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GLOBAL CROSS SECTIONS AND THE DENSITIES OF MAXIMAL INVARIANTS[1]

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This paper generalizes some results of Wijsman concerning the calculation of the density of a maximal invariant. The idea of the technique is to represent the sample space as a product space, one factor Z being a global cross section, i.e., essentially a set that intersects each orbit in a unique point, and the other factor being a coset space of the invariance group. Integration over the invariance group then gives the distribution of the identity function on Z which is a maximal invariant.

Part I of the paper gives sufficient conditions for the technique to be applicable, while Part II exhibits the technique along with an example. Part II is on a more elementary level than Part I and may be understood without a reading of Part I.

1. Introduction and summary. In this paper we present a method of obtaining an integral form of the density of a maximal invariant and prove some existence theorems which show that the method is applicable under general conditions. The paper is divided into two parts: the first part, which exhibits the existence results, requires a greater knowledge of Differential Geometry; understanding of the second part, which presents the technique and gives an example, is free of a reading of the initial section.

In order to sketch the ideas and motivation of the paper, let (X, A, P), $P = \{P_\theta, \theta \in \Theta\}$ be a model for a parametric testing problem, $H_1 : \theta \in \Theta_1$ vs. $H_2 : \theta \in \Theta_2$, Θ_i disjoint, $\Theta_1 \cup \Theta_2 = \Theta$. The P_θ are dominated by a σ-finite measure μ. Further, let the problem be invariant under the group of transformations G. For the various definitions, see Lehmann [8], Chapter 6. In this situation the statistician often invokes the principle of invariance, that is, he restricts his attention to test functions that are invariant, i.e., $\phi(x) = \phi(gx)$, $\forall x \in X$, $\forall g \in G$. The usual method then followed is to find a maximal invariant function ψ on X, i.e., a function that is measurable, invariant and such that if $\psi(x) = \psi(y)$, then $x = gy$ for some $g \in G$. Calling the sets, $Gx = \{gx \mid g \in G\}$ orbits, we can say that ψ distinguishes among orbits. Now the statistician must find the distribution of ψ under the various P_θ. Stein [9] was the first to suggest that this could be done by integration over the invariance group.

Wijsman [11] introduced the use, in addition to integration over the group, of global cross sections. Basically, his approach consists of representing the sample space X as a product space $Y \times Z$ where Y is a copy of the orbits and Z is a subset of X that intersects each orbit in a unique point. Any invariant function can be

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considered as a function on Z and the distributions induced on Z by the P_θ are marginal distributions. If X is a submanifold of E^p, Wijsman [11] has given conditions under which such a procedure can be carried out. Aside from some analyticity and algebraic conditions on G and X, Wijsman's results require Z to be generated by a Lie group H that is at least normal in the Lie group GH. In this paper we derive similar results without requiring the existence of the second group.

I. EXISTENCE THEOREMS

2. Preliminaries. In order to follow this part of the paper the reader will have to be familiar with some elementary concepts of Differential Geometry, Lie Groups and Transformation Groups. The necessary definitions can be found in, e.g., Cohn [3], Chevalley [2], and Wijsman [11]. Our definition of an analytic manifold is that of Cohn with the added restrictions that the analytic manifold have countably many components and be a submanifold of Euclidean space. The conditions imply that an analytic manifold has a topology that is second countable (Chevalley [2], Lemma 4, page 97). A, a subset of an analytic manifold, is said to have measure zero if for every chart h and its coordinate neighborhood U, $h(U \cap A)$ is a set of Lebesgue measure zero.

Let f be an analytic map from the analytic manifold M to the analytic manifold N. f is an analytic diffeomorphism if it is also one-to-one, onto and f^{-1} is analytic. A point in M at which the rank of f, i.e., the rank of the Jacobian matrix of f, is less than the dimension of N is called a critical point. A point $n \in N$ such that $f^{-1}(n)$ contains at least one critical point is called a critical value. The following is a specialization of a theorem of Sard to be found in Sternberg [10].

THEOREM 1 [Sard]. *If f is an analytic map from the analytic manifold M to the analytic manifold N, the set of critical values of f is a set of measure zero.*

An easy application of Sard's theorem gives:

LEMMA 1. *If f is a one-to-one, onto, analytic map from the analytic manifold M to the analytic manifold N, then the dimension of M equals the dimension of N.*

We shall restrict our attention to Lie groups that are analytic subgroups of the general linear group, i.e., the Lie groups shall be representable as matrix groups. Furthermore, if a Lie group G and an analytic manifold M are a Lie transformation group, the action shall be matrix multiplication of vectors.

The reader should also be familiar with the concepts of tangent space, analytic differential forms, the effects of mappings on these entities, and with the notion of an exponential map. Also, it is assumed that he knows that with respect to a continuous differential form an integral of continuous functions with compact support can be constructed as is done in Chevalley [2]. Using the Daniell approach, this fact allows the construction of a Baire measure on the manifold consistent with the original differential form. If a set is Baire measurable, it is straightforward to show that measure zero with respect to the Baire measure and with repect to the previously given definition are equivalent. We shall denote a differential form on a manifold M by ω_M and the corresponding measure by μ_M. If M is a group or coset space, it is tacitly assumed that the differential form and measure are left invariant.

3. Analytic diffeomorphism theorem. The following theorem gives conditions under which an analytic manifold (in applications, the sample space) can be considered the product space of a coset space and a submanifold.

Let (G, X) be a Lie transformation group with the restrictions mentioned above, then

THEOREM 2. *If Z is a submanifold of X such that*

(1) *each point of Z has the same isotropy group, say H, and*
(2) *Z intersects each G-orbit exactly once,*

then there exists an open submanifold, Z^*, of Z and an open submanifold of X, call it X^*, such that $X-X^*$ is a closed set of measure zero and the map $f: Y \times Z \to X$ defined by $f(\bar{g}, z) = gz$, restricted to $Y \times Z^*$, is an analytic diffeomorphism onto X^*, where $Y = G/H$.

PROOF. Let π be the natural map, $\pi: G \to Y$. f is well defined since, if $g_1 = g_2 h$, $g_1, g_2 \in G$, $h \in H$, then $g_1 z = g_2 hz = g_2 z$, $z \in Z$. It is also one-to-one since $g_1 z_1 = g_2 z_2$ implies $g_2^{-1} g_1 z_1 = z_2$ which implies that $g_1 \in g_2 H$ and $z_1 = z_2$ by condition (2). (2) also implies that f is onto.

In addition to showing that f is one-to-one and onto, we wish to show that it is analytic. Let $(L_{r+1}, \cdots, L_s)$ be a basis for the tangent space of H at e, the identity, and $(L_1, \cdots, L_r)$ a complimentary set in the tangent space of G at e. $(L_1, \cdots, L_s)$ is then a basis for the tangent space of G at e. Furthermore, this choice means that $(L_1, \cdots, L_r)$ is a basis for the tangent space of Y at $\pi(e)$, the image of the identity. Using this basis,

$$g \exp \textstyle\sum_{i=1}^r a_i L_i \exp \sum_{i=r+1}^s a_i L_i \to (a_1, \cdots, a_r, a_{r+1}, \cdots, a_s)$$

is a canonical chart near $g \in G$ and

$$g \exp \textstyle\sum_{i=1}^r a_i L_i \to (a_1, \cdots, a_r)$$

is a canonical chart near $\pi(g) \in Y$. Since X is a submanifold of E^p, Z is also and so the elements of Z have coordinates $z(t) = (z_1(t), \cdots, z_p(t))'$ in terms of some orthogonal coordinate system in E^p where $(t_1, \cdots, t_q) = (t)$ is a coordinate system on some open set in Z. The $z_i(t)$ are analytic functions of (t), since Z is a submanifold of E^p. The range values of the map from $G \times Z$ onto X in terms of the coordinate system (a, t) are of the form

$$g \exp \textstyle\sum_{i=1}^r a_i L_i \exp \sum_{i=r+1}^s a_i L_i \cdot z(t)$$

which equals $g \exp \sum_{i=1}^r a_i L_i \cdot z(t)$ since $\exp \sum_{i=r+1}^s a_i L_i \cdot z(t) = z(t)$, $\exp \sum_{i=r+1}^s a_i L_i$ being an element of H. This fact means that the range values of $f: Y \times Z \to X$ are $g \exp \sum_{i=1}^r a_i L_i \cdot z(t)$ and so f is analytic. In the above we have used the fact that group multiplication and group-manifold action is also matrix multiplication.

So, since f is also analytic, by Lemma 1, the dimension of $Y \times Z$ equals the dimension of X. Call this common dimension n. By Sard's theorem, f has rank n at almost every range value in X. The set of points, A, in $Y \times Z$ at which the rank of f is n is an open set since it is the set on which the Jacobian of f is nonzero. Thus, by the Inverse Function Theorem, f is a diffeomorphism on A and so $f(A)$, call it X^*, is open in X. $X - X^*$ is a closed set of measure zero.

The nature of A and X^* is seen more clearly if we note that f is equivariant under the actions of G, i.e., $f(ghH, z) = ghz = gf(hH, z)$, $g, h \in G$. Since the action of any element $g \in G$ on Y is an analytic diffeomorphism (Chevalley [2], page 111), the action $g:(y, z) \to (gy, z)$ on $Y \times Z$ is a diffeomorphism. Now, if f is a diffeomorphism in a neighborhood of a point (y_1, z_1), then it is one at each point (y_2, z_1) as can be seen by considering the action of $g \in G$ where $y_2 = gy_1$. Then f near (y_2, z_1) is the composition of the following diffeomorphisms,

$$(y, z) \to_{g^{-1}} (g^{-1}y, z) \to_f f(g^{-1}y, z) = g^{-1}f(y, z) \to_g f(y, z).$$

Thus, A has the form $Y \times Z^*$ where Z^* is open in Z since $Y \times Z^*$ is open and the projection map onto Z is an open map. X^* is a union of orbits. ▯

The above theorem gives us f as a diffeomorphism at almost all points of X. Wijsman [11] has shown that, under the conditions of the theorem, if Z is generated by a Lie transformation group, call it K, i.e., $Kx_0 = Z$ for some $x_0 \in X$, and K or G is normal in the Lie transformation group GK, then f is a diffeomorphism on all of $Y \times Z$. The question arises as to whether the almost everywhere statement in the conclusion of the theorem is necessary. That it is necessary even in the case that Z is generated by a group K, provided that GK is not a group, can be seen by the following example:

Let $X = \{(1, y, z)' \mid -\infty < y, z < \infty\}$ and let G be the group of lower triangular 3×3 matrices $\{(g_{ij}(s)), -\infty < s < \infty\}$ with 1's on the main diagonal, $g_{21} = g_{32} = s$ and $g_{31} = s^2/2$. Let K be the group of lower triangular 3×3 matrices, $\{(k_{ij}(t)), -\infty < t < \infty\}$, $k_{11} = k_{22} = 1$, $k_{33} = e^t$, $k_{21} = t$, $k_{31} = e^t - t - 1$ and $k_{32} = e^t - 1$. If we let $x_0 = (1, 0, 0)'$, and $Z = Kx_0 = \{(1, t, e^t - t - 1)', -\infty < t < \infty\}$, one can check with an application of the Law of the Mean that the hypotheses of Theorem 2 are satisfied. Since the isotropy groups of both G and K are trivial, (s, t) is a chart on $G \times Kx_0$. At $(s, t) = (0, 0)$, $d/(ds)g(s)x_0 = (0, 1, 0)'$ and $d/(dt)k(t)x_0 = (0, 1, 0)'$ and so we do not have a diffeomorphism at $(0, 0)$. Z could be chosen as $\{(1, 0, t)' \mid -\infty < t < \infty\}$ and we would have a diffeomorphism everywhere. Whether this choice is always possible seems to be unknown.

4. Measure and differential forms. In our applications X is an orientable, analytic manifold with a non-zero differential form of maximal order (Chevalley [2], page 158 ff.). This differential form on X allows one to construct an integral for continuous functions with compact support. Using the Daniell approach one can then extend the integral and finally construct a Baire measure on X. Since we will be "factoring" X into the product of two analytic manifolds, Y and Z, we would like to have measures on Y and Z such that the product measure on $Y \times Z$

is equivalent to the measure on X. A method of calculating the Radon–Nikodym derivative is also desirable. If H is compact, $Y = G/H$ has a natural measure inherited from G, the Lie group, in the following way: let μ be "the" Haar measure on G and $\pi: G \to Y$, the natural map; we define $\mu_Y = \mu\pi^{-1}$ on Y by $\mu_Y(A) = \mu\pi^{-1}(A) = \mu(\pi^{-1}(A))$, A being a Baire set on Y. μ_Y is a regular measure (Halmos [5], Theorem G, page 228), invariant under the actions of G and unique up to a multiplicative constant (Helgason [6], Theorem 1.7, page 369). Although Haar measure and our other invariant measures are unique only up to a multiplicative constant, we shall use the definite article in referring to them. To emphasize that a measure or differential form is invariant under a group G, we shall call it G-invariant.

In general, even though Y is an analytic manifold, μ_Y is not induced by a G-invariant differential form, e.g., $SO(3)/O(2)$ = the projective plane (Helgason [6] page 369). Intuitively the problem seems to be that the π, the natural map, may join the pieces of G together in a nonorientable way in forming Y. If this is the case, we can hope to put a differential form on small enough open subsets of Y by restricting the domain of π.

First we shall show that we can restrict our attention to G_0, the component of the identity, e, of G. H is a compact subgroup of G and $H_1 = G_0 \cap H$. H_1 is a compact subgroup of G_0 and so of G. π is the natural map, $\pi: G \to G/H$ and π_0, the natural map, $\pi_0: G_0 \to G_0/H_1$. π and π_0 are open and continuous. The set $G_0 H$ is an open subgroup of G, since G_0 is open and a normal subgroup of G (Cohn [3], Theorems 2.4.1 and 2.8.3). Now, $\pi(G_0) = G_0 H/H$ and $\pi^{-1}(\pi(g)) \cap G_0 = gH_1$, $g \in G_0$. π_0 partitions G_0 in the same way, so that there exists a one-to-one, onto map f such that $f: G_0 H/H \to G_0/H_1$, $f \circ \pi|_{G_0} = \pi_0$ and $\pi|_{G_0} = f^{-1} \circ \pi_0$. Since $\pi|_{G_0}$ and π_0 are continuous and open, f is a homeomorphism and so induces a one-to-one, onto map of the collection of Baire sets of $G_0 H/H$ onto the Baire sets of G_0/H_1. If μ is the Haar measure on G, its restrictions to the open subgroups $G_0 H$ and G_0 are also Haar measures for these groups. Let us denote the $G_0 H$-invariant measure induced on $G_0 H/H$ by μ_2 and the G_0-invariant measure on G_0/H_1 by μ_1. Because of the equivariance of f under G_0, i.e., $gf(x) = f(gx)$, $g \in G_0$, $x \in G_0 H/H$, $\mu_2 f^{-1}$ and μ_1 are G_0-invariant measures on G_0/H_1 and so $\mu_2 f^{-1} = k\mu_1$, for some $k > 0$, due to the uniqueness of the invariant measure. It is also true that $k\mu_1 f = \mu_2 f^{-1} f = \mu_2$ and so f can also be used to identify the invariant measures on $G_0 H/H$ and G_0/H_1 if we identify the two coset spaces. It is clear that μ_2 restricted to $G_0 H/H$ can be used to generate μ_2 on G/H uniquely because of G-invariance and second countability. Thus, once we have obtained the G_0-invariant measure on G_0/H_1 we can generate uniquely the G-invariant measure on G/H.

Thus, we shall assume that G is connected. Since this does not imply that H_1 (now called H) is connected, we must still be careful. While, in general, the G-invariant measure on G/H is not generated by a continuous differential form, we shall show that this measure can be generated by a continuous differential form on an open neighborhood of $\pi(e)$.

Let H_0 be the identity component of H and $\pi_0: G \to G/H_0$, the natural map. Being a closed subset of a compact set, H_0 is compact in H and G. Since H_0 is connected, Helgason's [6] Lemma 1.5 and Proposition 1.6, pages 367–368, show that G/H_0 has a G-invariant differential form, ω, that generates the G-invariant measure, $\mu\pi_0{}^{-1}$.

To find a differential form that generates the invariant measure even on an open subset of G/H and its relation to ω on G/H_0 requires more work. Let θ be the map, $\theta: G/H_0 \to G/H$, $\theta(gH_0) = gH$, $g \in G$, so that $\pi = \theta \circ \pi_0$. In the following we follow the notation of Chevalley [2], Proposition 4, page 58. We can choose an open, connected neighborhood V (in G) of e such that $V^{-1} \cap H \subset H_0$. Let $\pi(V) = W$. Choose a collection Δ of distinct representatives of H_0-cosets on H so that $H = \sum_{\delta \in \Delta} \delta H_0$, where addition means disjoint union. Since H is compact, Δ is a finite collection and so we can write $H = \sum_{i=1}^{k} \delta_i H_0$, $\delta_1 = e$. In the proof of Proposition 4, Chevalley shows that the sets $U_i = \pi_0(V\delta_i)$ are disjoint so that $\theta^{-1}(W) = \pi_0 \pi^{-1}(W) = \pi_0(VH) = \pi_0(V \sum \delta_i H_0) = \pi_0(\sum V\delta_i H_0) = \sum \pi_0(V\delta_i H_0)$ or

$$\theta^{-1}(W) = \sum \pi_0(V\delta_i) = \sum U_i. \tag{1}$$

Chevalley shows that the U_i are the components of $\theta^{-1}(W)$. $\pi(e) \in U_1$ and we shall write U for U_1. Furthermore, Proposition 4 states that $\theta\,|_{U_i}: U_i \to W$ is a homeomorphism. It is useful to show that it is a diffeomorphism and we shall do this for U. Let $(L_1, \cdots, L_n)$ be a basis for the tangent space of G at e such that $(L_{m+1}, \cdots, L_n)$ is a basis for the tangent space of H (and so H_0) at e. We can assume that V is small enough so that

$$\exp \textstyle\sum_{i=1}^{m} a_i L_i \exp \sum_{i=m+1}^{n} a_i L_i \to (a_1, \cdots, a_n)$$

is a chart on V. Then,

$$\pi_0(\exp \textstyle\sum_{i=1}^{m} a_i L_i \exp \sum_{i=m+1}^{n} a_i L_i) \to (a_1, \cdots, a_m)$$

is a chart on U and

$$(\exp \textstyle\sum_{i=1}^{m} b_i L_i \exp \sum_{i=m+1}^{n} b_i L_i) \to (b_1, \cdots, b_m)$$

is a chart on W. In terms of these charts, $b_i = \theta^i\,|_U(a_1, \cdots, a_m)$, $i = 1, \cdots, m$, and, by definition of θ, $b_i = a_i$ so that in terms of these charts $\theta\,|_U$ is the identity function and so analytic.

Denoting the diffeomorphism $\theta^{-1}\,|_U$ by ψ, we define the differential form ω_ψ on W by $\omega_\psi = \delta\psi(\omega)$. We want to exhibit the relation between ω_ψ and $\mu\pi^{-1}$. Let D be a measurable subset of W and let $V_D = V \cap \pi^{-1}(D)$. Then, $\pi(V_D) = D$ since $\pi(V \cap \pi^{-1}(D)) \subset \pi\pi^{-1}(D) = D$ and since for any $d \in D$, there is a $g \in V$ such that $\pi(g) = d$ implying that $g \in V \cap \pi^{-1}(D)$ so that $\pi(V \cap \pi^{-1}(D)) \supset D$. Replacing in (1) V by V_D and W by D (only the fact that $V^{-1}V \cap H \subset H_0$ is used to establish (1) and V_D has the same property), we get

$$\theta^{-1}(D) = \sum \pi_0(V_D\, \delta_i) \tag{2}$$

and so $\pi^{-1}(D) = \pi_0{}^{-1}\theta^{-1}(D) = \sum V_D \delta_i H_0$. Thus,

$$\mu(\pi^{-1}(D)) = \sum \pi(V_D \delta_i H_0) = \sum \mu(V_D H_0 \delta_i) = k\mu(V_D H_0) = k\mu\pi_0{}^{-1}(\pi_0(V_D)),$$

the second equality occurring because $\delta_i \in H$ and H_0 is normal in H and the third because H is compact and thus the modular function is equal to 1. To write $\pi_0(V_D)$ more conveniently, observe that the ith term on the right in (2), $\pi_0(V_D\delta_i) \subset \pi_0(V\delta_i) = U_i$. Thus, $\theta^{-1}(D) \cap U_i = \pi_0(V_D\delta_i)$ and, in particular, $\pi_0(V_D) = \theta^{-1}(D) \cap U = \psi(D)$, $(\psi = \theta^{-1}|_U)$. Thus,

$$\mu\pi^{-1} = k\mu\pi_0{}^{-1}\psi. \tag{3}$$

Since ω generates $\mu\pi_0{}^{-1}$, $\omega_\psi = \delta\psi(\omega)$ and if μ_ψ is the measure generated by ω_ψ on W, then it is easy to show that $\mu_\psi = \mu\pi_0{}^{-1}\psi$. Comparing this equation with (3) gives us

$$\mu\pi^{-1} = k\mu_\psi. \tag{4}$$

Thus, $k\omega_\psi$ generates the G-invariant measure, $\mu\pi^{-1}$, on W. In the future we shall write ω_ψ for $k\omega_\psi$. Due to the G-invariance, the measure on the subsets of W determines the $\mu\pi^{-1}$ measure on all measurable subsets of G/H.

We summarize these results in the following theorem.

THEOREM 3. *Let G be a Lie group with countably many components and Haar measure, μ, and let H be a compact subgroup of G with π being the natural map, $\pi: G \to G/H$. There exists an open neighborhood of $\pi(e) \in G/H$ on which the G-invariant measure, $\mu\pi^{-1}$, on G/H is generated by a differential form.*

Now let us return to the situation where $Y \times Z$ is diffeomorphic to X. We show that in this situation where an orientable analytic manifold can be represented as a product manifold of two analytic manifolds, the factor manifolds are necessarily orientable.

LEMMA 2. *Let U, V, and W be analytic manifolds such that $U \times V$ is diffeomorphic to W. Furthermore, let ω_W be a nonzero analytic differential form of maximal order on W. Then, U and V also possess such analytic differential forms, say, ω_U and ω_V, and*

$$\omega_W = \delta(\cdot,\cdot)\omega_U \wedge \omega_V, \qquad \textit{where } \delta \textit{ is analytic and } \delta > 0.$$

PROOF. Let the dimensions of U, V, and W be n, m, and p, respectively, so that $n+m=p$. Let u_0 be an arbitrary but fixed point of U with $(u_1, \cdots, u_n)$ a coordinate system around u_0. If v_0 is a point of V, let $(v_1, \cdots, v_m)$ be a coordinate system around v_0. Thus, $(u_1, \cdots, u_n, v_1, \cdots, v_m)$ can be regarded as a coordinate system around (u_0, v_0) in W. We can express ω_W as

$$\omega_W = h(\cdot,\cdot)\,du_1 \wedge \cdots \wedge du_n \wedge dv_1 \wedge \cdots \wedge dv_m$$

in terms of this coordinate system. We define a differential form on V in a neighborhood of v_0 by

$$\omega_V = h(u_0,\cdot)\,dv_1 \wedge \cdots \wedge dv_m.$$

Since $h(u_0,\cdot)$ is analytic and nonzero by the properties of ω_W, ω_V is a nonzero analytic differential form near v_0. ω_V is defined on all V by piecing together the above forms defined in a neighborhood of each point of V. To see that this definition is consistent, let $(v_1, \cdots, v_m)$ and $(v_1', \cdots, v_m')$ be two coordinate systems with common domains of definition on V and let

$$dv_1 \wedge \cdots \wedge dv_m = J(\cdot)\, dv_1' \wedge \cdots \wedge dv_m'$$

where J is the Jacobian of the transformation from $(v_1', \cdots, v_m')$ to $(v_1, \cdots, v_m)$. Being a change of coordinates, J is nonzero in the domain. The corresponding change of coordinates in W from $(u_1, \cdots, u_n, v_1', \cdots, v_m')$ to $(u_1, \cdots, u_n, v_1, \cdots, v_m)$ gives us

$$\text{(5)} \qquad du_1 \wedge \cdots \wedge du_n \wedge dv_1 \wedge \cdots \wedge dv_m = J(\cdot, \cdot)\, du_1 \wedge \cdots \wedge du_n \wedge dv_1' \wedge \cdots \wedge dv_m'$$

where J can be and is identified with the Jacobian given above. Now,

$$\omega_W = h_1(\cdot, \cdot)\, du_1 \wedge \cdots \wedge du_n \wedge dv_1 \wedge \cdots \wedge dv_m$$

and

$$\omega_W = h_2(\cdot, \cdot)\, du_1 \wedge \cdots \wedge dv_n \wedge dv_1' \wedge \cdots \wedge dv_m'.$$

By (5), $h_1(\cdot, \cdot)J(\cdot) = h_2(\cdot, \cdot)$ and so, in particular, $h_1(u_0, \cdot)J(\cdot) = h_2(u_0, \cdot)$ giving

$$\begin{aligned}\omega_V(\cdot) = h_1(u_0, \cdot)\, dv_1 \wedge \cdots \wedge dv_m &= h_1(u_0, \cdot)\, J(\cdot)\, dv_1' \wedge \cdots \wedge dv_m' \\ = h_2(u_0, \cdot)\, dv_1' \wedge \cdots \wedge dv_m'.\end{aligned}$$

Thus, the definition of ω_V is consistent on V.

By symmetry, a nonzero analytic differential form, ω_W, exists on W.

$\omega_U \wedge \omega_V$ and ω_W are both nonzero analytic differential forms of order p on W. Since at each point, the forms are a one-dimensional vector space, $\omega_W = \delta(\cdot, \cdot)\omega_U \wedge \omega_V$ with δ being analytic since the forms are. By possibly changing the sign of ω_W on some or all W components, δ can be chosen positive on all W and so the lemma is proved.

Note. Throughout the proof the analytic differential forms can be replaced by continuous differential forms with the only change in the conclusion being that δ is continuous. □

The above lemma implies that if X has a nonzero analytic differential form so do $Y = G/H$ and Z.

II. METHOD

This section of the paper describes a method of finding the density of a maximal invariant. While references are made to Part I, a reading of that part is not necessary for an understanding of the technique. The method is first described for the case that X, the sample space, is a submanifold of Euclidean space and G is a matrix Lie transformation group. We then specialize to X being an open subset of Euclidean space as the technique is simpler in this most common situation.

5. Method. Let X, the sample space, be a submanifold of Euclidean space E^p with p, a density (integrable function is sufficient) with respect to a σ-finite measure μ_X, μ_X generated by a nonzero analytic differential form ω_X. Let G be an analytic, e.g., closed, subgroup of the general linear group $GL(p, R)$ on E^p such that $GX = X$.

We wish to find a maximal invariant under G and its distribution under p. Let Z be a submanifold of X that intersects each G-orbit in a unique point and has the same compact isotropy subgroup at each point, i.e., $\{g \in G \mid gz = z\} = H$, $\forall z \in Z$. Clearly, the identity map on Z is a maximal invariant. Furthermore, if $Y = G/H$, $\pi: G \to Y$ being the natural map, $Y \times Z$ and X are isomorphic under the map $(\pi(g), z) \to gz$. In fact, $Y \times Z$ and X are diffeomorphic a.e. (μ_X) (Part I, Theorem 2).

We will now see how to represent the integral, $\int_X p(x)\mu_X(dx)$, as an iterated integral on $Y \times Z$. The integration over Y will give us the marginal distribution on Z, i.e., the density of a maximal invariant. Let μ_Z be a measure on Z generated by a nonzero analytic differential form ω_Z and let $\mu_Y(= \mu_G \pi^{-1})$ be the left invariant measure on Y. If ω_G and μ_G are the left invariant form and measure on G, then μ_Y is generated in a neighborhood of $\pi(e)$ by the differential form ω_Y (Part I, Theorem 3) and so in a neighborhood of $\pi(g)$ by $\delta g^{-1}(\omega_Y)$. Thus $\omega_X = f(y, z)\,\delta g^{-1}(\omega_Y) \wedge \omega_Z$ in a neighborhood of (y, z), $y = \pi(g)$. So $\mu_X = |f(y, z)|\,\mu_Y \mu_Z$. f is essentially a Jacobian that will be zero only on the null set where $Y \times Z$ and X are not diffeomorphic.

Now

$$\int_X p(x)\,\mu_X(dx) = \int_Z \int_Y p(y, z)\,|f(y, z)|\,\mu_Y(dy)\,\mu_Z(dz) \tag{6}$$

where μ_X, μ_Y and μ_Z are Baire measures, p is Baire measurable, $p(x) = p(y, z)$ and x is the image of (y, z). If p is a density,

$$\int_Y p(y, z)\,|f(y, z)|\,\mu_Y(dy) \tag{7}$$

is the density with respect to μ_Z of the identity function on Z. Since integration over G is usually more symmetric, we note that if $h: Y \to R$ is integrable (μ_Y), then

$$\int_Y h\mu_Y(dy) = \int_G h \circ \pi \mu_G(dg)$$

(Lehmann [8], Lemma 2, page 38.) Writing h for $h \circ \pi$, (7) becomes

$$\int_G p(gz)\,|f(g, z)|\,\mu_G(dg) \tag{8}$$

where $gz = x$ and $f(g, z) = f(\pi(g), z)$.

In many applications X is an open subset of E^p and then $f(y, z)$ has a simple form. Let μ_X be Lebesgue measure. At (y, z), $\mu_X(dx) = |f(y, z)|\,\mu_Y(dy)\mu_Z(dz)$ and, at (gy, z), $\mu_X(g\,dx) = |f(gy, z)|\,\mu_Y(dy)\mu_Z(dz)$ using the invariance of μ_Y. Since $\mu_X(g\,dx) = |g|\,\mu_X(dx)$, where $|g|$ is the Jacobian of the linear map, $g: X \to X$, $|g|\,|f(y, z)| = |f(gy, z)|$. Letting $y = \pi(e) = \bar{e}$ and then $\bar{g} = \pi(g) = y$, we obtain $|g|\,|f(\bar{e}, z)| = |f(y, z)|$. The choice of the representative of $\pi^{-1}(y)$ is immaterial since $g_1 \in g_2 H$ implies that $|g_1| = |g_2|$, H being compact. Writing $f(z)$ for $f(\bar{e}, z)$, (7) becomes

$$f(z) \int_Y p(y, z)\,|g|\,\mu_Y(dy). \tag{9}$$

In order to evaluate $f(z)$ we make some calculations in a neighborhood of $(\bar{e}, z)$. Let $(x_1, \cdots, x_p)$ be the coordinates on X; $(y_1, \cdots, y_r)$, those on Y; and $(z_1, \cdots, z_s)$, those on Z. Also let $\omega_X = \Pi\, dx_i$, $\omega_Y = k\Pi\, dy_i$, and $\omega_Z = h(z)\Pi\, dz_i$ at $(\bar{e}, z)$. Since $\omega_X = \pm f(z)\omega_Y \wedge \omega_Z$ at $(\bar{e}, z)$, we see that $kf(z)h(z)$ is the Jacobian of the change of coordinates, $(x_1, \cdots, x_p)$ to $(y_1, \cdots, y_r, z_1, \cdots, z_s)$.

These last remarks show that we have essentially found a nice change of coordinates; nice in the sense that the Jacobian need only be evaluated at special points and that integration need only be carried out over some of the coordinates. We also note that the factoring of $|f(y, z)|$ into $|g|\,|f(\bar{e}, z)|$ occurred because of the invariance of μ_Y and the fact that the Radon–Nikodym derivative, $\mu(g\,dx)/\mu(dx)$, at x did not depend on the z "coordinate" of x. Thus in cases where, for example, X is a coset space of G^*, μ_X is the invariant measure and G is a subgroup of G^*, e.g., rotations of the sphere, similar results are to be expected.

Returning to the previous paragraph, we now wish to write (9) in terms of integration over G. Assuming μ_G is known and that μ_H is Haar measure on H, we need to find k in the expression, $\omega_Y = k\Pi\, dy_i$ at $\bar{e}$. If l is μ_G integrable, then

$$\int_G l(g)\,\mu_G(dg) = \int_Y \int_H l(gh)\,\mu_H(dh)\,\mu_Y(dy) \tag{10}$$

where $\mu_Y = \mu_G \pi^{-1}$ and $\mu_H(H) = 1$ (Helgason [6], Theorem 1.7, page 369). In a small enough neighborhood V of $e \in G$, where $(Y \cap V) \times (H \cap V)$ is diffeomorphic to V (Chevalley [2], pages 109–110), (10) implies that $\mu_G = \mu_Y \mu_H$ and so, in a possibly smaller neighborhood, $\omega_G = \pm\omega_Y \wedge \omega_H$ allowing us to evaluate k. Note that we do not need to know the form of ω_Y. (9) now becomes

$$f(z)\int_G p(gz)\,|g|\,\mu_G(dg). \tag{11}$$

We now exhibit an example of the procedure.

6. The distribution of the roots of $|A - \lambda B| = 0$, A and B positive definite. Let X be the collection of the ordered pairs of $p \times p$ positive definite matrices (A, B) such that the eigenvalues of AB^{-1} are distinct. The group G acting on X is the general linear group, $GL(p, R)$, with the action being $C: (A, B) \to (CAC', CBC')$, $C \in G$. A candidate for the role of Z is the set of all (D, I_p) where $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$, $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$. $(\lambda_1, \cdots, \lambda_p)$ is the chart we shall use on Z.

In order to determine $Y = G/H$ we first need to know H. If $C \in H$, then $(CDC', CC') = (D, I)$ for all (D, I) and so C is orthogonal and $CDC' = D$. Since $DCD^{-1} = C$, $c_{ij} = \lambda_i c_{ij} \lambda_j^{-1}$ and, since $\lambda_i \neq \lambda_j$ unless $i = j$, $c_{ij} = 0$ if $i \neq j$. So C is diagonal with the diagonal elements being $+1$'s and -1's. If C has this form, clearly $C \in H$. To see that we have a one-to-one mapping from the coordinates of $Y \times Z$ to those of X we show that $gz_1 = z_2$ implies that $g \in H$. Let $C \in G$, $(D_1, I) \in Z$, $(D_2, I) \in Z$ and $(CD_1C', CC') = (D_2, I)$. The last equation implies that C is orthogonal and that D_1 and D_2 have the same eigenvalues. Since the λ_i are ordered, $D_1 = D_2$. That the map is onto, i.e., $GZ = X$, is seen by letting $(A, B) \in X$ and $C = T\Omega$ where $TT' = B$ and Ω is orthogonal with $\Omega' T^{-1} A T'^{-1} \Omega = D$. Then $(CDC', CC') = (A, B)$ since the eigenvalues of $T^{-1}AT'^{-1}$ and AB^{-1} are the same. Thus, $GZ = X$ since the other inclusion is trivial.

On X our chart consists of the lower left triangular elements of A and B and Lebesgue measure is $\mu_X(dx) = \prod_{i \geqq j} da_{ij} db_{ij}$. On Z the chart is $(\lambda_1, \cdots, \lambda_p)$, $\lambda_1 > \cdots > \lambda_p > 0$ and the measure is $\mu_Z(dz) = \prod_{i=1}^p d\lambda_i$. On G, the chart at C is $(c_{11}, \cdots, c_{pp})$, all the elements of the matrix C. Referring, if necessary, to Deemer and Olkin [4], it is easily seen that $\mu_G(dg) = |C|^{-p} \Pi\, dc_{ij}$. Since H is a finite group, we can identify a neighborhood of $\pi(e)$ in Y with a neighborhood of e in G and thus use the same chart there. Furthermore, since H contains 2^p elements, each element of Y is the image of 2^p elements of G and so $\mu_Y(dy) = 2^p |C|^{-p} \Pi\, dc_{ij}$ near $\pi(e)$. Thus $k = 2^p$.

To evaluate $f(z)$ we use the equations $A = CDC'$, $B = CC'$ and so at $(\pi(e), z)$, $dA = (dC)D + dD + D\,dC'$, $dB = dC + dC'$. Thus

$$\prod_{i \geqq j} da_{ij}\, db_{ij} = 2^p \prod_{i>j} (\lambda_j - \lambda_i)\, \Pi\, dc_{ij} \prod_{i=1}^p d\lambda_i$$

at $(\pi(e), z)$. Since $h(z) = 1$, $f(z) = \prod_{i>j}(\lambda_j - \lambda_i)$. As $|g| = |C|^{2(p+1)}$ (Deemer and Olkin [4]) and if p is a density on X, (11) shows that the density of $(\lambda_1, \cdots, \lambda_p)$ with respect to $\Pi\, d\lambda_i$ is

$$\prod_{i>j}(\lambda_j - \lambda_i) \int_G |C|^{2(p+1)} p(gz)\, \Pi\, dc_{ij}, \qquad \lambda_1 > \cdots > \lambda_p > 0.$$

A common situation occurs when $p(x)$ is the product of two Wishart densities with different population covariance matrices. Because of the action of G on the parameter space, we can, without loss of generality, let one of the parameter covariance matrices be the identity and the other diagonal. So, let A have a Wishart (n, p, T) distribution with $T = \operatorname{diag}(\theta_1, \cdots, \theta_p)$, $\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_p > 0$ and B, an independent Wishart (m, p, I) distribution. The joint density on (A, B) is

$$p(A, B) = k(p, n, m) |T|^{-(n/2)} |A|^{\frac{1}{2}(n-p-1)} |B|^{\frac{1}{2}(m-p-1)} \operatorname{etr} -\tfrac{1}{2}(AT^{-1} + B),$$

where

$$k(p, n, m) = \tfrac{1}{2}^{\frac{1}{2}(n+m)p} \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^p [\Gamma(\tfrac{1}{2}(n-i+1))\Gamma(\tfrac{1}{2}(m-i+1))].$$

As a function on GZ,

$$p(gz) = k(p, n, m) |T|^{-\frac{1}{2}n} |C|^{m+n-2(p+1)} |D|^{\frac{1}{2}(n-p-1)} \operatorname{etr} -\tfrac{1}{2}(CDC'\,T^{-1} + CC').$$

Not writing the constant, $k(p, n, m)$, the marginal density, i.e., the density of D, with respect to $\prod_{i=1}^p d\lambda_i$ is

$$(12) \qquad (\prod_{i=1}^p \lambda_i)^{\frac{1}{2}(n-p-1)} \prod_{i>j} (\lambda_j - \lambda_i) (\prod_{i=1}^p \theta_i)^{-\frac{1}{2}n} \cdot \int_G |C|^{m+n} \operatorname{etr} -\tfrac{1}{2}(CDC'\,T^{-1} + CC') |C|^{-p} \prod_c dc_{ij}.$$

In the special case that $T = \sigma^2 I$, the density of D, (12), becomes

$$(13) \qquad \sigma^{-np} (\prod_{i=1}^p \lambda_i)^{\frac{1}{2}(n-p-1)} \prod_{i>j} (\lambda_j - \lambda_i) \cdot \int_G |C|^{m+n} \operatorname{etr} -\tfrac{1}{2}(C(\sigma^{-2}D + I)C') |C|^{-p} \prod_c dc_{ij}.$$

Making the transformation, $C \to C(\sigma^{-2}D+I)^{\frac{1}{2}} = R$, the integral becomes

$$|I+\sigma^{-2}D|^{-\frac{1}{2}(m+n)} \int_G |R|^{m+n} \operatorname{etr} -\tfrac{1}{2}RR' \, |R|^{-p} \prod_R dr_{ij}$$
$$= |I+\sigma^{-2}D|^{-\frac{1}{2}(m+n)} \pi^{\frac{1}{2}p^2} 2^{\frac{1}{2}(m+n)p} \prod_{i=1}^{p} [\Gamma(\tfrac{1}{2}(m+n-i+1)/\Gamma(\tfrac{1}{2}(p-i+1))]$$

and so the density (13) becomes

$$\prod_{i=1}^{p} \frac{\Gamma(\frac{1}{2}(m+n-i+1))\,\lambda_i^{\frac{1}{2}(n-p-1)}(1+\lambda_i/\sigma^2)^{-\frac{1}{2}(m+n)}}{\Gamma(\frac{1}{2}(n-i+1))\,\Gamma(\frac{1}{2}(m-i+1))\,\Gamma(\frac{1}{2}(p-i+1))} \pi^{\frac{1}{2}p} \sigma^{-np} \prod_{i>j} (\lambda_j - \lambda_i)$$

which is to be found with $\sigma^2 = 1$ in Anderson [1] page 315. The distribution in the general case (12) has been found by Constantine in terms of hypergeometric functions of a matrix argument and is listed in James [7], page 484.

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