + a_2\sum_{j=0}^{n-1} d_{j-1}\eta_j\\ &\leqq a_2(\textstyle\sum_{j=0}^{n-1} d_j\eta_j^{\,2})^{\frac{1}{2}} + a_2\sum_{j=0}^{n-1} d_{j-1}\eta_j\\ &= O\left(\frac{1}{n^2}\right) + a_2\sum_{j=0}^{n-1} d_{j-1}\eta_j\,.\end{aligned} \tag{3.25}$$

Let $I_1 = \{i \mid h(x) > \varepsilon \text{ on } [t_i, t_{i+1}]\}$ for some fixed $\varepsilon > 0$ and let $I_2 = \{0, 1, \cdots, n-1\} - I_1$. For n sufficiently large, the uniform continuity of h ensures that $h < 2\varepsilon$ on $[t_i, t_{i+1}]$ for $i \in I_2$ and that $h < 3\varepsilon$ on $[t_i, t_{i+1}]$ for $i+1 \in I_2$. If $j \in I_1$ then

$$d_{j-1}\eta_j = d_{j-1}d_j\textstyle\int_{t_j}^{t_{j+1}}|\varphi(x)|\,dx \leqq d_{j-1}d_j^{\,2}\max|\varphi| \leqq d_{j-1}\max|\varphi|\,1/\varepsilon^2 n^2 \tag{3.26}$$

while if $j \in I_2$ then

$$\begin{aligned}d_{j-1}\eta_j &\leqq (d_{j-1}+d_j)^2\textstyle\int_{t_{j-1}}^{t_{j+1}}|\varphi(x)|\,dx\\ &\leqq (d_{j-1}+d_j)^{\frac{1}{2}}\dfrac{4B^{\frac{1}{2}}}{n^2}\left(\displaystyle\int_{t_{j-1}}^{t_{j+1}}\frac{\varphi^2}{h^4}\right)^{\frac{1}{2}}\end{aligned} \tag{3.27}$$

if n is sufficiently large. Therefore

$$\begin{aligned}\textstyle\sum_{j=0}^{n-1} d_{j-1}\eta_j &= \textstyle\sum_{I_1} d_{j-1}\eta_j + \sum_{I_2} d_{j-1}\eta_j\\ &\leqq \frac{\max|\varphi|}{\varepsilon^2 n^2}\sum_{I_1} d_{j-1} + \sum_{I_2}(d_{j-1}+d_j)^{\frac{1}{2}}\frac{4B^{\frac{1}{2}}}{n^2}\left(\int_{t_{j-1}}^{t_j}\frac{\varphi^2}{h^4}\right)^{\frac{1}{2}}\\ &\leqq O\left(\frac{1}{n^2}\right) + \frac{4B^{\frac{1}{2}}}{n^2}\left(\sum_{I_2}(d_{j-1}+d_j)\right)^{\frac{1}{2}}\left(\sum_{I_2}\int_{t_{j-1}}^{t_{j+1}}\frac{\varphi^2}{h^4}\right)^{\frac{1}{2}}\\ &\leqq O\left(\frac{1}{n^2}\right) + \frac{4(2B)^{\frac{1}{2}}}{n^2}\left(2\int_{h<3\varepsilon}\frac{\varphi^2}{h^4}\right)^{\frac{1}{2}} = O\left(\frac{1}{n^2}\right).\end{aligned} \tag{3.28}$$

(3.24), (3.25) and (3.28) give

$$|J_2\psi(0)A'(0)| = o\left(\frac{1}{n^4}\right). \tag{3.29}$$

We turn next to the last term on the right-hand side of (3.15).

From (3.17), (3.9), (3.7) and (3.23) it is easy to get that $|J_2\psi(x)| = o(1/n^2)$. Then using Assumption C,

$$\begin{aligned}\left|\textstyle\int_0^1 J_2\psi(t)\langle (f-g)', K_+^{0,2}(\cdot,t)\rangle\,dt\right| &\leqq c\,\|(f-g)'\|\textstyle\int_0^1|J_2\psi(t)|\,dt\\ &= o\left(\frac{1}{n^2}\right)\|(f-g)'\|\,.\end{aligned} \tag{3.30}$$

The remaining term on the right-hand side of (3.15) is $\int_0^1 (J_1\psi)^2$. When it is found below that $\int_0^1 (J_1\psi)^2 = c/n^4 + o(1/n^4)$, (3.29) and (3.30) will give

$$\|(f-g)'\|^2 \sim \int_0^1 (J_1\psi)^2. \tag{3.31}$$

Recalling the definition of I_1 following (3.25), we write

$$\int_0^1 (J_1\psi)^2 = \textstyle\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (J_1\psi)^2 = \sum_1 \int_{t_j}^{t_{j+1}} (J_1\psi)^2 + \sum_2 \int_{t_j}^{t_{j+1}} (J_1\psi)^2 \tag{3.32}$$

where $\sum_1$ denotes the sum over those j for which $j-1$, j and $j+1$ are all in I_1, $\sum_2$ denotes the sum over the remaining j. We attack first $\sum_2$ in (3.32) using (3.19) to get

$$|J_1\psi(x)|^2 \leqq a_1{}^2(d_j+d_{j+1})^2\left(\int_{t_j}^{t_{j+2}} |\varphi(x)|\,dx\right)^2 \tag{3.33}$$

and then, relying once more on (3.22),

$$\begin{aligned} &\textstyle\sum_2 \int_{t_j}^{t_{j+1}} |J_1\psi(x)|^2\,dx \\ &\leqq \textstyle\sum_2 a_1{}^2(d_j+d_{j+1})^3\left(\int_{t_j}^{t_{j+2}} |\varphi(x)|\,dx\right)^2 \\ &\leqq a_1{}^2\frac{2^4B}{n^4}\sum_2 \int_{t_j}^{t_{j+2}} \frac{\varphi^2}{h^4} \leqq 2a_1{}^2\frac{2^4B}{n^4}\int_{h<4\varepsilon} \frac{\varphi^2}{h^4} = O\left(\frac{1}{n^4}\right)o(\varepsilon). \end{aligned} \tag{3.34}$$

For j appearing in $\sum_1$, $h(x) > \varepsilon$ on $[t_{j-1}, t_{j+2}]$. Then it is easy to obtain from (2.16) that

$$d_{j-1}, d_j, d_{j+1} = O\left(\frac{1}{n}\right); \qquad \frac{d_{j-1}}{d_j} = 1+o(1); \qquad \frac{d_{j+1}}{d_j} = 1+o(1). \tag{3.35}$$

Hence for $t \in [t_j, t_{j+1}]$

$$\begin{aligned} U_j(t) &= \varphi(t_j)[(t_{j+1}-t)-d_j/2]+o(1)\,d_j \\ L_j &= \varphi(t_j)(d_j+d_{j-1})^{-1}\,d_j{}^3/6+o(1)\,d_j{}^2 = \varphi(t_j)\,d_j{}^2/12+o(1)\,d_j{}^2 \\ L_{j+1} &= \varphi(t_{j+1})(d_j+d_{j+1})^{-1}\,d_{j+1}^3/6+o(1)\,d_{j+1}^2 = \varphi(t_j)\,d_j{}^2/12+o(1)\,d_j{}^2 \end{aligned} \tag{3.36}$$

so that for $x \in [t_j, t_{j+1}]$ (see (3.16))

$$J_1\psi(x) = \varphi(t_j)[d_j{}^2/12-(t_{j+1}-x)(x-t_j)/2]+o(1/n^2). \tag{3.37}$$

Then (3.37) gives for j in $\sum_1$

$$\int_{t_j}^{t_{j+1}} (J_1\psi)^2 = \frac{1}{720}\varphi^2(t_j)\,d_j{}^5+o\left(\frac{1}{n^5}\right). \tag{3.38}$$

(2.16) and the mean value theorem allow us to write $d_j = [nh(\theta_j)]^{-1}$ for $\theta_j \in (t_j, t_{j+1})$ and therefore

$$\begin{aligned} \textstyle\sum_1 \int_{t_j}^{t_{j+1}} (J_1\psi)^2 &= \textstyle\sum_1 \left[\frac{1}{720}\varphi^2(t_j)\, d_j^5 + o\left(\frac{1}{n^5}\right)\right] \\ &= \frac{1}{720n^4} \textstyle\sum_1 \frac{\varphi^2(t_j)}{h^4(\theta_j)} d_j + o\left(\frac{1}{n^4}\right) \\ &= \frac{1}{720n^4} \textstyle\sum_1 \int_{t_j}^{t_{j+1}} \frac{\varphi^2}{h^4} + o\left(\frac{1}{n^4}\right) \\ &\leqq \frac{1}{720n^4} \int_{h>\varepsilon} \frac{\varphi^2}{h^4} + o\left(\frac{1}{n^4}\right). \end{aligned} \tag{3.39}$$

Letting $\varepsilon \to 0$ in (3.39) and (3.34) and using the results in (3.32) we find

$$n^4 \int_0^1 (J_1\psi)^2 \leqq \frac{1}{720} \int_0^1 \frac{\varphi^2}{h^4} + o(1). \tag{3.40}$$

The remark at (3.31), in conjunction with the one at (3.6), then concludes the proof of Theorem 4.

With Theorem 4 proved, we assert that Theorem 2 for the case $k = 1$ remains true if every $P_{1,T_n} f_j$ which appears there is replaced by $P_{T_n} f_j$. In particular, the lower bound portion of Theorem 2 requires no further proof since one has always

$$\liminf_{n\to\infty} n^4 \textstyle\sum_{j=1}^p a_j \|f_j - P_{T_n} f_j\|^2 \geqq \liminf_{n\to\infty} n^4 \textstyle\sum_{j=1}^p a_j \|f_j - P_{1,T_n} f_j\|^2. \tag{3.41}$$

The second portion of the modified Theorem 2 then follows from Theorem 4.

Returning again to the regression problem with regression functions $f_1, \cdots, f_J$ having associated functions $\varphi_1, \cdots, \varphi_J$ satisfying (**), Theorems 4.1 through 4.9 of [2] and their corollaries now hold in an improved sense when $k = 1$. We can claim, for example, the asymptotic optimality of sampling according to $h \sim (\phi' A^{-1} M A^{-1} \phi)^{1/5}$ for the minimization problem $\operatorname{tr} A_T{}^{-1} M$ mentioned at (3.1) and (3.3). As a second example, we give the analogue to the Corollary to Theorem 4.2 of [2]: for the problem of minimizing the generalized variance $\det A_T{}^{-1}$, the regular sequence $\{T_n^*\}$ generated by the density $h \sim (\phi' A^{-1} \phi)^{1/5}$ is asymptotically optimum in the sense that

$$\frac{\det A_{T_n^*}^{-1} - \det A^{-1}}{\inf_{T \in \mathscr{D}_n} [\det A_T{}^{-1} - \det A^{-1}]} \to 1 \qquad \text{as} \quad n \to \infty. \tag{3.42}$$

REFERENCES

[1] SACKS, JEROME and YLVISAKER, DONALD (1966). Designs for regression problems with correlated errors. *Ann. Math. Statist.* **37** 66–89.

[2] SACKS, JEROME and YLVISAKER, DONALD (1968). Designs for regression problems with correlated errors; many parameters. *Ann. Math. Statist.* **39** 49–69.

[3] SACKS, JEROME and YLVISAKER, DONALD (1970). Statistical designs and integral approximation. To appear in the Proceedings of the 12th Biennial Seminar of the Canadian Mathematical Congress.

[4] WAHBA, GRACE (1970). A note on the regression design problems of Sacks and Ylvisaker (Unpublished).