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QUADRATIC SUBSPACES AND COMPLETENESS

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1. Introduction and summary. Since the notion of completeness for a family of distributions was introduced by Lehmann and Scheffé [7], a problem of interest has been to determine conditions under which a complete sufficient statistic exists for a family of multivariate normal distributions. One approach to this problem, first formulated for a completely random model in some work by Graybill and Hultquist [2] and extended to a mixed linear model by Basson [1], has a basic assumption of commutativity for certain pairs of matrices. In the present paper some of the commutativity conditions and an associated eigenvalue condition assumed in the theorems on completeness in both [1] and [2] are replaced by the weaker requirement of a quadratic subspace. These subspaces, i.e., quadratic subspaces, are introduced and briefly investigated in Section 2 and are found to possess some rather interesting mathematical properties. The existence of $\mathscr{A}$-best estimators (e.g., [12]) is also examined for several situations; and it is found that the usual estimators in the weighting factors for the recovery of interblock information in a balanced incomplete block design (treatments fixed and blocks random) have an optimal property when the number of treatments is equal to the number of blocks.

Throughout the paper $(\mathscr{A}, (-,-))$ denotes the FDHS (finite-dimensional Hilbert space or finite-dimensional inner product space) of $n \times n$ real symmetric matrices with the trace inner product. The notation $Y \sim N_n(X\beta, \sum_{i=1}^m \nu_i V_i)$ means that Y is an $n \times 1$ random vector distributed according to a multivariate normal distribution with expectation $X\beta$ and covariance matrix $\sum_{i=1}^m \nu_i V_i$; and for such a random vector the following is assumed:

(a) X is a known $n \times p$ matrix and β is an unknown vector of parameters ranging over $\Omega_1 = R^p$.

(b) Each $V_i (i = 1, 2, \cdots, m)$ is a known $n \times n$ real symmetric matrix, $V_m = I$, and $\nu = (\nu_1, \cdots, \nu_m)'$ is a vector of unknown parameters ranging over a subset Ω_2 of R^m.

(c) The set Ω_2 contains a non-void open set in R^m and $\sum_{i=1}^m \nu_i V_i$ is a positive definite matrix for each $\nu \in \Omega_2$.

(d) The parameters ν and β are functionally independent so that the entire parameter space is $\Omega = \Omega_1 \times \Omega_2$.

For the special case when $X = 0$ the notation $Y \sim N_n(0, \sum_{i=1}^m \nu_i V_i)$ is used and for this situation the parameter space Ω reduces to Ω_2.

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The notation and terminology in the following sections is generally consistent with the usage in [12]. The adjoint of a linear operator **T** is denoted by **T*** and the transpose of a matrix A is denoted by A′. Additionally, the unique Moore–Penrose generalised inverse of a matrix A is denoted by A^+, and as in [12], only real finite-dimensional linear spaces are considered.

2. Quadratic subspaces. In the following definition the notion of a quadratic subspace of the vector space of real symmetric matrices $\mathscr{A}$ is introduced.

DEFINITION. A subspace $\mathscr{B}$ of $\mathscr{A}$ with the property that $B \in \mathscr{B}$ implies $B^2 \in \mathscr{B}$ is said to be a quadratic subspace of $\mathscr{A}$.

Since quadratic subspaces are used in the sequel, an investigation of some elementary properties associated with these subspaces is considered in this section. The special case when the matrices in a quadratic subspace commute is briefly considered near the end of this section, and such a subspace of $\mathscr{A}$ is referred to as a commutative quadratic subspace. Verification for most of the succeeding lemmas and observations is straightforward, and so proofs are generally omitted. More details regarding verification, however, may be found in [10].

LEMMA 1. *Let $\mathscr{B}$ be a subspace of $\mathscr{A}$ and let $\mathscr{B}_1$ be an arbitrary spanning set for $\mathscr{B}$, then the following conditions are all equivalent:*

(1) (a) $A \in \mathscr{B} \Rightarrow A^2 \in \mathscr{B}$,

(b) $A, B \in \mathscr{B}_1 \Rightarrow (A+B)^2 \in \mathscr{B}$,

(c) $A, B \in \mathscr{B}_1 \Rightarrow AB+BA \in \mathscr{B}$, and

(*d*) $A \in \mathscr{B} \Rightarrow A^k \in \mathscr{B}$ *for each finite integer* $k \geqq 1$.

LEMMA 2. *Let $\mathscr{B}$ be a quadratic subspace of $\mathscr{A}$ and let $A \neq 0$ be a member of $\mathscr{B}$. If $\lambda_1, \lambda_2, \cdots, \lambda_r$ are the distinct nonzero eigenvalues of A and $P_1, P_2, \cdots, P_r$ are the corresponding symmetric, idempotent, and pairwise orthogonal* (*i.e.*, $P_iP_j = 0$ *for* $i \neq j$) *matrices such that*

$$A = \sum_{i=1}^r \lambda_i P_i,$$

then $P_1, P_2, \cdots, P_r$ are all members of the subspace $\mathscr{B}$.

PROOF. Define two matrices Δ and Δ_1 as follows:

$$\Delta = \begin{bmatrix} \lambda_1 & \lambda_1{}^2 & \cdots & \lambda_1{}^r \\ \lambda_2 & \lambda_2{}^2 & \cdots & \lambda_2{}^r \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \lambda_r & \lambda_r{}^2 & \cdots & \lambda_r{}^r \end{bmatrix} \quad \text{and} \quad \Delta_1 = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1{}^{r-1} \\ 1 & \lambda_2 & \cdots & \lambda_2{}^{r-1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 1 & \lambda_r & \cdots & \lambda_r{}^{r-1} \end{bmatrix}.$$

The determinant of Δ_1 is a Vandermode determinant and so

$$|\Delta| = (\textstyle\prod_i \lambda_i)(\prod_{i<j}(\lambda_i - \lambda_j)).$$

Since the λ_i's are nonzero and distinct, it is clear that $|\Delta| \neq 0$, which implies that Δ is a nonsingular matrix. Let α be an arbitrary vector in R^r and let $\beta = \Delta^{-1}\alpha$, then

$$\begin{aligned}\textstyle\sum_{i=1}^r \beta_i A^i &= \textstyle\sum_{i=1}^r \beta_i(\sum_{k=1}^r \lambda_k{}^i P_k) \\ &= \textstyle\sum_{k=1}^r(\sum_{i=1}^r \beta_i\lambda_k{}^i)P_k = \sum_{k=1}^r \alpha_k P_k.\end{aligned}$$

From (1.d) it is clear that $A, A^2, \cdots, A^r$ are in the subspace $\mathscr{B}$; thus, $\sum_{i=1}^r \beta_i A^i = \sum_{k=1}^r \alpha_k P_k$ is an element of $\mathscr{B}$. Since α is arbitrary it follows that $P_1, \cdots, P_r$ are all members of $\mathscr{B}$.

Suppose that $\mathscr{B}$ is a quadratic subspace of $\mathscr{A}$, then some properties which follow immediately from Lemmas 1 and 2 are the following:

(2) (a) $A \in \mathscr{B} \Rightarrow A^+ \in \mathscr{B}$.
(b) $A \in \mathscr{B} \Rightarrow AA^+ \in \mathscr{B}$.
(c) $A, B \in \mathscr{B} \Rightarrow \mathrm{ABA} \in \mathscr{B}$.
(d) There exists a basis for $\mathscr{B}$ consisting of symmetric idempotent matrices.
(e) $A \in \mathscr{B} \Rightarrow \{\mathrm{ABA}: B \in \mathscr{B}\}$ is a quadratic subspace of $\mathscr{A}$.

In (2.b) note that AA^+ is the unique symmetric idempotent matrix with the same column space as the matrix A.

We now wish to show that a quadratic subspace $\mathscr{B}$ contains an identity element, i.e., a matrix π such that $\pi B = B$ for all $B \in \mathscr{B}$. To show this it is convenient to first consider two lemmas. In the following, the notation $\underline{R}(-)$ denotes the range (column space) of the indicated linear operator (matrix) and $\underline{N}(-)$ denotes the null space.

LEMMA 3. *Let* **A** *and* **B** *be linear operators between two real* FDHS's *such that* $\mathbf{AB}^* = 0$, *then* $\underline{R}(\mathbf{A}+\mathbf{B}) = \underline{R}(\mathbf{A})+\underline{R}(\mathbf{B})$.

LEMMA 4. *To each pair of matrices A and B in a quadratic subspace $\mathscr{B}$ there is a matrix $T \in \mathscr{B}$ such that $\underline{R}(A)+\underline{R}(B) = \underline{R}(T)$.*

PROOF. Let $A, B \in \mathscr{B}$ and let $T = P+NB^2N$ where $N = I-P$ and P is the symmetric idempotent matrix such that $\underline{R}(P) = \underline{R}(A)$. Since

$$\underline{R}(A)+\underline{R}(B) = \underline{R}(A) \oplus \underline{R}(NB) = \underline{R}(P) \oplus \underline{R}(NB^2N),$$

it is clear from Lemma 3 that $\underline{R}(A)+\underline{R}(B) = \underline{R}(T)$. Observe that

$$T = P+NB^2N = P+B^2-(PB^2+B^2P)+PB^2P.$$

Since each term in this expanded form of T is in $\mathscr{B}$, it is clear that $T \in \mathscr{B}$. Thus, the statement is established.

Suppose that $\mathscr{B}$ is a quadratic subspace of $\mathscr{A}$ and that $B_1, B_2, \cdots, B_m$ is a spanning set for $\mathscr{B}$. By consecutively applying Lemma 4, there exist matrices $T_1, T_2, \cdots, T_m$ in $\mathscr{B}$ such that

$$\underline{R}(B_1)+\underline{R}(B_2)+\cdots+\underline{R}(B_i) = \underline{R}(T_i) \quad \text{for} \quad i = 1, 2, \cdots, m.$$

Let $\pi = T_m T_m^+$ for one such T_m, then by (2.b) it follows that $\pi \in \mathscr{B}$ and it is clear that $\pi B = B$ for all $B \in \mathscr{B}$. Thus, π is an identity element in $\mathscr{B}$. Some properties associated with the matrix π are the following:

(3) (a) The matrix π is unique; that is, if $\pi_1 B = B$ for all $B \in \mathscr{B}$ and $\pi_1 \in \mathscr{B}$, then $\pi_1 = \pi$.

(b) $\pi B = B\pi$ for all $B \in \mathscr{B}$.

(c) $A \in \mathscr{B}$ and $\text{rank}(A) = \text{rank}(\pi) \Rightarrow \underline{R}(A) = \underline{R}(\pi)$.

(d) $A \in \mathscr{B}$ and $\text{rank}(A) = \text{rank}(\pi) \Rightarrow \{ABA : B \in \mathscr{B}\} = \mathscr{B}$.

These properties are immediate to verify.

Consider now the case when $\mathscr{B}$ is a commutative quadratic subspace of $\mathscr{A}$. The implications of commutativity are not investigated in detail; however, two points may be mentioned. Since the elements in $\mathscr{B}$ commute, it follows from (1.c) that $A, B \in \mathscr{B}$ implies that $AB = BA \in \mathscr{B}$. Using the usual definitions of matrix addition and multiplication the class $\mathscr{B}$ becomes a ring, and since it is also a linear space it follows that $\mathscr{B}$ is an algebra. Thus, under the usual definitions of matrix addition and multiplication a commutative quadratic subspace is a real commutative algebra with an identity element. Before passing on to a particular characterization of a commutative quadratic subspace, it is convenient to state the following lemma.

LEMMA 5. *Let $\mathscr{R} = \{R_1, R_2, \cdots, R_k, R\}$ be $k+1$ nonzero symmetric idempotent matrices such that R commutes with $R_1, R_2, \cdots, R_k$; $R \notin \text{sp}\,\{R_1, R_2, \cdots, R_k\}$; and $R_1, R_2, \cdots, R_k$ are pairwise orthogonal. Define*

$$P_i = RR_i \qquad i = 1, 2, \cdots, k$$

$$P_{i+k} = R_i - RR_i \qquad i = 1, 2, \cdots, k$$

and

$$P_{2k+1} = R - R(R_1 + R_2 + \cdots + R_k).$$

It follows that each of the matrices in $\mathscr{P} = \{P_1, P_2, \cdots, P_{2k+1}\}$ is symmetric and idempotent; the matrices in $\mathscr{P}$ are pairwise orthogonal; there exist at least $k+1$ linearly independent matrices in $\mathscr{P}$; and $\text{sp}\mathscr{R} \subset \text{sp}\mathscr{P}$.

Let $\mathscr{B}$ be a commutative quadratic subspace of $\mathscr{A}$ and by (2.c) let $B_1, B_2, \cdots, B_k$ be a basis for $\mathscr{B}$ such that each B_i is idempotent. From Lemma 5 it is clear that from the matrices

$$B_1B_2, \quad B_1 - B_2, \quad B_2 - B_1B_2, \quad B_3, \cdots, B_k,$$

we may select a basis $P_1, P_2, \cdots, P_k$ such that $P_1P_2 = 0$. By considering the matrices

$$P_1P_3, \quad P_2P_3, \quad P_1 - P_1P_3, \quad P_2 - P_2B_3, \quad P_3 - (P_1P_3 + P_2P_3), \quad P_4, \cdots, P_k,$$

we may select another basis $Q_1, Q_2, \cdots, Q_k$ for $\mathscr{B}$ such that $Q_1Q_2 = Q_1Q_3 = Q_2Q_3 = 0$. It is clear that this process may be continued until we obtain a basis

$R_1, R_2, \cdots, R_k$ for $\mathscr{B}$ such that $R_iR_j = 0$ for $i \neq j$. From this observation the necessity part of the following lemma is established.

LEMMA 6. *A necessary and sufficient condition for a subspace $\mathscr{B}$ to be a commutative quadratic subspace is the existence of a basis $R_1, R_2, \cdots, R_k$ for $\mathscr{B}$ such that each R_i is idempotent and such that $R_iR_j = 0$ for $i \neq j$. Moreover, apart from the indexing such a basis for a commutative quadratic subspace is unique.*

PROOF. Necessity follows from the preceding paragraph and the sufficiency is obvious; thus, consider the uniqueness statement. Let $R_1, \cdots, R_k$ be a basis for a commutative quadratic subspace $\mathscr{B}$ such that each R_i is idempotent and such $R_iR_j = 0$ for $i \neq j$. Suppose $P_1, P_2, \cdots, P_k$ is another such basis for $\mathscr{B}$. Let m be fixed and let $\{\alpha_i\}$ and $\{\beta_{hj}\}$ be the unique sets of real numbers such that $P_m = \sum_i \alpha_i R_i$ and such that $R_h = \sum_j \beta_{hj} P_j$ for $h = 1, \cdots, k$. It follows that

$$\alpha_h R_h = P_m R_h = \beta_{hm} P_m \quad \text{for} \quad h = 1, 2, \cdots, k.$$

Since the R_i's are linearly independent, $P_m \neq 0$, and the α_h's and the β_{hm}'s must be zero or one it follows that $P_m = R_h$ for some h. Since this argument holds for $m = 1, 2, \cdots, k$ the uniqueness statement is established.

3. Completeness and best quadratic estimators. Let $Y \sim N_n(0, \sum_{i=1}^m \nu_i V_i)$ and let $\bar{\mathscr{A}} = \{(A, U): A \in \mathscr{A}\}$ where U denotes the random variable YY'. Let $\mathbf{W}$ and $\mathbf{\Sigma}_T$ for $T \in \mathscr{A}$ be linear operators into $\mathscr{A}$ defined by

$$\alpha \in R^m \Rightarrow \mathbf{W}\alpha = \textstyle\sum_{i=1}^m \alpha_i V_i,$$
$$A \in \mathscr{A} \Rightarrow \mathbf{\Sigma}_T A = TAT.$$

For $\nu \in \Omega_2$ it is easily seen that the expectation of U is $\mathbf{W}\nu$ and that $\mathbf{\Sigma}_\nu$, the covariance operator of U, is $2\mathbf{\Sigma}_{W\nu}$. That is, $E[(A, U)] = (A, \mathbf{W}\nu)$ and Cov $[(A, U), (B, U)] = (A, \mathbf{\Sigma}_\nu B) = 2(A, \mathbf{W}\nu B\mathbf{W}\nu)$. By assumption Ω_2 contains a non-void open set in R^m; thus, for $Y \sim N_n(0, \sum_{i=1}^m \nu_i V_i)$ it follows that

$$\mathscr{E} = \text{sp}\,\{\mathbf{W}\nu : \nu \in \Omega_2\} = \text{sp}\,\{V_1, V_2, \cdots, V_m\},$$
$$\mathscr{V} = \text{sp}\,\{\mathbf{\Sigma}_\nu : \nu \in \Omega_2\} = \text{sp}\,\{\mathbf{\Sigma}_V : V \in \mathscr{E}\}.$$

Expressions for the expectation and the covariance operator of U when Y has a nonzero mean are omitted; however, such expressions may be found in [9] where a discussion is given concerning locally best quadratic and locally best linear plus quadratic estimators when $Y \sim N_n(X\beta, \sum_i \nu_i V_i)$.

THEOREM 1. *Suppose that $Y \sim N_n(0, \sum_{i=1}^m \nu_i V_i)$. To each $\mathscr{A}$-estimable function there exists an $\mathscr{A}$-best estimator if and only if*

$$\underline{R}(\mathbf{W}) = \text{sp}\{V_1, V_2, \cdots, V_m\}$$

is a quadratic subspace of $\mathscr{A}$.

PROOF. Since $I \in \mathscr{E} = \underline{R}(\mathbf{W})$ it follows that the identity operator $\mathbf{\Sigma}_\mathrm{I}$ on $\mathscr{A}$ is in $\mathscr{V}$. From Corollary 5.2 in [12] it follows that an $\bar{\mathscr{A}}$-best estimator exists for each

$\mathscr{A}$-estimable function if and only if $\underline{R}(\mathbf{W})$ is an invariant subspace of $\boldsymbol{\Sigma}_A$ for all $A \in \underline{R}(\mathbf{W})$. That is, if and only if

$$A \in \underline{R}(\mathbf{W}) \Rightarrow \boldsymbol{\Sigma}_A[\underline{R}(\mathbf{W})] \subset \underline{R}(\mathbf{W}),$$

which is equivalent to

$$A, B \in \underline{R}(\mathbf{W}) \Rightarrow ABA \in \underline{R}(\mathbf{W}).$$

Since $I \in \underline{R}(\mathbf{W})$ this last condition implies that $\underline{R}(\mathbf{W})$ is a quadratic subspace. Conversely, if $\underline{R}(\mathbf{W})$ is a quadratic subspace, then (2.c) implies the desired result.

If $Y \sim N_n(0, \sum_{i=1}^m v_i V_i)$ and $\underline{R}(\mathbf{W})$ is a quadratic subspace of $\mathscr{A}$, then Theorem 1 implies that $\mathscr{A}$-best estimators exist for all $\mathscr{A}$-estimable functions. It seems natural to ask if these $\mathscr{A}$-best estimators are uniformly minimum variance unbiased estimators. Before answering this question some preliminary results are needed concerning an expression for the inverse of the matrix $\mathbf{W}v$ when $v \in \Omega_2$ and $\underline{R}(\mathbf{W})$ is a quadratic subspace of $\mathscr{A}$.

Assume that $\underline{R}(\mathbf{W})$ is a quadratic subspace of $\mathscr{A}$. From (2.a) it is clear that for each A $\in \underline{R}(\mathbf{W})$ there is a conditional inverse for A which is also in $\underline{R}(\mathbf{W})$. Thus, for each vector α there is a vector γ such that $(\mathbf{W}\alpha)(\mathbf{W}\gamma)(\mathbf{W}\alpha) = \mathbf{W}\alpha$. An explicit expression for a vector γ satisfying this last expression is given in Appendix II of [10]; for present considerations, however, it is more convenient to utilize another representation. In the remainder of this paragraph suppose that the matrices $V_1, V_2, \cdots, V_m$ are linearly independent, then $\mathbf{W}^*\mathbf{W}$ is an invertible linear operator on R^m and $\mathbf{W}\eta = \mathbf{W}\alpha$ if and only if $\eta = \alpha$. Let Ω^* be the collection of all $\alpha \in R^m$ such that $(\mathbf{W}\alpha)^{-1}$ exists. For each $\alpha \in \Omega^*$ let $\theta(\alpha) = (\theta_1(\alpha), \cdots, \theta_m(\alpha))'$ denote some vector in R^m such that

$$(\mathbf{W}\alpha)^{-1} = \textstyle\sum_{i=1}^m \theta_i(\alpha) V_i = \mathbf{W}\theta(\alpha).$$

Such a vector always exists since $\underline{R}(\mathbf{W})$ is a quadratic subspace of $\mathscr{A}$, and the linear independence of $V_1, \cdots, V_m$ implies that $\theta(\alpha)$ is unique. Thus, θ is a well defined function from Ω^* into R^m. Moreover, θ is a one to one mapping of Ω^* onto Ω^* and considering θ as a function from Ω^* to Ω^* it is easily verified that $\theta^{-1} = \theta$. An expression for $\theta(\alpha)$ involving $(\mathbf{W}\alpha)^{-1}$ is given by

$$\alpha \in \Omega^* \Rightarrow \theta(\alpha) = (\mathbf{W}^*\mathbf{W})^{-1}\mathbf{W}^*(\mathbf{W}\alpha)^{-1}.$$

In the following two lemmas some properties of the set Ω^* and of the function θ are investigated. A proof of Lemma 7 may be found in Appendix I of [10]. Note also that the matrices $A_1, \cdots, A_m$ in Lemma 7 need not be linearly independent.

LEMMA 7. *Let $A_1, \cdots, A_m$ be real $n \times n$ symmetric matrices such that $A_m = I$ and let ψ denote the collection of all vectors $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)'$ such that $\sum_{i=1}^m \alpha_i A_i$ is non singular. The set ψ is both an open and a dense set in R^m.*

LEMMA 8. *Suppose that the matrices $V_1, V_2, \cdots, V_m$ used to define the linear operator $\mathbf{W}$ are linearly independent and that Ω^* and θ are defined as in the paragraph preceding Lemma 7. Let Ω^* be given the relative topology of R^m, then the function $\theta:\Omega^* \to \Omega^*$ is one to one, continuous, and onto. Moreover, the function $\theta:\Omega^* \to R^m$ is an open mapping, i.e., $\theta[O]$ is an open set in R^m for every open set O in Ω^*.*

PROOF. It was previously noted that θ is one to one and onto Ω^*. Let $\mathscr{A}$ be given the topology induced by the norm.

$$A \in \mathscr{A} \Rightarrow ||A|| = \sup\{(a'A^2a)^{\frac{1}{2}} : a'a \leqq 1\};$$

and let $\mathscr{I}$, the matrices $A \in \mathscr{A}$ such that A^{-1} exists, be given the relative topology of $\mathscr{A}$. It is well known (e.g., Theorem 9.8 in Rudin [8]) that the function from $\mathscr{I}$ onto $\mathscr{I}$ which takes A into A^{-1} is continuous. Since the functions $\mathbf{W}:\Omega^* \to \mathscr{I}$ and $(\mathbf{W}^*\mathbf{W})^{-1}\mathbf{W}^*:\mathscr{I} \to R^m$ are also continuous, it follows that

$$\theta(\alpha) = (\mathbf{W}^*\mathbf{W})^{-1}\mathbf{W}^*(\mathbf{W}\alpha)^{-1}$$

is continuous since it is the composition of continuous functions. Let O be an open set in Ω^*. Since θ is one to one, continuous, and onto Ω^*, it follows that $\theta[O]$ is open in Ω^*. However, by Lemma 7 Ω^* is an open set in R^m and hence $\theta[O]$ is an open set in R^m.

THEOREM 2. *Let $Y \sim N_n(X\beta, \sum_{i=1}^m \nu_i V_i)$ and suppose the following two conditions are satisfied:*

(4) (a) $\underline{R}(\mathbf{W}) = \text{sp}\{V_1, V_2, \cdots, V_m\}$ *is a quadratic subspace of* $\mathscr{A}$,
and (b) $\underline{R}(X)$ *is an invariant subspace of* V_i *for* $i = 1, \cdots, m$;
then the vector statistics $\mathbf{W}^*U = (Y'V_1Y, \cdots, Y'Y)'$ *and* $X'Y$ *are jointly a complete sufficient statistic.*

PROOF. We assume, without loss in generality, that the matrices $V_1, V_2, \cdots, V_m$ are linearly independent and that rank $(X) = p$. Let θ and Ω^* be defined as in Lemma 8. For each $\nu \in \Omega_2$ and $\beta \in R^p$ the density of Y is of the form

$$\begin{aligned} &h(\nu, \beta) \exp\{-\tfrac{1}{2}(y-X\beta)'(\textstyle\sum_{i=1}^m \nu_i V_i)^{-1}(y-X\beta)\} \\ &= h^*(\nu, \beta) \exp\{\textstyle\sum_{i=1}^m -(\frac{1}{2})\theta_i(\nu)y'V_iy + \beta'X'(\sum_{i=1}^m \theta_i(\nu)V_i)X(X'X)^{-1}(X'y)\} \\ &= h^*(\nu, \beta) \exp\{\textstyle\sum_{i=1}^m \phi_i(\nu)y'V_iy + [\delta(\nu, \beta)]'(X'y)\}, \end{aligned}$$

where $\phi_i(\nu) = -(\frac{1}{2})\theta_i(\nu)$ and $\delta(\nu, \beta) = (X'X)^{-1}X'(\sum_{i=1}^m \theta_i(\nu)V_i)X\beta$. Let

$$\Omega' = \{(\delta_1(\nu, \beta), \cdots, \delta_p(\nu, \beta), \phi_1(\nu), \cdots, \phi_m(\nu))' : \nu \in \Omega_2, \beta \in R^p\}.$$

For a fixed $\nu \in \Omega_2$ the matrix $(X'X)^{-1}X'(\sum_{i=1}^m \theta_i(\nu)V_i)X$ is non singular so that as β ranges over R^p it is clear that $\delta(\nu, \beta)$ also ranges over R^p. Thus

$$R^p \times (\phi_1(\nu), \cdots, \phi_m(\nu))' \in \Omega'$$

and so $\Omega' = R^p \times \phi[\Omega_2]$ where $\phi = -(\frac{1}{2})\theta$. Since there is a non-void open set in Ω_2 and since θ is an open mapping from Ω^* into R^m, it follows that there is a non-void open set in $\phi[\Omega_2]$. Hence from Section 4.3 in Lehmann [6] it follows that $\mathbf{W}^*U$ and $X'Y$ are jointly a complete sufficient statistic.

Suppose that $Y \sim N_n(X\beta, \sum_{i=1}^m \nu_i V_i)$ and that the conditions in (4) are satisfied. Let P denote the symmetric idempotent matrix such that $\underline{R}(P) = \underline{R}(X)$ and let $\mathbf{T}$ denote the linear operator $\mathbf{\Sigma}_N\mathbf{W}$ where $N = I-P$, i.e.,

$$\alpha \in R^m \Rightarrow \mathbf{T}\alpha = \mathbf{\Sigma}_N\mathbf{W}\alpha = \textstyle\sum_{i=1}^m \alpha_i N V_i N.$$

Since (4.b) is satisfied it is clear that $V_iN = NV_i = NV_iN$ for $i = 1, 2, \cdots, m$; thus, for $\alpha \in R^m$

$$\text{(5)} \qquad \mathbf{T}^*\mathbf{T}\alpha = \sum_{k=1}^{m} \alpha_k \begin{bmatrix} (NV_kN, NV_1N) \\ \cdot \\ \cdot \\ \cdot \\ (NV_kN, NV_mN) \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{tr}(V_1NV_1) \cdots \operatorname{tr}(V_1NV_m) \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \operatorname{tr}(V_mNV_1) \cdots tr(V_mNV_m) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m \end{bmatrix}.$$

From this matrix expression for $\mathbf{T}^*\mathbf{T}\alpha$ it follows from Theorem 3 in [11] that a parametric function $\sum_k \lambda_k \nu_k$ is $\mathscr{A}$-estimable if and only if there exists an α such that $\mathbf{T}^*\mathbf{T}\alpha = \lambda$; and for such an α an unbiased estimator for $\sum_k \lambda_k \nu_k$ is given by

$$(\mathbf{T}\alpha, U) = \sum_k \alpha_k Y'NV_kNY = \sum_k \alpha_k Y'(V_k - PV_kP)Y.$$

Moreover, this last expression shows that $(\mathbf{T}\alpha, U)$ is a function of the complete sufficient statistic and hence $(\mathbf{T}\alpha, U)$ is a uniformly minimum variance unbiased estimator for $\sum_k \lambda_k \nu_k$. It may also be noted that the vector statistics $\mathbf{T}^*U$ and PY are independent and are jointly a complete sufficient statistic.

Now suppose that $Y \sim N_n(X\beta, \sum_i \nu_i V_i)$ and that $\underset{\sim}{R}(X)$ is not necessarily an invariant subspace of $V_i (i = 1, 2, \cdots, m)$. For $\alpha \in R^m$ the form of $\mathbf{T}^*\mathbf{T}\alpha$ is

$$\text{(6)} \qquad \mathbf{T}^*\mathbf{T}\alpha = \begin{bmatrix} \operatorname{tr}(NV_1NV_1N) \cdots \operatorname{tr}(NV_1NV_mN) \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \operatorname{tr}(NV_mNV_1N) \cdots \operatorname{tr}(NV_mNV_mN) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_m \end{bmatrix}.$$

It is no longer necessarily true that $\sum_k \lambda_k \nu_k$ is $\mathscr{A}$-estimable if and only if there exists an α such that $\mathbf{T}^*\mathbf{T}\alpha = \lambda$. However, for the linear space of random variables

$$= \{(A, U): A \in \mathscr{A}, \quad AX = 0\},$$

the existence of an α such that $\mathbf{T}^*\mathbf{T}\alpha = \lambda$ is both a necessary and sufficient condition for $\sum_k \lambda_k \nu_k$ to be $\mathscr{N}$-estimable. We now consider conditions under which $\mathscr{N}$-best estimators exist. By considering only the subspace $\mathscr{N}$ instead of the entire space $\mathscr{A}$ we may utilize our previous results. In addition, there is at least some justification for restricting attention to the subspace $\mathscr{N}$ when searching for unbiased quadratic estimators for parametric functions of the form $\sum_k \lambda_k \nu_k$. One criterion which leads to considering only the subspace $\mathscr{N}$ is to require that the variance of a quadratic form $Y'AY$ be independent of the parameter β. Hsu [3] used this criterion and showed that by requiring Var $(Y'AY)$ to be independent of β implied that $AX = 0$. Hsu considered what is generally referred to as a fixed linear model and he did not assume normality; however, the result remains true for the present situation. Another criterion which leads to considering only the subspace $\bar{\mathscr{N}}$ is to require that a quadratic estimator of the form $Y'AY$ be invariant under the class of all transformations of the form $Y+X\alpha$ for arbitrary α. This criterion seems to be a reasonable requirement of estimators for linear functions of the parameters in the covariance matrix. The class of distributions

$$\{N_n(X\beta, \textstyle\sum_i v_i V_i) : \beta \in R^p,\ v \in \Omega_2\}$$

also remains invariant under this class of transformations.

THEOREM 3. *Suppose that* $Y \sim N_n(X\beta, \sum_{i=1}^m v_i V_i)$. *Let* $q = n - \text{rank}(X)$ *and let* Q *be an* $n \times q$ *matrix such that* $\underline{R}(Q) = \underline{N}(X')$ *and such that* $Q'Q = I_q$. *To each* $\mathscr{N}$*-estimable function there exists a* $\mathscr{N}$*-best estimator if and only if*

$$\mathscr{B} = \text{sp}\{Q'V_1Q, \cdots, Q'V_mQ\}$$

is a quadratic subspace of $\mathscr{A}^q$ *where* $\mathscr{A}^q$ *denotes the collection of real* $q \times q$ *matrices.*

PROOF. Observe that $Q'Y \sim N_q(0, \sum_{i=1}^m v_i Q'V_iQ)$ and that $Q'V_mQ = I_q$. For the subspace of quadratic estimators

$$\bar{\mathscr{A}}^q = \{Y'QAQ'Y : A \in \mathscr{A}^q\},$$

Theorem 1 implies that to each $\bar{\mathscr{A}}^q$-estimable function there exists an $\bar{\mathscr{A}}^q$-best estimator if and only if $\mathscr{B}$ is a quadratic subspace of $\mathscr{A}^q$. Since

$$\{QAQ' : A \in \mathscr{A}^q\} = \{A : A \in \mathscr{A},\quad AX = 0\},$$

the desired result follows.

COROLLARY 3.1. *Let* $Y \sim N_n(X\beta, \sum_{i=1}^m v_i V_i)$ *and suppose that* Q *and* $\mathscr{B}$ *are defined as in Theorem* 3. *If* $\mathscr{B}$ *is a quadratic subspace of* $\mathscr{A}^q$, *then the class of distributions induced by the vector random variable*

$$(Y'QQ'V_1QQ'Y, \cdots, Y'QQ'V_mQQ'Y) = (Y'NV_1NY, \cdots, Y'NY)$$

is complete.

For a random vector $Y \sim N_n(X\beta, \sum_i v_i V_i)$ it was noted previously that a necessary and sufficient condition for $\sum_k \lambda_k v_k$ to be $\bar{\mathscr{N}}$-estimable is the existence of an α such that $\mathbf{T}^*\mathbf{T}\alpha = \lambda$. It is also clear that if α is a vector such that $\mathbf{T}^*\mathbf{T}\alpha = \lambda$, then an unbiased estimator for $\sum_k \lambda_k v_k$ is given by

$$(\mathbf{T}\alpha, U) = \textstyle\sum_k \alpha_k Y'NV_kNY.$$

Thus, if the conditions in Corollary 3.1 are satisfied and $\sum_k \lambda_k v_k$ is $\bar{\mathscr{N}}$-estimable, then the best (minimum variance unbiased) estimator for $\sum_k \lambda_k v_k$ within the class of all unbiased estimators which are only a function of the random vector $Q'Y$

is given by $(\mathbf{T}\alpha, U)$ where α is any vector such that $\mathbf{T}^*\mathbf{T}\alpha = \lambda$. In addition, if the conditions in Theorem 2 (i.e., conditions (4)) are satisfied, then $(\mathbf{T}\alpha, U)$ is the uniformly minimum variance unbiased estimator for $\sum_k \lambda_k \nu_k$.

4. An example. Consider a mixed linear model corresponding to a balanced incomplete block design (Kempthorne [5], Chapter 26) with treatments fixed and blocks random. Let t, b, k, r, and λ denote the parameters associated with the design and express the $n = rt = bk$ observations $Y_{ij} = \mu + \gamma_i + \tau_j + e_{ij}$ in matrix form by

$$Y = \mu 1 + X\tau + B\gamma + e.$$

Assume that $\gamma \sim N_b(0, \sigma_b^2 I)$, $e \sim N_n(0, \sigma^2 I)$, γ and e independent; then

$$Y \sim N_n(\mu 1 + X\tau, \sigma_b^2 BB' + \sigma^2 I).$$

Let $A = X'B$ denote the incidence matrix, let J_j^i denote an $i \times j$ matrix consisting of all ones, and suppose that the parameters μ, τ, σ_b^2, and σ^2 are all unknown. For the case $k = t$, i.e., a balanced complete block, it is clear that $BB'X = BJ_t^b = J_t^n$ so that $\underset{\sim}{R}(X)$ is an invariant subspace of BB'; and since $(BB')^2 = kBB'$, it follows from Theorem 2 that there exists a complete sufficient statistic. Thus, let us now consider the more interesting case when $k < t$.

For $k < t$ it is no longer true that $\underset{\sim}{R}(X)$ is an invariant subspace of BB'. If $\underset{\sim}{R}(X)$ were an invariant subspace of BB', then best linear unbiased estimators would exist, i.e., independent of the parameters σ_b^2 and σ^2. Moreover, for the mixed linear model we are considering, a minimal sufficient statistic that is not complete was exhibited by Hultquist and Graybill [4]. For these reasons we investigate if Theorem 3 may be utilized.

Let Q be an $n \times q (q = n - t)$ matrix such that $\underset{\sim}{R}(Q) = \underset{\sim}{N}(X')$ and such that $Q'Q = I_q$. To utilize Theorem 3 it must be true that $\mathscr{B} = \mathrm{sp}\{Q'BB'Q, I_q\}$ is a quadratic subspace. In [12] a similar situation is considered and from the results in [12] it follows that $\mathscr{B}$ is a quadratic subspace if and only if $Q'BB'Q$ has no more than two distinct eigenvalues. For a balanced incomplete block design it is well known that

$$AA' = (r - \lambda)I_t + \lambda J_t^t.$$

Since $k < t$ implies that $r > \lambda$, it is clear that the eigenvalues of AA' are the nonzero real numbers $(r - \lambda)$ and $(r - \lambda) + \lambda t = rk$. Thus, the matrix

$$B'QQ'B = B'(I - (1/r)XX')B = kI_b - (1/r)A'A$$

has the eigenvalues $k - (r - \lambda)/r = \lambda t/r$ and $k - rk/r = 0$ when $b = t$ and the eigenvalues $\lambda t/r$, k, and zero when $b > t$. Since the matrix $Q'BB'Q$ is always singular, it follows that the eigenvalues of $Q'BB'Q$ are $\lambda t/r$ and zero when $b = t$ and $\lambda t/r$, k, and zero when $b > t$. Thus, from Theorem 3 it follows that $\mathscr{N}$-best estimators exist for each $\mathscr{N}$-estimable function if and only if $b = t$; and from Corollary 3.1 it follows that the class of distributions induced by the random vector $(Y'QQ'BB'QQ'Y, Y'QQ'Y)$ is complete when $b = t$.

Assume now that $b = t$ and let us find the form of the equations $\mathbf{T^*T}\alpha = \delta$. Since $N = QQ'$, it is clear from (6) that

$$\mathbf{T^*T}\alpha = \begin{bmatrix} \operatorname{tr}[(Q'BB'Q)^2] & \operatorname{tr}(Q'BB'Q) \\ \operatorname{tr}(Q'BB'Q) & \operatorname{tr}(I_q) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

The eigenvalue $(r-\lambda)$ of the matrix AA' has a multiplicity of $(t-1)$ and so the eigenvalue $\lambda t/r$ of the matrix $Q'BB'Q$ also has a multiplicity of $(t-1)$. Moreover, when $b = t$ the only nonzero eigenvalue of $Q'BB'Q$ is $\lambda t/r$. Thus, noting that $(t-1)\lambda t/r = n-t = q$, it follows that

$$\mathbf{T^*T}\alpha = q\begin{bmatrix} \lambda t/r & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

Upon solving the equations $\mathbf{T^*T}\alpha = \delta$ and substituting the solution into the expression for $(\mathbf{T}\alpha, U)$, we obtain

$$(7) \qquad (\mathbf{T}\alpha, U) = (\lambda t f)^{-1}(\lambda t\delta_2 - r\delta_1)Y'[N-(r/\lambda t)NBB'N]Y \\ +(\delta_1/q)Y'[(r/\lambda t)NBB'N]Y,$$

where f equals the degrees of freedom for the intrablock error mean square in the usual analysis of variance table, i.e., $f = n-2t+1$. Thus, as an estimator for $\delta_1\sigma_b^2+\delta_2\sigma^2$ the statistic given in (7) is the one with minimum variance among all unbiased estimators which are a function only of the random vector $Q'Y$. It might also be noted that $(\mathbf{T}\alpha, U)$ is a linear combination of the usual intrablock error sum of squares and the sum of squares for blocks eliminating treatments. To see this, note that when $b = t$, the matrix $Q'BB'Q$ has $\lambda t/r$ as its only nonzero eigenvalue. This implies that $E_b = (r/\lambda t)NBB'N$ and $E = N-E_b$ are both symmetric idempotent matrices. Thus, from the relationship

$$\underset{\sim}{R}(X)+\underset{\sim}{R}(B) = \underset{\sim}{R}(X) \circ \underset{\sim}{R}(NB),$$

it is clear that $Y'E_bY$ is the sum of squares for blocks eliminating treatments and $Y'EY$ is the intrablock error sum of squares. From this observation we note that when $b = t$ the pertinent quantities used for estimating weights in the recovery of interblock information (e.g., Kempthorne [5], Chapter 26) have the property that the class of distributions corresponding to the vector $(Y'E_bY, Y'EY)$ is complete.

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