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SOME PROPERTIES OF A NORMAL PROCESS NEAR A LOCAL MAXIMUM

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1. Summary. Consider a stationary normal process $\xi(t)$ with mean zero and the covariance function $r(t)$. Properties of the sample functions in the neighborhood of zeros, upcrossings of very high levels, etc. have been studied by, among others, Kac and Slepian, 1959 [4] and Slepian, 1962 [11]. In this paper we shall study the sample functions near local maxima of height u, especially as $u \to -\infty$, and mainly use similar methods as [4] and [11].

Then it is necessary to analyse carefully what is meant by "near a maximum of height u." In Section 2 we derive the "ergodic" definition, i.e. the definition which is possible to interpret by the aid of relative frequencies in a single realisation. This definition has been treated previously by Leadbetter, 1966 [5], and it turns out to be related to Kac and Slepian's horizontal window definition.

In Section 3 we give a representation of $\xi(t)$ near a maximum as the difference between a non-stationary normal process and a deterministic process, and in Section 4 we examine these processes as $u \to -\infty$. We have then to distinguish between two cases.

A: *Regular case.* $r(t) = 1 - \lambda_2 t^2/2 + \lambda_4 t^4/4! - \lambda_6 t^6/6! + o(t^6)$ as $t \to 0$, where the positive λ_{2k} are the spectral moments. Then it is proved that if $\xi(t)$ has a maximum of height u at $t = 0$ then, as $u \to -\infty$,

$$(\lambda_2\lambda_6 - \lambda_4^2)(\lambda_4 - \lambda_2^2)^{-1}\{\xi((\lambda_2\lambda_6 - \lambda_4^2)^{-\frac{1}{2}}(\lambda_4 - \lambda_2^2)^{\frac{1}{2}} t|u|^{-1}) - u\}$$
$$\sim |u|^{-3}\{t^4/4! + \omega(\lambda_4 - \lambda_2^2)^{\frac{1}{2}}\lambda_2^{-\frac{1}{2}} t^3/3! - \zeta(\lambda_4 - \lambda_2^2)\lambda_2^{-1} t^2/2\}$$

where ω and ζ are independent random variables (rv), ω has a standard normal distribution and ζ has the density $z\exp(-z)$, $z > 0$.

Thus, in the neighborhood of a very low maximum the sample functions are fourth degree polynomials with positive t^4-term, symmetrically distributed t^3-term, and a negatively distributed t^2-term but without t-term.

B: *Irregular case.* $r(t) = 1 - \lambda_2 t^2/2 + \lambda_4 t^4/4! - \lambda_5 |t|^5/5! + o(t^5)$ as $t \to 0$, where $\lambda_5 > 0$. Now

$$\xi(tu^{-2}) - u \sim |u|^{-5}\{\lambda_2\lambda_5(\lambda_4 - \lambda_2^2)^{-1}|t|^3/3! + (2\lambda_5)^{\frac{1}{2}}\omega(t) - \zeta(\lambda_4 - \lambda_2^2)\lambda_2^{-1} t^2/2\}$$

where $\omega(t)$ is a non-stationary normal process whose second derivative is a Wiener process, independent of ζ which has the density $z\exp(-z)$, $z > 0$.

The term $\lambda_5|t|^5/5!$ "disturbs" the process in such a way that the order of the distance which can be surveyed is reduced from $1/|u|$ (in Case A) to $1/|u|^2$.

The results are used in Section 5 to examine the distribution of the wave-length and the crest-to-trough wave-height, i.e., the amplitude, discussed by, among

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others, Cartwright and Longuet–Higgins, 1956 [1]. One hypothesis, sometimes found in the literature, [10], states that the amplitude has a Rayleigh distribution and is independent of the mean level. According to this hypothesis the amplitude is of the order $1/|u|$ as $u \to -\infty$ while the results of this paper show that it is of the order $1/|u|^3$.

2. Definition of maximum and conditional distributions. Let $\{\xi(t), -\infty < t < \infty\}$, where t is called the "time," be a real, stationary, separable, normal process with zero mean, unit variance and covariance function $r(t)$. Let $\lambda_{2k} = \int_0^\infty x^{2k}\, dF(x)$ be the spectral moments and assume $\lambda_4 < \infty$. Then, with probability one, the process has a continuous sample derivative and we can define a second derivative in quadratic mean (see Cramér and Leadbetter [2] Chapter 4). We shall consider some sample function properties in the neighborhood of a time t_0, given that "$\xi(t)$ has a local maximum of height u at $t = t_0$". As the last event has probability zero we can regard it as a limit of a sequence of events with non-zero probability. Using ideas from [4], we introduce, for $h, h' > 0$, the following events, where for the sake of simplicity from now on we put $t_0 = 0$:

$A(h, h')$: "$\xi'(s)$ has a downcrossing of zero at some time s

$$= s_0 \in (-h', 0) \quad \text{and} \quad u < \xi(s_0) < u + h."$$

(A function is said to have a downcrossing of zero at $x = x_0$ if, for some $d > 0$, $f(x) \geqq 0$ for $x \in (x_0 - d, x_0)$ and $f(x) \leqq 0$ for $x \in (x_0, x_0 + d)$.)

Now let $\tau = (t_1, \cdots, t_n)'$ be a vector of given (positive or negative) times, put $\mathbf{x} = (x_1, \cdots, x_n)'$ and $\mathbf{y} = (y_1, \cdots, y_n)'$ and define the following probability densities and conditional probability densities.

$$\begin{array}{lll} p(u) & \text{for} & \xi(0) \\ p(v \mid u) & \text{for} & \xi'(0) \mid \xi(0) = u \\ p(u, v, z) & \text{for} & \xi(0), \xi'(0), \xi''(0) \\ p(z \mid u, v) & \text{for} & \xi''(0) \mid \xi(0) = u, \xi'(0) = v \\ p_n(\mathbf{y}, u, v, z) & \text{for} & \xi(t_1), \cdots, \xi(t_n), \xi(0), \xi'(0), \xi''(0) \\ p_n(\mathbf{y} \mid u, v, z) & \text{for} & \xi(t_1), \cdots, \xi(t_n) \mid \xi(0) = u, \xi'(0) = v, \xi''(0) = z. \end{array} \tag{1}$$

Furthermore, let

$$p_\tau(\mathbf{y}, u) = \int_{-\infty}^0 |z| p_n(\mathbf{y}, u, 0, z)\, dz \Big/ \int_{-\infty}^0 |z| p(u, 0, z)\, dz. \tag{2}$$

By using $A(h, h')$ as an approximating condition for a local maximum of height u at $t = 0$, we arrive at a conditional distribution, which, following Kac and Slepian, we call "the conditional distribution given maximum in vertical-horizontal window sense" (= vh-sense).

THEOREM 1.

$$\lim_{h\to 0}\lim_{h'\to 0} P(\xi(t_i) \leqq x_i,\ i = 1, \cdots, n \mid A(h, h')) = \int_{y_i \leqq x_i} p_\tau(\mathbf{y}, u)\, d\mathbf{y} = F_\tau(\mathbf{x}, u). \tag{3}$$

The proof follows the lines in [5] with the modification that we have introduced the process values at the times t_i, and it is deferred to the Appendix. (A heuristic proof is obtained by using Rice's classical differential method [9].)

Now we have

$$p(u, 0, z) = p(u)p(0 \mid u)p(z \mid u, 0) \quad \text{and} \quad p_n(\mathbf{y}, u, 0, z) = p(u)p(0 \mid u)p(z \mid u, 0)p_n(\mathbf{y} \mid u, 0, z)$$

so we have common factors $p(u)$ and $p(0 \mid u)$ in the nominator and denominator in (2) which are not involved in the integration. Cancelling these factors and introducing the density

$$q(z, u) = |z| p(z \mid u, 0) / \int_{-\infty}^{0} |\zeta| p(\zeta \mid u, 0)\, d\zeta \qquad (z < 0), \tag{4}$$

we can write the density for the distribution in (3) in the following simple form.

$$p_\tau(\mathbf{x}, u) = \int_{-\infty}^{0} q(z, u) p_n(\mathbf{x} \mid u, 0, z)\, dz. \tag{5}$$

To justify our choice of $A(h, h')$ in the definition of a maximum we give the following ergodic frequency interpretation of the distribution $F_\tau(\mathbf{x}, u)$ in Theorem 1.

Let $\xi(t)$ be a realisation of the process in $0 \leqq t \leqq T$ and put

$N_T(u, h)$ = the number of local maxima t of $\xi(\cdot)$ in $[0, T]$ with $u < \xi(t) < u + h$;

$N_T(\mathbf{x}, u, h)$ = the number of local maxima t of $\xi(\cdot)$ in $[0, T]$ with $u < \xi(t) < u + h$ and $\xi(t + t_i) \leqq x_i$, $i = 1, \cdots, n$.

The appropriate sample distribution is then the quotient $N_T(\mathbf{x}, u, h)/N_T(u, h)$ and we have

THEOREM 2. *If the process is ergodic, then with probability one*

$$\lim_{h\to 0}\lim_{T\to\infty} N_T(\mathbf{x}, u, h)/N_T(u, h) = F_\tau(\mathbf{x}, u).$$

The proof is deferred to the Appendix.

We see that if we want to make frequency statements about a single realisation, observed over a long time, we have to use the distribution (5), i.e., vh-sense. Other definitions of a maximum will give results similar to those considered here. Since it is impossible to give a natural ergodic frequency interpretation of these definitions we do not give details.

3. Explicit formulas for a normal process. We shall now make full use of the normality of the process and derive an explicit representation of the process given a maximum of height u at $t = 0$. We begin with some definitions.

$$S_{11} = \begin{pmatrix} 1 & 0 & -\lambda_2 \\ 0 & \lambda_2 & 0 \\ -\lambda_2 & 0 & \lambda_4 \end{pmatrix}$$

(6) $$S_{22} = \begin{pmatrix} 1 & r(t_2-t_1) \cdots & r(t_n-t_1) \\ r(t_1-t_2) & 1 \quad \cdots & r(t_n-t_2) \\ \vdots & \vdots & \vdots \\ r(t_1-t_n) & r(t_2-t_n) \cdots & 1 \end{pmatrix}$$

$$S_{12} = \begin{pmatrix} r(t_1) & \cdots & r(t_n) \\ -r'(t_1) & \cdots & -r'(t_n) \\ r''(t_1) & \cdots & r''(t_n) \end{pmatrix}$$

$$S_{21} = S'_{12}$$

(7) $$(G_1 G_2 G_3) = S_{21} S_{11}^{-1}$$

In (7) the $n \mid 1$ matrices G_1, G_2, and G_3 are the columns in $S_{21}S_{11}^{-1}$. As the function r is even, S_{22} is symmetric.

$$A(t) = (\lambda_4 r(t) + \lambda_2 r''(t))/(\lambda_4 - \lambda_2^2)$$

(8) $$B(t) = (\lambda_2 r(t) + r''(t))/(\lambda_4 - \lambda_2^2)$$

$$C(s, t) = r(s-t) - \{\lambda_2(\lambda_4 - \lambda_2^2)\}^{-1}\{\lambda_2\lambda_4 r(s)r(t) + \lambda_2^2 r(s)r''(t) + (\lambda_4 - \lambda_2^2)r'(s)r'(t) + \lambda_2^2 r''(s)r(t) + \lambda_2 r''(s)r''(t)\}.$$

The densities in (5) can now be evaluated.

LEMMA 1. (a) $p_n(\mathbf{x} \mid u, 0, z)$ *is the probability density of an n-variate normal variable with mean and covariance matrix*

(9) $$m = u \cdot G_1 + z \cdot G_3$$

$$S_{2\cdot 1} = S_{22} - S_{21}S_{11}^{-1}S_{12}.$$

(b)

(10) $$q(z, u) = -z \exp(-(z + \lambda_2 u)^2/2(\lambda_4 - \lambda_2^2))/k(u) \qquad (z < 0),$$

where $k(u) = \int_0^\infty z \exp(-(z - \lambda_2 u)^2/2(\lambda_4 - \lambda_2^2))\, dz$.

PROOF. As the process is normal with covariance function $r(t)$ it is known that the $(n+3)$-variate rv $(\xi(0), \xi'(0), \xi''(0), \xi(t_1), \cdots, \xi(t_n))'$ has a normal distribution with mean zero and with the covariance matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}.$$

Now, by a well-known property of the normal distribution (Rao [8], page 441), $(\xi(t_1), \cdots, \xi(t_n) \mid \xi(0) = u, \xi'(0) = 0, \xi''(0) = z)'$ has an n-variate normal distribution with mean and covariance matrix

$$m = S_{21}S_{11}^{-1}(u, 0, z)'$$

$$S_{2\cdot 1} = S_{22} - S_{21}S_{11}^{-1}S_{12}.$$

This proves (a).

Part (b) follows in a similar way from the fact that $(\xi''(0) \mid \xi(0) = u, \xi'(0) = 0)$ has a normal distribution with mean $-\lambda_2 u$ and variance $\lambda_4 - \lambda_2^2$ so that

$$p(z \mid u, 0) = (2\pi(\lambda_4 - \lambda_2^2))^{-\frac{1}{2}} \exp(-(z + \lambda_2 u)^2 / 2(\lambda_4 - \lambda_2^2)).$$

LEMMA 2. *The elements of* $G_1, G_3, S_{2\cdot 1}$ *are given by* $(G_1)_\nu = A(t_\nu)$, $(G_3)_\nu = B(t_\nu)$, $(S_{2\cdot 1})_{\nu\mu} = C(t_\nu, t_\mu)$.

The proof is straightforward.

Now it is convenient to determine the characteristic function (ch.f.) of the probability density (5). Put $\mathbf{s} = (s_1, \cdots, s_n)'$. Then we have the ch.f.

$$\varphi_\tau^u(\mathbf{s}) = \int \exp(i \cdot \mathbf{s}'\mathbf{x}) p_\tau(\mathbf{x}, u)\, d\mathbf{x}$$

$$= \int_{-\infty}^{0} q(z, u) \{\int \exp(i \cdot \mathbf{s}'\mathbf{x}) p_n(\mathbf{x} \mid u, 0, z)\, d\mathbf{x}\}\, dz.$$

Lemma 1 and the ch.f. for a normal distribution give

$$\varphi_\tau^u(\mathbf{s}) = \int_{-\infty}^{0} q(z, u) \exp(i \cdot \mathbf{s}'(u \cdot G_1 + z \cdot G_3) - \tfrac{1}{2}\mathbf{s}'S_{2\cdot 1}\mathbf{s})\, dz \tag{11}$$

$$= \exp(iu \cdot \mathbf{s}'G_1 - \tfrac{1}{2}\mathbf{s}'S_{2\cdot 1}\mathbf{s}) \cdot \int_{-\infty}^{0} q(z, u) \exp(iz \cdot \mathbf{s}'G_3)\, dz.$$

Here the first factor is the ch.f. of an n-variate normal variable with mean $u \cdot G_1$ and covariance matrix $S_{2\cdot 1}$, while the second factor is the ch.f. of an n-variate rv $\zeta \cdot G_3$ where ζ has the probability density $q(\cdot, u)$. Thus we find that $(\xi(t_1), \cdots, \xi(t_n))'$ can be written as a sum of two independent n-variate rv's, a normal variable $(\Delta_1(t_1), \cdots, \Delta_1(t_n))'$ with $E(\Delta_1(t_k)) = u \cdot (G_1)_k = u \cdot A(t_k)$ and $\mathrm{Cov}(\Delta_1(t_j), \Delta_1(t_k)) = (S_{2\cdot 1})_{jk} = C(t_j, t_k)$ and a singular variable $(\Delta_2(t_1), \cdots, \Delta_2(t_n))'$ with $\Delta_2(t_k) = \zeta \cdot (G_3)_k = \zeta \cdot B(t_k)$ where ζ has the probability density $q(\cdot, u)$. Changing sign in $q(\cdot, u)$ and replacing $\Delta_1(t)$ by $uA(t) + \Delta_1(t)$ we get

THEOREM 3. *Given a maximum of height* u *at* $t = 0$, $\xi(t)$ *has the same finite-dimensional distributions as the process* $uA(t) + \Delta_1(t) - \Delta_2(t)$ *where* Δ_1 *and* Δ_2 *are two independent stochastic processes,* Δ_1 *a non-stationary zero-mean normal process with the covariance function* $C(s, t)$ *and* Δ_2 *a deterministic process given by* $\Delta_2(t) = \zeta B(t)$. *The* rv ζ *has the probability density*

$$z \exp(-(z - \lambda_2 u)^2) / 2(\lambda_4 - \lambda_2^2)) / k(u) \qquad (z > 0).$$

4. The process in a neighborhood of a very low maximum. In Theorem 3 we saw how a maximum of height u affects the sample functions. Now we shall study this influence in a special case when it can be suspected to be particularly strong, viz.

when $u \to -\infty$. It seems plausible that after a very low maximum the sample function will soon turn upwards again and therefore we can expect the process to be very well determined in a neighborhood of the maximum. The strength of this influence depends on the covariance function near $t = 0$. Therefore let

$$\xi^*(t, u) = c_1 |u|^a(\xi(c_2 t|u|^{-b}) - u)$$

be a normalization of the process $\xi(t)$ conditioned by a maximum (in vh-sense) of height u at $t = 0$. Here a, b, c_1, and c_2 are positive constants to be chosen later.

LEMMA 3.

$$\begin{aligned} u^2k(u) &= u^2\int_0^\infty z \exp(-(z-\lambda_2 u)^2/2(\lambda_4-\lambda_2{}^2))\,dz \\ &= \exp(-\lambda_2{}^2u^2/2(\lambda_4-\lambda_2{}^2))\{(\lambda_4-\lambda_2{}^2)^2/\lambda_2{}^2+o(1)\} \qquad u\to-\infty. \end{aligned}$$

PROOF.

$$\begin{aligned} \int_0^\infty z\exp(-(z-\lambda_2u)^2/2(\lambda_4-\lambda_2{}^2))\,dz &= (\lambda_4-\lambda_2{}^2)\exp(-\lambda_2{}^2u^2/2(\lambda_4-\lambda_2{}^2) \\ &\quad +(2\pi)^{\frac{1}{2}}\lambda_2u(\lambda_4-\lambda_2{}^2)^{\frac{1}{2}}\Phi(\lambda_2u(\lambda_4-\lambda_2{}^2)^{-\frac{1}{2}}) \end{aligned}$$

where Φ is the standardized normal distribution function. As we shall let $u \to -\infty$ we can use a well-known expansion of Φ (see e.g. Feller [3], page 193):

$$\begin{aligned} &(2\pi)^{\frac{1}{2}}\Phi(\lambda_2u(\lambda_4-\lambda_2{}^2)^{-\frac{1}{2}}) \\ &\quad = \exp(-\lambda_2{}^2u^2/2(\lambda_4-\lambda_2{}^2))\{-(\lambda_4-\lambda_2{}^2)^{\frac{1}{2}}/\lambda_2u+(\lambda_4-\lambda_2{}^2)^{\frac{3}{2}}/\lambda_2{}^3u^3+o(u^{-4})\} \end{aligned}$$

which will give the result.

To be able to examine $A(t)$, $B(t)$, and $C(s, t)$ when $s, t \to 0$ we must make some further assumptions about $r(t)$ for small t.

CASE A. Let $r^{\mathrm{IV}}(t) = \lambda_4 - \lambda_6 t^2/2 + o(t^2)$, $t \to 0$ so that $r(t) = 1-\lambda_2 t^2/2+\lambda_4 t^4/4! - \lambda_6 t^6/6! + o(t^6)$. With probability one, this process has a continuous second derivative with the nice covariance function $\lambda_4 - \lambda_6 t^2/2 + o(t^2)$ and it can be expected to behave very regularly. It will be found that the spectral moments affect the distributions only through the following functions.

$$\text{(12)} \qquad \begin{aligned} \alpha &= (\lambda_2\lambda_6-\lambda_4{}^2)/(\lambda_4-\lambda_2{}^2) \\ \beta &= (\lambda_4-\lambda_2{}^2)/\lambda_2. \end{aligned}$$

LEMMA 4. *As* $s, t \to 0$

$$\begin{aligned} A(t) &= 1-\alpha t^4/4!+o(t^4) \\ B(t) &= t^2/2+o(t^2) \\ C(s, t) &= \alpha\beta s^3t^3/3!3!+o((s, t)^6). \end{aligned}$$

In the last expression $o((s, t)^6) = \sum_{j+k=6} o(s^jt^k)$.

PROOF. $A(t)$ and $B(t)$ are simple consequences of the above expansion of $r(t)$. $C(s, t)$ is somewhat more tedious to handle and the details are omitted.

It will now turn out that the finite-dimensional distributions of the normed process

$$\xi_A^*(t, u) = \alpha |u|^3 \{\xi(\alpha^{-\frac{1}{4}} t |u|^{-1}) - u\} \tag{13}$$

can be expressed in terms of a stochastic polynomial defined by

$$\Gamma_A(t) = t^4/4! + \omega\beta^{\frac{1}{2}} t^3/3! - \zeta\beta t^2/2 \tag{14}$$

where ω and ζ are independent rv's with the probability densities $(2\pi)^{-\frac{1}{2}} \exp(-w^2/2)$ and $z\exp(-z)$, $z > 0$ respectively. We then have the following theorem where $\mathscr{L}(\xi_n) \to \mathscr{L}(\xi)$ means convergence in law.

THEOREM 4. *For any set of times* $(t_i)_{i=1}^n$

$$\mathscr{L}(\xi_A^*(t_i, u), i = 1, \cdots, n) \to \mathscr{L}(\Gamma_A(t_i), i = 1, \cdots, n) \quad \textit{when} \quad u \to -\infty.$$

PROOF. Consider the n-variate rv $-u^3(\xi(t_1/|u|) - u, \cdots, \xi(t_n/|u|) - u)'$. Using the ch.f. (11) and Lemma 2 we can write the ch.f. of this variable as

$$\begin{aligned} &\exp(iu^4 \textstyle\sum s_\nu) \varphi^u_{\tau/|u|}(-u^3 \cdot \mathbf{s}) \\ &\quad = \exp(iu^4 \textstyle\sum s_\nu - iu^4 \sum s_\nu A(t_\nu/|u|) - \frac{1}{2} u^6 \sum s_\nu s_\mu C(t_\nu/|u|, t_\mu/|u|)) \\ &\qquad \cdot \textstyle\int_{-\infty}^0 q(z, u) \exp(-izu^3 \sum s_\nu B(t_\nu/|u|))\, dz = F_1 F_2, \qquad \text{say.} \end{aligned} \tag{15}$$

Lemma 4 gives

$$\begin{aligned} u^4 \textstyle\sum s_\nu - u^4 \sum s_\nu A(t_\nu/|u|) &= u^4 \textstyle\sum s_\nu - u^4 \sum s_\nu (1 - \alpha t_\nu^4/4! u^4 + o(u^{-4})) \\ &= \alpha \textstyle\sum s_\nu (t_\nu^4/4! + o(1)) \to \alpha \cdot \mathbf{s}'(t_1^4/4! \cdots t_n^4/4!)'. \end{aligned}$$

In the same way we get

$$\begin{aligned} u^6 \textstyle\sum s_\nu s_\mu C(t_\nu/|u|, t_\mu/|u|) &= u^6 \textstyle\sum s_\nu s_\mu (\alpha\beta t_\nu^3 t_\mu^3/3!3! u^6 + o(u^{-6})) \\ &\to \alpha\beta \cdot \mathbf{s}'(t_1^3/3! \cdots t_n^3/3!)'(t_1^3/3! \cdots t_n^3/3!)\mathbf{s}. \end{aligned}$$

Hence we find that F_1 tends towards the ch.f. of an n-variate normal variable with mean and covariance matrix

$$\alpha(t_1^4/4! \cdots t_n^4/4!)' \quad \text{and} \quad \alpha\beta(t_1^3/3! \cdots t_n^3/3!)'(t_1^3/3! \cdots t_n^3/3!).$$

This is a singular variable and it can be expressed as

$$\alpha(t_1^4/4! \cdots t_n^4/4!)' + \omega(\alpha\beta)^{\frac{1}{2}}(t_1^3/3! \cdots t_n^3/3!)'$$

where ω is a univariate normal variable with mean 0 and variance 1. To simplify the factor F_2 in (15) we use Lemma 4 which gives

$$u^2 \textstyle\sum s_\nu B(t_\nu/|u|) = u^2 \sum s_\nu (t_\nu/2u^2 + o(u^{-2})) = \sum s_\nu (t_\nu^2/2 + o(1))$$

so that

$$F_2 = \textstyle\int_{-\infty}^0 q(z, u) \exp(-izu \sum s_\nu (t_\nu^2/2 + o(1))\, dz.$$

Recalling the representation (10) of q and Lemma 3 we substitute z for uz, $(u < 0)$, expand the exponent in q and get

$$F_2 = \int_0^\infty z\{u^2k(u)\}^{-1}\exp(-(z/u+\lambda_2 u)^2/2(\lambda_4-\lambda_2^2))$$
$$\cdot\exp(-iz\sum s_\nu(t_\nu^2/2+o(1)))\,dz$$
$$= \int_0^\infty z\exp(-z^2/2u^2(\lambda_4-\lambda_2^2))\cdot\exp(-z\beta^{-1})$$
$$\cdot\exp(-iz\sum s_\nu(t_\nu^2/2+o(1)))\,dz\{\beta^2+o(1)\}^{-1}$$
$$\to \int_0^\infty \beta^{-2}z\exp(-z\beta^{-1})\exp(-iz\sum s_\nu t_\nu^2/2)\,dz.$$

(Dominated convergence makes it possible to pass to the limit under the integral sign.)

This is the ch.f. of an n-variate rv $-\zeta\beta(t_1^2/2\cdots t_n^2/2)'$ where ζ has the probability density $z\exp(-z)$, $z > 0$.

Since the limit-ch.f. is factorized, ω and ζ are independent and we have derived the limit distribution of $|u|^3\{\xi(t/|u|)-u\}$. The proof is complete if we change the scale in the appropriate way.

We can observe that this choice of norming constants, $a = 3$ and $b = 1$, is the only one which leads to a non-trivial limit of F_1.

CASE B. Modifying the assumptions about $r(t)$ we require $r^{\mathrm{IV}}(t) = \lambda_4-\lambda_5|t|+o(t)$ with $\lambda_5 > 0$, so that $r(t) = 1-\lambda_2 t^2/2+\lambda_4 t^4/4!-\lambda_5|t|^5/5!+o(t^5)$, $t\to 0$. Now the process has a second derivative with a covariance function of the same form as that of a normal Markov process. Then $\xi(t)$ can be expected to behave more irregularly than in Case A. Indeed, we must choose $a = 5$ and $b = 2$ to get a non-trivial limit of $\xi^*(t, u)$ when $u\to -\infty$.

The result can be expressed in terms of a non-stationary, zero-mean, normal process with the covariance function

$$\begin{aligned} R(s,t) &= (-|s-t|^5+s^5+t^5-5s^4t-5st^4+10s^3t^2+10s^2t^3)/2\cdot 5! && \text{when } s \text{ and } t > 0,\\ &= 0 && \text{when } s\cdot t < 0,\\ &= (-|s-t|^5-s^5-t^5+5s^4t+5st^4-10s^3t^2-10s^2t^3)/2\cdot 5! && \text{when } s \text{ and } t < 0. \end{aligned}$$

It is easily proved that such a process can be represented as

$$\omega(t) = \int_0^t\left(\int_0^s \varepsilon(\tau)\,d\tau\right)ds \tag{16}$$

where $\{\varepsilon(t), t\in R\}$ is a normed Wiener-process with $\varepsilon(0) = 0$. We now proceed as in Case A and arrive at the following lemma, the proof of which is omitted.

LEMMA 5. *As* s, $t\to 0$

$$A(t) = \lambda_2\lambda_5(\lambda_4-\lambda_2^2)^{-1}|t|^3/3!+o(t^3)$$
$$B(t) = t^2/2+o(t^2)$$
$$C(s,t) = 2\lambda_5 R(s,t)+o((s,t)^5).$$

The limit-distributions of the normalized, conditioned process

$$\xi_B^*(t,u) = |u|^5\{\xi(t/u^2)-u\}$$

can be expressed in terms of the process

$$\Gamma_B(t) = \lambda_2\lambda_5(\lambda_4-\lambda_2^2)^{-1}|t|^3/3!+(2\lambda_5)^{\frac{1}{2}}\omega(t)-\zeta\beta t^2/2$$

where $\omega(t)$ is given by (16) and the rv ζ is independent of the process $\omega(t)$ and has the probability density $z\exp(-z)$, $z>0$. The constant β is given by (12).

THEOREM 5. *For any set of times* $(t_i)_{i=1}^n$

$$\mathscr{L}(\xi_B^*(t_i,u), i=1,\cdots,n) \to \mathscr{L}(\Gamma_B(t_i), i=1,\cdots,n) \quad \textit{when} \quad u\to-\infty.$$

PROOF. The ch.f. of the n-variate rv $-u^5(\xi(t_1/u^2)-u, \cdots, \xi(t_n/u^2)-u)'$ is

$$\begin{aligned}&\exp(iu^6\textstyle\sum s_\nu)\varphi^u_{\tau/u^2}(-u^2\cdot \mathrm{s})\\ &\quad= \exp(iu^6\textstyle\sum s_\nu - iu^6\sum s_\nu A(t_\nu/u^2)-\frac{1}{2}u^{10}\sum s_\nu s_\mu C(t_\nu/u^2, t_\mu/u^2))\\ &\qquad\cdot\textstyle\int_{-\infty}^0 q(z,u)\exp(-izu^5\sum s_\nu B(t_\nu/u^2))\,dz = H_1H_2, \qquad \text{say}.\end{aligned}$$

If we introduce the notation $\gamma = \lambda_2\lambda_5(\lambda_4-\lambda_2^2)^{-1}$ and use Lemma 5 we get the following limits for the terms in H_1:

$$\begin{aligned}u^6\textstyle\sum s_\nu - u^6\sum s_\nu A(t_\nu/u^2) &= u^6\textstyle\sum s_\nu(\gamma|t_\nu|^3/3!u^6+o(u^{-6}))\\ &\to \gamma\cdot \mathrm{s}'(|t_1|^3/3!\cdots|t_n|^3/3!)' \qquad \text{and}\\ u^{10}\textstyle\sum s_\nu s_\mu C(t_\nu/u^2, t_\mu/u^2) &= u^{10}\textstyle\sum s_\nu s_\mu(2\lambda_5 R(t_\nu,t_\mu)u^{-10}+o(u^{-10}))\\ &\to 2\lambda_5\textstyle\sum s_\nu s_\mu R(t_\nu,t_\mu).\end{aligned}$$

Hence H_1 tends to the ch.f. of an n-variate normal variable with mean and covariance matrix

$$\lambda_2\lambda_5(\lambda_4-\lambda_2^2)^{-1}(|t_1|^3/3!\cdots|t_n|^3/3!)' \quad\text{and}\quad 2\lambda_5(R(t_\nu,t_\mu)).$$

This gives the first two terms of the limit process Γ_B. The last term follows from H_2 as in Theorem 4.

5. Applications. Let the process $\xi(t)$ have a maximum of height u at $t=t_1$ and the next minimum at $t=t_2$. Put

$$\tau_u = t_2-t_1; \qquad \delta_u = \xi(t_1)-\xi(t_2).$$

Then we call τ_u the wave-length and δ_u the (crest-to-trough) wave-height. We shall consider the ergodic distribution of these variables given a maximum at $t_1=0$. In order to describe their limit-distribution as $u\to-\infty$ we define the corresponding quantities for the stochastic polynomial $\Gamma_A(t)$ given by (14), i.e.,

$$\begin{aligned}\tau &= \text{the unique positive zero of } \Gamma_A'(t)\\ \delta &= -\Gamma_A(\tau).\end{aligned}$$

Under the conditions of Theorem 4 we then have the following theorem.

THEOREM 6.

$$\mathscr{L}(\alpha^{\frac{1}{2}}|u|\tau_u, \alpha|u|^3\delta_u) \to \mathscr{L}(\tau, \delta) \qquad as \quad u \to -\infty.$$

OUTLINE OF PROOF. A natural way to prove this theorem is to expand the process $\Delta_1(t)$ of Theorem 3 in a power series around the origin and use the expansions of Lemma 4 for the functions $A(t)$ and $B(t)$. In this way we can approximate $\xi_A{}^*(t, u)$ by a polynomial and then it is not difficult to prove the convergence stated. A complete proof, using a function space model for Δ_1, can be found in [7].

The general distribution of the wave-length and wave-height is not easily evaluated. However Theorem 6 is strong enough to disprove a hypothesis sometimes found in the literature [10]. According to this the wave-height $\xi(t_1)-\xi(t_2)$ and the mean-level $(\xi(t_1)+\xi(t_2))/2$ are independent rv's with the probability densities $\sigma_1{}^{-2}x\exp(-x^2/2\sigma_1{}^2)$, $x > 0$ and $(2\pi\sigma_2{}^2)^{-\frac{1}{2}}\exp(-x^2/2\sigma_2{}^2)$ respectively, where σ_1 and σ_2 depend on the spectral moments, i.e., are Rayleigh- and normal-distributed rv's. A simple consequence of the hypothesis is that $|u|\,\delta_u$ has an asymptotic gamma-distribution when $u \to -\infty$. Since this conflicts with Theorem 6 the hypothesis is wrong.

APPENDIX

PROOF OF THEOREM 1 AND THEOREM 2.

ASSUMPTIONS. $\{\xi(t),\ t \in R\}$ is a real, stationary, separable, normal process with fourth spectral moment $\lambda_4 < \infty$. With probability one its sample functions are continuously differentiable and the second derivative exists in quadratic mean. The distribution of $(\xi(0), \xi'(0), \xi''(0), \xi(t), \xi'(t), \xi''(t))'$ is non-singular.

Let h and h' be positive numbers, $\tau = (t_1, \cdots, t_n)'$ and $\mathbf{x} = (x_1, \cdots, x_n)'$ given vectors, and define the events

$$A(h, h')\text{: “}\xi'(s) \text{ has a downcrossing of zero at some time } s = s_0 \in (-h', 0) \text{ and } u < \xi(s_0) < u+h\text{”},$$

$$A(\mathbf{x}, h, h')\text{: “}A(h, h') \text{ and } \xi(t_i) < x_i,\ i = 1, \cdots, n\text{”},$$

$$B(\mathbf{x}, h, h')\text{: “}A(h, h') \text{ and } \xi(s_0+t_i) < x_i,\ i = 1, \cdots, n\text{”}.$$

Let the number of local maxima be

$$N(u, h) = N_1(u, h) = \text{the number of local maxima } t \text{ of } \xi(\cdot) \text{ in } [0, 1] \text{ with } u < \xi(t) < u+h$$

$$N(\mathbf{x}, u, h) = N_1(\mathbf{x}, u, h) = \text{the number of local maxima } t \text{ of } \xi(\cdot) \text{ in } [0, 1] \text{ with } u < \xi(t) < u+h \text{ and } \xi(t+t_i) \leqq x_i,\quad i = 1, \cdots, n.$$

Since the process has a continuous distribution we can replace the non-strict inequalities $\xi(t_i) < x_i$ in $A(\mathbf{x}, h, h')$ by strict ones and obtain

(A1) $$P(\xi(t_i) \leqq x_i, i = 1, \cdots, n \mid A(h, h')) = P(A(\mathbf{x}, h, h'))/P(A(h, h')).$$

In order to prove Theorem 1 we have to determine the limit

$$\lim_{h' \to 0} P(A(\mathbf{x}, h, h'))/P(A(h, h')).$$

LEMMA A1.

$$\lim_{h' \to 0} (P(A(\mathbf{x}, h, h')) - P(B(\mathbf{x}, h, h')))/h' = 0.$$

LEMMA A2.

$$\lim_{h' \to 0} P(B(\mathbf{x}, h, h'))/h' = E(N(\mathbf{x}, u, h)),$$

$$\lim_{h' \to 0} P(A(h, h'))/h' = E(N(u, h)).$$

LEMMA A3.

$$\lim_{h' \to 0} P(A(\mathbf{x}, h, h'))/P(A(h, h')) = E(N(\mathbf{x}, u, h))/E(N(u, h)).$$

Lemma A3 is a direct combination of Lemma A1 and Lemma A2.

PROOF OF LEMMA A1. For arbitrary events A and B we have $|P(A) - P(B)| \leqq P(A-B) + P(B-A)$. Hence

(A2) $$|P(A(\mathbf{x}, h, h')) - P(B(\mathbf{x}, h, h'))| \leqq P(A(\mathbf{x}, h, h') - B(\mathbf{x}, h, h')) + P(B(\mathbf{x}, h, h') - A(\mathbf{x}, h, h')).$$

Now the event $A(\mathbf{x}, h, h') - B(\mathbf{x}, h, h')$ implies that $\xi'(\cdot)$ has a downcrossing of zero in $(-h', 0)$ and that, for some $i = 1, \cdots, n$, $\xi(\cdot)$ has a downcrossing of the level x_i in the interval $(t_i - h', t_i)$. Let us introduce the event

$A_i(h')$: "$A(h, h')$ and $\xi(\cdot)$ has a downcrossing of the level x_i for some t in $(t_i - h', t_i)$".

Then we have

(A3) $$P(A(\mathbf{x}, h, h') - B(\mathbf{x}, h, h')) \leqq P(\textstyle\sum_{i=1}^n A_i(h')) \leqq \sum_{i=1}^n P(A_i(h')).$$

If we can establish that $P(A_i(h')) = o(h')$ we have proved that $P(A(\mathbf{x}, h, h') - B(\mathbf{x}, h, h'))/h' \to 0$ as $h' \to 0$; a similar discussion of the second term in (A2) gives Lemma A1. But a little reflection shows that $A_i(h') \subseteq E_1 \cup E_2 \cup E_3$ where the events E_1, E_2, E_3 are defined as follows.

E_1: "$\xi'(-h') > 0 > \xi'(0)$ and $\xi(t_i - h') > x_i > \xi(t_i)$,"

E_2: "$\xi'(\cdot)$ has more than one zero in $[-h', 0]$,"

E_3: "$\xi(\cdot)$ has more than one crossing of x_i in $[t_i - h', t_i]$."

Since the streams of zeros and crossings are regular the probabilities $P(E_2)$ and $P(E_3)$ are $o(h')$ as $h' \to 0$. The probability $P(E_1)$ is obtained by integrating the density of $\xi'(-h')$, $\xi'(0)$, $\xi(t_i-h')$, $\xi(t_i)$ over a certain region in R^4, and since we have assumed these densities to be non-singular it is not difficult to prove that $P(E_1)/h' \to 0$ as $h' \to 0$. The details are omitted.

PROOF OF LEMMA A2. Both limits are direct consequences of Korolyook's theorem for stationary streams. By Dobrushin's lemma these streams of down-crossings are obviously regular, and hence their intensities are equal to the mean number of events per time unit. This is the content of Lemma A2. (For Korolyook's theorem and Dobrushin's lemma, see [2], 3.8.)

LEMMA A4. *For a normal process we have*

$$E(N(\mathbf{x}, u, h)) = \int_u^{u+h} \int_{y_i \leq x_i} \int_{-\infty}^{0} |z| \, p_n(\mathbf{y}, \eta, 0, z) \, dz \, d\mathbf{y} \, d\eta$$

where $p_n(\mathbf{y}, u, v, z)$ is the density of $\xi(t_1), \cdots, \xi(t_n), \xi(0), \xi'(0), \xi''(0)$.

PROOF. Define

$N'(\mathbf{x}, u, h) =$ the number of local maxima t of $\xi(\cdot)$ in $[0, 1]$

with $u < \xi(t) < u+h$ and $\xi(t+t_i) < x_i$, $i = 1, \cdots, n$.

Dropping $\mathbf{x}$, u, h we have to prove that $E(N) = E(N')$ and that $E(N')$ is given by the asserted formula. We first prove that $E(N-N') = 0$.

Since the process has a continuous derivative every crossing of a given level is, with probability one, either a downcrossing or an upcrossing, and a little reflection shows that

$$0 \leq N - N' \leq \sum_1^n D_i + \sum_1^n U_i$$

where D_i and U_i denote the number of times a downcrossing zero of the derivative is followed by an x_i-crossing (down- and up- respectively) of the process after exactly the time t_i. Such events are very rare and all D_i and U_i have zero expectation, which will now be proved for D_i.

Let $a_k = k/2^m$, $k = 0, \cdots, 2^m - 1$, be a division of the interval [0, 1], and put

$$\chi_k = 1 \quad \text{if} \quad \xi'(\alpha_k) > 0 > \xi'(\alpha_{k+1}) \quad \text{and}$$
$$\xi(\alpha_k + t_i) > x_i > \xi(\alpha_{k+1} + t_i)$$
$$= 0 \quad \text{otherwise;}$$
$$Z_m = \sum_{k=0}^{2m-1} \chi_k \, .$$

The process is stationary and therefore

$$E(Z_m) = \sum E(\chi_k) = 2^m E(\chi_0) = 2^m P(\chi_0 = 1)$$

where $P(\chi_0 = 1) = P(\xi'(0) > 0 > \xi'(2^{-m})$ and $\xi(t_i) > x_i > \xi(t_i + 2^{-m}))$. In the same way as in the proof of Lemma A1 it can be shown that $2^m P(\chi_0 = 1) \to 0$ as $m \to \infty$

which gives that $\lim E(Z_m) = 0$. Now, almost certainly and for m sufficiently large, there is one positive χ_k for every time contributing to D_i which implies $D_i \leqq \liminf Z_m$. It follows that $E(D_i) = 0$, and we have proved that $E(N) = E(N')$.

With the same technique as above we can now calculate $E(N'(\mathbf{x}, u, h))$. Define

$$\begin{aligned} \psi_k &= 1 \quad \text{if} \quad \xi'(\alpha_k) > 0 > \xi'(\alpha_{k+1}) \quad \text{and} \quad u < \xi(\alpha_k) < u+h \quad \text{and} \\ &\qquad\qquad \xi(\alpha_k + t_i) < x_i, \qquad i = 1, \cdots, n, \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

$$V_m = \sum_{k=0}^{2m-1} \psi_k .$$

It is then easy to prove that

$$E(V_m) = 2^m P(\psi_0 = 1) \to \int_u^{u+h} \int_{y_i \leqq x_i} \int_{-\infty}^0 |z| \, p_n(\mathbf{y}, \eta, 0, z) \, dz \, d\mathbf{y} \, d\eta$$

and that $\lim V_m = N'$ with dominated convergence. (In that proof the strict inequalities in N' in contrast to the non-strict in N are essential.) Then

$$E(N(\mathbf{x}, u, h)) = E(N'(\mathbf{x}, u, h)) = \lim E(V_m)$$

which together with the expression above proves the lemma.

Since $N(\mathbf{x}, u, h) \uparrow N(u, h)$ as $\min_i(x_i) \to \infty$ (a.c.) we have $E(N(u, h)) = \lim E(N(\mathbf{x}, u, h))$. Then the following lemma is a direct consequence of Lemma A4.

LEMMA A5.

$$E(N(u, h)) = \int_u^{u+h} \int_{-\infty}^0 |z| \, p(\eta, 0, z) \, dz \, d\eta.$$

PROOF OF THEOREM 1. Lemma A4 and Lemma A5, in combination with Lemma A3 and the statement (A1), give

$$\text{(A4)} \quad \begin{aligned} &\lim_{h' \to 0} P(\xi(t_i) \leqq x_i, i = 1, \cdots, n \mid A(h, h')) \\ &= \int_u^{u+h} \int_{y_i \leqq x_i} \int_{-\infty}^0 |z| \, p_n(\mathbf{y}, \eta, 0, z) \, dz \, d\mathbf{y} \, d\eta / \int_u^{u+h} \int_{-\infty}^0 |z| \, p(\eta, 0, z) \, dz \, d\eta. \end{aligned}$$

When $h \to 0$, the right-hand side of (A4) tends to

$$F_\tau(\mathbf{x}, u) = \int_{y_i \leqq x_i} p_\tau(\mathbf{y}, u) \, d\mathbf{y}$$

where p_τ is given by (2), and Theorem 1 follows.

PROOF OF THEOREM 2. If the process is ergodic, then with probability one

$$\lim_{T \to \infty} N_T(\mathbf{x}, u, h)/T = E(N(\mathbf{x}, u, h)), \qquad \lim_{T \to \infty} N_T(u, h)/T = E(N(u, h)),$$

and hence

$$\lim_{T \to \infty} N_T(\mathbf{x}, u, h)/N_T(u, h) = E(N(\mathbf{x}, u, h))/E(N(u, h)).$$

Since the left-hand side of the last relation is exactly the quotient (A4) and tends to $F_\tau(\mathbf{x}, u)$ as $h \to 0$, Theorem 2 follows.

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