*The Annals of Mathematical Statistics*
1971, Vol. 42, No. 2, 809–811

# ON THE DISTRIBUTION OF LINEAR COMBINATIONS OF NON-CENTRAL CHI-SQUARES

BY DAVID A. HARVILLE

*Aerospace Research Laboratories*

Press [1] expressed the distribution of an arbitrary linear combination of non-central chi-square variates as a mixture of distributions of weighted differences between pairs of central chi-squares. The distributions appearing in the mixture depend on the coefficients in the linear combination. Here, by modifying Press's results, we obtain a mixture representation not exhibiting that property.

Following Press, let $\chi^2_{m,d}$ denote a non-central chi-square variate, having $m$ degrees of freedom and non-centrality parameter $d$, whose probability density function is given by

$$p(x) = [x^{(m-2)/2}/2^{m/2}\pi^{\frac{1}{2}}]\exp[-(d^2+x)/2] \cdot \sum_{j=0}^{\infty}[(xd^2)^j\Gamma(j+\tfrac{1}{2})/(2j)!\Gamma(j+m/2)],$$

for $x > 0$ and zero otherwise. (Press's paper contains an error in that $p(x)$ there should be defined as above. That change is necessary for the validity of his results.) Denote by $f_m(x)$ and $F_m(x)$ the probability density function and the cumulative distribution function, respectively, of $\chi^2_{m,0}$. We use $p^{(\alpha,\beta)}_{m,n}(x)$ to represent the pdf of $\alpha\chi^2_{m,0}+\beta\chi^2_{n,0}$, where $\alpha > 0$, $\beta > 0$, and $\chi^2_{m,0}$ and $\chi^2_{n,0}$ are independent. The corresponding cdf will be designated $P^{(\alpha,\beta)}_{m,n}(x)$.

Define

$$\begin{aligned} U &= \alpha[\chi^2_{m_0,d_0}+\textstyle\sum_{i=1}^r a_i\chi^2_{m_i,d_i}], \\ V &= \beta[\chi^2_{n_0,g_0}+\textstyle\sum_{j=1}^s b_j\chi^2_{n_j,g_j}], \\ T &= U-V, \\ U^* &= \alpha\textstyle\sum_{i=1}^r a_i\chi^2_{m_i,d_i}, \\ V^* &= \alpha\textstyle\sum_{j=1}^s b_j\chi^2_{n_j,g_j}, \quad \text{and} \\ T^* &= U-V^*, \end{aligned}$$

where $\alpha > 0$, $\beta > 0$, $a_i \geqq 1$, $b_j \geqq 1$, for all $i$ and $j$, and all the chi-square variates are independent. Take $K_0(r)$, $K_1(r)$, $K_2(r)$, $\cdots$, to be a sequence of constants which depend on $(m_i, d_i, a_i)$, $i = 1, \cdots, r$, and which are given explicitly in Press's Theorem 2.1B. The sequence $K_0{}^*(s)$, $K_1{}^*(s)$, $K_2{}^*(s)$, $\cdots$, is defined in the same way in terms of $(n_j, g_j, b_j)$, $j = 1, \cdots, s$. Set

$$q_0 = (\textstyle\prod_{j=1}^r a_j{}^{-m_j/2})\exp[-(\tfrac{1}{2})\sum_{j=0}^r d_j{}^2],$$
$$q_0{}^* = (\textstyle\prod_{j=1}^s b_j{}^{-n_j/2})\exp[-(\tfrac{1}{2})\sum_{j=0}^s g_j{}^2],$$

Received March 25, 1970.

and, for $i = 1, 2, \cdots$,

$$q_i = \sum_{j=0}^{i} [(d_0^2/2)^{i-j} \exp(-d_0^2/2)/(i-j)!] K_j(r),$$
$$q_i^* = \sum_{j=0}^{i} [(g_0^2/2)^{i-j} \exp(-g_0^2/2)/(i-j)!] K_j^*(s).$$

Press has shown in a series of theorems that, if $F(x)$ and $f(x)$ denote the cdf and pdf of $U$ and if $H(t)$ and $h(t)$ represent the cdf and pdf of $T$, then

$$\begin{aligned} F(x) &= \textstyle\sum_{i=0}^{\infty} q_i F_{M+2i}(x/\alpha), \\ f(x) &= \textstyle\sum_{i=0}^{\infty} (q_i/\alpha) f_{M+2i}(x/\alpha), \\ h(t) &= \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i q_j^* p_{M+2i,N+2j}^{(\alpha,\beta)}(t), \quad \text{and} \\ H(t) &= \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i q_j^* P_{M+2i,N+2j}^{(\alpha,\beta)}(t), \end{aligned} \tag{1}$$

where $M = \sum_{i=0}^{r} m_i$ and $N = \sum_{j=0}^{s} n_j$. Also, $q_i > 0$, $q_j^* > 0$, for all $i$ and $j$, and $\sum_{i=0}^{\infty} q_i = \sum_{j=0}^{\infty} q_j^* = 1$.

By making straightforward modifications in Press's proofs of his Theorems 2.1, 3.2, 3.3, and 4.1 and noting that $p_{i,j}^{(1,1)}(x) = p_{j,i}^{(1,1)}(-x)$, the following result on the distributions of $U^*$ and $T^*$ was obtained.

THEOREM. *If $F^*(x)$, $H^*(t)$, and $Z^*(t)$ denote the* cdf's *of $U^*$, $T^*$, and $-T^*$, respectively, and if $f^*(x)$, $h^*(t)$, and $z^*(t)$ represent the corresponding* pdf's, *then*

$$\begin{aligned} F^*(x) &= \textstyle\sum_{i=0}^{\infty} K_i(r) F_{M^*+2i}(x/\alpha), \\ f^*(x) &= \textstyle\sum_{i=0}^{\infty} [K_i(r)/\alpha] f_{M^*+2i}(x/\alpha), \end{aligned} \tag{2}$$

$$\begin{aligned} h^*(t) &= \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i [K_j^*(s)/\alpha] p_{M+2i,N^*+2j}^{(1,1)}(t/\alpha), \\ H^*(t) &= \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i K_j^*(s) P_{M+2i,N^*+2j}^{(1,1)}(t/\alpha), \\ z^*(t) &= \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i [K_j^*(s)/\alpha] p_{N^*+2j,M+2i}^{(1,1)}(t/\alpha) \quad \textit{and} \\ Z^*(t) &= \textstyle\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i K_j^*(s) P_{N^*+2j,M+2i}^{(1,1)}(t/\alpha), \end{aligned} \tag{3}$$

*where* $M^* = \sum_{i=1}^{r} m_i$, $M = M^* + m_0$, *and* $N^* = \sum_{j=1}^{s} n_j$. *Also,* $K_i(r) > 0$, $K_j^*(s) > 0$, *for all $i$ and $j$, and* $\sum_{i=0}^{\infty} K_i(r) = \sum_{j=0}^{\infty} K_j^*(s) = 1$.

$H^*(t)$ and $Z^*(t)$ were previously obtained by Robinson [2] for the special case where $d_0 = d_1 = \cdots = d_r = g_1 = \cdots = g_s = 0$; i.e., the case where $T^*$ is a linear combination of *central* chi-square variates.

It is clear that, not only does an arbitrary linear combination of non-central chi-square variates have a representation of the same form as $T$, but also that it must have a representation of the same form as either $T^*$ or $-T^*$. Moreover, mixtures like (2) or (3) have an advantage over one like (1) in that the distributions appearing in (1), i.e. the $P_{i,j}^{(\alpha,\beta)}(\cdot)$, are determined by the coefficients in our linear combination of non-central chi-squares, while those appearing in (2) and (3), the

$P_{i,j}^{(1,1)}(\cdot)$, do not vary with changes in those coefficients. If tables of $P_{i,j}^{(1,1)}(x)$ were prepared, (2) and (3) could be used to make percentage point calculations of the distribution of any linear combination of non-central chi-squares whatsoever.

## REFERENCES

[1] PRESS, S. J. (1966). Linear combinations of non-central chi-square variates. *Ann. Math. Statist.* **37** 480–487.

[2] ROBINSON, J. (1965). The distribution of a general quadratic form in normal variates. *Austral. J. Statist.* **7** 110–114.