The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 578–593

MATCHMAKING[1]

BY MORRIS H. DEGROOT, PAUL I. FEDER, AND PREM K. GOEL

Carnegie–Mellon University and Yale University

1. Introduction and summary. Matching problems are discussed in many elementary probability books, such as Feller (1968). In one version of the problem, as described by Hodges and Lehmann (1964), the photographs of n film stars are paired randomly with n photographs of the same stars taken when they were babies, and the distribution of the number of correct matches is derived. In this paper we shall study the same problem when the photographs are paired on the basis of various measurements that are made on them, rather than randomly.

For example, suppose that r different facial measurements are made on the photograph of each star and that s facial measurements are made on each baby photograph. By comparing these measurements, it will typically be possible to devise a method for pairing the photographs that will yield a larger number of correct matches than would be obtained from random pairing. In fact, the procedures that will be developed in this paper can be regarded as formalizations of the heuristic procedures that a person follows when he pairs the photographs on the basis of perceived resemblances. In other versions of the same problem, dental records of parents are to be matched with dental records of their children, or measurements made on the chest X-rays of n individuals are to be matched with other medical records of these same individuals.

The problems described here are related in principle to problems of document linkage that have been treated in the statistical literature [see, e.g., DuBois (1969) and the references given there] but the models and methods that are used here seem to be new and unrelated to the models and methods that have previously been used in such problems.

For any positive integer k, we shall let R^k denote the space of all k-dimensional vectors $z = (z_1, \cdots, z_k)$, where $-\infty < z_i < \infty$ for $i = 1, \cdots, k$.

Now let T denote an r-dimensional random vector ($r \geqq 1$), let U denote an s-dimensional random vector ($s \geqq 1$), and suppose that T and U have some specified joint distribution over the space R^{r+s}. We shall assume that a random sample of n vectors $(t_1, u_1), \cdots, (t_n, u_n)$ has been drawn from this joint distribution. It is assumed, however, that before the values in this sample can be observed, each vector (t_i, u_i) in the sample is broken into two separate vectors, namely the vector t_i with r components and the vector u_i with s components.

The vectors $t_1, \cdots, t_n$ are then observed in some random order, say $v_1, \cdots, v_n$ and the vectors $u_1, \cdots, u_n$ are observed in some independent random order, say $w_1, \cdots, w_n$. As a result of this randomization, it is not known how the vectors

Received June 22, 1970.

[1] This research was supported in part at Carnegie–Mellon University by the National Science Foundation under grant GP-8824 and in part at Yale University by the Army, Navy, Air Force, and NASA under a contract administered by the Office of Naval Research.

$v_1, \cdots, v_n$ and the vectors $w_1, \cdots, w_n$ were paired in the original sample. It is assumed that a priori (i.e., before the specific values of $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are observed) all $n!$ ways of pairing $v_1, \cdots, v_n$ with $w_1, \cdots, w_n$ are equally likely to reproduce the original sample.

The observed vectors $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ will be called the values of a *broken random sample* from the specified joint distribution of T and U. The general problem to be considered here is that of pairing the observed vectors $v_1, \cdots, v_n$ with the observed vectors $w_1, \cdots, w_n$ in order to reproduce as many of the vectors (t_i, u_i) from the original sample as possible.

The application of this model to the problem of matching the photographs of n individuals with their baby photographs or matching two sets of medical records of n individuals should be clear. The important assumption that we have made here is that the observations for the n given individuals can be regarded as the values in a random sample of size n from some larger population of individuals for which the probability distribution is known.

In this paper we shall assume that the joint distribution of T and U can be represented by a joint pdf f of the following form:

$$f(t, u) = \alpha(t)\beta(u)\, e^{\gamma(t)\delta(u)} \qquad t \in R^r, \quad u \in R^s, \tag{1.1}$$

where α, β, γ, and δ are arbitrary real-valued functions of the indicated vectors.

If either $r = 1$ or $s = 1$, and if the joint distribution of T and U is a multivariate normal distribution, then their joint pdf will be of the form (1.1). A multivariate normal distribution of this type is undoubtedly the most important special case of (1.1). In particular, if both t and u are one-dimensional, the pdf of every bivariate normal distribution is of the form (1.1).

Another example of a bivariate pdf of the form (1.1) is

$$\begin{aligned} f(t, u) &= t\, e^{-t(1+u)} && t > 0, \quad u > 0, \\ &= 0 && \text{otherwise.} \end{aligned}$$

We shall now present a summary of the specific problems that will be considered in this paper and some of the results that will be obtained.

In Section 2, the problem of pairing the vectors $v_1, \cdots, v_n$ with the vectors $w_1, \cdots, w_n$ in order to maximize the probability of a completely correct set of n matches is considered. It is shown that this probability is maximized if the values of $\gamma(v_1), \cdots, \gamma(v_n)$ are ordered from smallest to largest, the values of $\delta(w_1), \cdots \delta(w_n)$ are similarly ordered, and corresponding terms in these two orderings are paired with each other. This solution is also the maximum likelihood solution for the problem of pairing $v_1, \cdots, v_n$ with $w_1, \cdots, w_n$.

In Section 3 this maximum likelihood solution is applied to the multivariate normal distribution and is shown to have a natural and intuitive interpretation in terms of regression.

In Section 4, we consider the problem of choosing a vector w_j from the set $w_1, \cdots, w_n$ in order to maximize the probability of correctly matching one specified vector v_i from the set $v_1, \cdots, v_n$. It is shown that if $\gamma(v_i)$ is the minimum or the maximum of the n values $\gamma(v_1), \cdots, \gamma(v_n)$, then v_i should be paired with a vector w_j for which $\delta(w_j)$ is a minimum or a maximum, respectively. For intermediate values of $\gamma(v_i)$, the solution is shown to be more complicated.

In Section 5, the problem of pairing the vectors $v_1, \cdots, v_n$ with the vectors $w_1, \cdots, w_n$ in order to maximize the expected number of correct matches is considered. Although the general solution of this problem is complicated, it is shown here again that the vector v_i for which $\gamma(v_i)$ is a minimum should always be paired with the vector w_j for which $\delta(w_j)$ is a minimum and the vector v_i for which $\gamma(v_i)$ is a maximum should always be paired with the vector w_j for which $\delta(w_j)$ is a maximum. In particular, it follows that when $n = 3$, the solution to this problem and the maximum likelihood solution are always identical.

In Section 6, sufficient conditions are given under which, for an arbitrary value of n, the maximum likelihood solution will also maximize the expected number of correct matches. The simplest and most striking sufficient condition given there, but also the most severe condition, is that

$$[\max_i \gamma(v_i) - \min_i \gamma(v_i)][\max_j \delta(w_j) - \min_j \delta(w_j)] \leqq 1.$$

Finally, in Section 7, some examples are given in which these sufficient conditions are not satisfied and the maximum likelihood solution does not maximize the expected number of correct matches.

2. The maximum likelihood solution. Let $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ be the values of a broken random sample from the distribution represented by the pdf f given in (1.1). We shall now consider the problem of pairing the vectors $v_1, \cdots, v_n$ with the vectors $w_1, \cdots, w_n$ in a way that maximizes the probability or, equivalently, the likelihood of a completely correct set of n matches. Let $\varphi = [\varphi(1), \cdots, \varphi(n)]$ denote an arbitrary permutation of the integers $1, \cdots, n$, and let Φ denote the set of all $n!$ possible permutations.

If in the original sample, the vector v_i was paired with the vector $w_{\varphi(i)}$, for $i = 1, \cdots, n$, then the value of the joint pdf for the entire sample would be

$$\prod_{i=1}^n f[v_i, w_{\varphi(i)}] = [\prod_{i=1}^n \alpha(v_i)][\prod_{i=1}^n \beta(w_{\varphi(i)})] \exp[\sum_{i=1}^n \gamma(v_i)\delta(w_{\varphi(i)})]. \tag{2.1}$$

Thus, for any permutation $\varphi \in \Phi$, the likelihood that the vectors in the original sample were paired in accordance with the permutation φ is given by (2.1). Therefore, the maximum likelihood solution is given by the permutation φ for which (2.1) is maximized.

For $i = 1, \cdots, n$, let $x_i = \gamma(v_i)$ and let $y_i = \delta(w_i)$. Without loss of generality, we shall assume that the vectors $v_1, \cdots, v_n$ and the vectors $w_1, \cdots, w_n$ have been indexed so that

$$x_1 \leqq x_2 \leqq \cdots \leqq x_n \quad \text{and} \quad y_1 \leqq y_2 \leqq \cdots \leqq y_n. \tag{2.2}$$

Let x denote the column vector whose transpose is $x' = (x_1, \cdots, x_n)$, let y denote the vector whose transpose is $y' = (y_1, \cdots, y_n)$, and for any permutation $\varphi \in \Phi$, let $y(\varphi)$ denote the vector whose transpose is $[y(\varphi)]' = [y_{\varphi(1)}, \cdots, y_{\varphi(n)}]$. Then, for any given values of $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$, it can be seen from (2.1) that the maximum likelihood solution is the permutation φ for which $x'y(\varphi)$ is maximized.

It is well known [see, e.g., Hardy, Littlewood, and Pólya (1967) page 261] that since the components of x and y are ordered in accordance with (2.2), then for all $\varphi \in \Phi$, $x'y(\varphi) \leqq x'y$. Thus, the maximum likelihood solution is the permutation $\varphi^* = (1, 2, \cdots, n)$ under which each component of x is paired with the corresponding component of y. In other words, v_i is paired with w_i for $i = 1, \cdots, n$.

Furthermore, since all $n!$ permutations in the set Φ have the same prior probability of having specified the pairings in the original sample, it follows from the likelihood function (2.1) that after the vectors $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ have been observed, the posterior probability $p(\varphi)$ of any particular permutation $\varphi \in \Phi$ is

$$p(\varphi) = \frac{e^{x'y(\varphi)}}{\sum_{\psi \in \Phi} e^{x'y(\psi)}}. \tag{2.3}$$

Because the prior probabilities are assumed to be equal, the permutation φ having the highest posterior probability is again the permutation $\varphi^* = (1, 2, \cdots, n)$. In other words the posterior probability of achieving a completely correct set of n matches is maximized by the maximum likelihood solution.

3. The maximum likelihood solutions for a multivariate normal distribution. Suppose now that the joint distribution of T and U from which the original random sample was drawn is a multivariate normal distribution. Suppose also that T is one-dimensional (i.e., $r = 1$) and that U is an s-dimensional random vector ($s \geqq 1$).

Let m_T and $\sigma_T{}^2$ denote the mean and variance of T, let m_U and σ_{UU} denote the $s \times 1$ mean vector and $s \times s$ covariance matrix of U, and let σ_{TU} denote the $1 \times s$ vector of covariances of T with each component of U. If the joint pdf $f(t, u)$ of T and U is written as the product of the conditional pdf of T given $U = u$ and the marginal pdf $g(u)$ of U, we obtain

$$\begin{aligned} f(t, u) &= (\text{const}) g(u) \exp\left\{-\frac{[t - m_T - \sigma_{TU}\sigma_{UU}^{-1}(u - m_U)]^2}{2\sigma_{T\mid U}^2}\right\} \\ &= \alpha(t)\beta(u)\exp\left\{\frac{1}{\sigma_{T\mid U}^2} t[m_T + \sigma_{TU}\sigma_{UU}^{-1}(u - m_U)]\right\}, \end{aligned} \tag{3.1}$$

where $\sigma_{T\mid U}^2 = \sigma_T{}^2 - \sigma_{TU}\sigma_{UU}^{-1}\sigma_{TU}'$.

Now suppose that $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are the values of a broken random sample from this distribution. For $i = 1, \cdots, n$, let

$$x_i = v_i \tag{3.2}$$

and let

$$y_i = \frac{1}{\sigma_{T\mid U}^2}[m_T + \sigma_{TU}\sigma_{UU}^{-1}(w_i - m_U)]. \tag{3.3}$$

If it is assumed that $x_1 \leqq \cdots \leqq x_n$ and $y_1 \leqq \cdots \leqq y_n$, then it follows from (3.1) and the results obtained in Section 2 that the maximum likelihood solution is to pair v_i with w_i for $i = 1, \cdots, n$, and that this same solution will maximize the probability of a completely correct set of n matches.

This solution has the following intuitive interpretation: For any permutation $\varphi \in \Phi$,

$$[x - y(\varphi)]'[x - y(\varphi)] = x'x + y'y - 2x'y(\varphi). \tag{3.4}$$

Therefore, maximizing the value of $x'y(\varphi)$ among all permutations φ is equivalent to minimizing the value of (3.4). In other words, the values of $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ should be paired so that the sum of the squared distances between the two numbers in each pair is minimized.

For the multivariate normal distribution,

$$E(T \mid U) = m_T + \sigma_{TU}\sigma_{UU}^{-1}(U - m_U)$$

and it is known that among all functions of U, the function $E(T \mid U)$ has the maximum correlation with T. Since $x_1, \cdots, x_n$ are the observed values of T in the sample, and $y_1, \cdots, y_n$ are the observed values of $E(T \mid U)$ except for the constant factor $1/\sigma_{T \mid U}^2$, the maximum likelihood solution requires pairing these values so that the sum of the squared distances between corresponding elements is minimized. This procedure maximizes the correlation between x and $y(\varphi)$.

In particular, suppose that both $r = 1$ and $s = 1$, so the random sample is drawn from a bivariate normal distribution, and let ρ denote the correlation coefficient of this distribution. If $\rho > 0$, then it follows from (3.2) and (3.3) that the maximum likelihood solution is to order the observed values in the broken random sample so that $v_1 \leqq \cdots \leqq v_n$ and $w_1 \leqq \cdots \leqq w_n$, and then to pair v_i with w_i for $i = 1, \cdots, n$. If $\rho < 0$, then the solution is to order the observed values so that $v_1 \leqq \cdots \leqq v_n$ and $w_1 \geqq \cdots \geqq w_n$ and then to pair v_i with w_i for $i = 1, \cdots, n$. Finally, if $\rho = 0$, then all permutations are equally likely.

4. Matching an individual observation. We shall suppose again that $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are the values of a broken random sample from a distribution with pdf f as given by (1.1) and that these values have been ordered so that the relations in (2.2) are satisfied. We shall now consider the problem of pairing the vector v_1 with one of the vectors $w_1, \cdots, w_n$ in order to maximize the probability of a correct match.

For $j = 1, \cdots, n$, let $\Phi(j)$ denote the subset of Φ containing all permutations $\varphi = [\varphi(1), \cdots, \varphi(n)]$ such that $\varphi(1) = j$. Thus, $\Phi(j)$ contains the $(n-1)!$ permutations which specify that v_1 is to be paired with w_j.

If $p(\varphi)$ is defined as in (2.3), then the probability p_{1j} that the pairing of v_1 and w_j yields a correct match is

$$p_{1j} = \textstyle\sum_{\varphi \in \Phi(j)} p(\varphi). \tag{4.1}$$

It follows from (2.3) that the probability p_{1j} is maximized by a value of j such that the following value π_j is maximized:

$$\pi_j = \sum_{\varphi \in \Phi(j)} e^{x'y(\varphi)}. \tag{4.2}$$

We shall show that π_j is always maximized when $j = 1$.

For any fixed value of $j (j = 2, \cdots, n)$, there is a one-to-one correspondence between the permutations in $\Phi(1)$ and the permutations in $\Phi(j)$, defined as follows: If φ is any permutation in $\Phi(1)$, then $\varphi(1) = 1$ and $\varphi(i) = j$ for some value of $i (i = 2, \cdots, n)$. To this permutation φ, we let correspond the permutation $\psi \in \Phi(j)$ such that $\psi(1) = j$, $\psi(i) = 1$, and $\psi(k) = \varphi(k)$ for all values of k except $k = 1$ and $k = i$. If φ and ψ correspond in this way, then

$$e^{x'y(\varphi)} - e^{x'y(\psi)} = c(e^{x_1 y_1 + x_i y_j} - e^{x_1 y_j + x_i y_1}), \tag{4.3}$$

where
$$c = \exp\left[\sum_{k \neq 1, i} x_k y_{\varphi(k)}\right].$$

Since $x_1 \leqq x_i$ and $y_1 \leqq y_j$, then $x_1 y_1 + x_i y_j \geqq x_1 y_j + x_i y_1$. Therefore, it follows from (4.3) that

$$e^{x'y(\varphi)} \geqq e^{x'y(\psi)}. \tag{4.4}$$

It can now be seen from (4.2) that in the computation of π_1 and π_j, each term in the summation over $\Phi(1)$ will be at least as large as the corresponding term in the summation over $\Phi(j)$. Hence, $\pi_1 \geqq \pi_j$.

It follows from this relation that in order to maximize the probability of matching v_1 correctly, it should always be paired with w_1. Of course, the same argument shows that in order to maximize the probability of matching w_1 correctly, it should always be paired with v_1.

A similar argument can be used to show that in order to maximize the probability of matching v_n correctly, it should be paired with w_n, and that in order to maximize the probability of matching w_n correctly, it should be paired with v_n.

In general, the solution is more complicated if $i \neq 1$ and $i \neq n$ and the vector v_i must be paired with one of the vectors $w_1, \cdots, w_n$. We shall not pursue the general solution here, but shall simply illustrate this remark by the following example, which shows that when trying to match the vector v_i correctly $(i = 2, 3, \cdots, n-1)$, it may be optimal to pair v_i with w_1.

Suppose that $n = 3$ and consider the problem of pairing v_2 with w_1, w_2, or w_3, when the values of x_1, x_2, x_3 and y_1, y_2, y_3 are $x_1 = x_2 = y_1 = 0$ and $x_3 = y_2 = y_3 = 1$. The two permutations in Φ such that v_2 is paired with w_1 are $\varphi_1 = (2, 1, 3)$ and $\varphi_2 = (3, 1, 2)$. For these two permutations,

$$e^{x'y(\varphi_1)} + e^{x'y(\varphi_2)} = 2e. \tag{4.5}$$

The two permutations in Φ such that v_2 is paired with w_2 are $\psi_1 = (1,2,3)$ and $\psi_2 = (3, 2, 1)$. For these two permutations

$$e^{x'y(\psi_1)} + e^{x'y(\psi_2)} = 1 + e. \tag{4.6}$$

Since $y_2 = y_3$ in this example, this same value $1+e$ will also be obtained if v_2 is paired with w_3. It follows that the probability of obtaining a correct match is greatest when v_2 is paired with w_1.

5. Maximizing the expected number of correct matches. We shall continue to suppose that $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are the values in a broken random sample from a distribution with pdf f as given by (1.1), and we shall now consider the problem of finding a permutation $\varphi \in \Phi$ for which the expected number of correct matches is maximized. Since the sample is drawn from an absolutely continuous distribution, we can assume without loss of generality that no two vectors $v_1, \cdots, v_n$ are equal and that no two vectors $w_1, \cdots, w_n$ are equal. It is convenient to assume also that no two values of $x_1, \cdots, x_n$ are equal and that no two values of $y_1, \cdots, y_n$ are equal. In other words, we shall assume that $x_1 < \cdots < x_n$ and that $y_1 < \cdots < y_n$.

For any permutation $\varphi \in \Phi$, let $M(\varphi)$ denote the expected number of correct matches when v_i is paired with $w_{\varphi(i)}$ for $i = 1, \cdots, n$. Also, for any two permutations φ and ζ in Φ, let $N(\varphi, \zeta)$ denote the number of values of i ($i = 1, \cdots, n$) such that $\varphi(i) = \zeta(i)$. Thus, $N(\varphi, \zeta)$ is the number of correct matches when the observations in the broken sample are paired according to φ and the vectors in the original sample were actually paired according to ζ. It then follows that for any permutation $\varphi \in \Phi$,

$$M(\varphi) = \sum_{\zeta \in \Phi} N(\varphi, \zeta) p(\zeta), \tag{5.1}$$

where $p(\zeta)$ is the posterior probability given by (2.3). It can be seen from (2.3) that finding a permutation φ that maximizes $M(\varphi)$ is equivalent to finding a permutation that maximizes the following value $V(\varphi)$:

$$V(\varphi) = \sum_{\zeta \in \Phi} N(\varphi, \zeta)\, e^{x'y(\zeta)}. \tag{5.2}$$

For any permutation $\varphi \in \Phi$, and any subset of permutations $B \subset \Phi$, it is convenient to let

$$S(\varphi, B) = \sum_{\zeta \in B} N(\varphi, \zeta)\, e^{x'y(\zeta)}. \tag{5.3}$$

Then, for any disjoint subsets B_0 and B_1 such that $B_0 \cup B_1 = \Phi$,

$$V(\varphi) = S(\varphi, \Phi) = S(\varphi, B_0) + S(\varphi, B_1).$$

Let φ be any fixed permutation in Φ, and suppose that there exist integers i and j with $i < j$ and $\varphi(i) > \varphi(j)$. Since we are still assuming that the relations (2.2) are satisfied, this assumption about φ means that there are values such that $x_i < x_j$ and $y_{\varphi(i)} > y_{\varphi(j)}$. Let ψ be the permutation in Φ such that $\psi(i) = \varphi(j)$, $\psi(j) = \varphi(i)$, and $\psi(k) = \varphi(k)$ for all other values of k. We shall now derive conditions under which $V(\psi) > V(\varphi)$.

Let Φ_0 be the subset of Φ containing all permutations ζ such that $\zeta(i) \neq \varphi(i)$, $\zeta(i) \neq \varphi(j)$, $\zeta(j) \neq \varphi(i)$, and $\zeta(j) \neq \varphi(j)$. It follows that $N(\psi, \zeta) = N(\varphi, \zeta)$ for each $\zeta \in \Phi_0$. Therefore, by (5.3),

$$S(\psi, \Phi_0) = S(\varphi, \Phi_0). \tag{5.4}$$

Next, let Φ_1 be the subset of Φ containing all permutations ζ such that $\zeta(i) = \varphi(i)$ and $\zeta(j) = \varphi(j)$. Similarly, let Φ_2 be the subset of Φ containing all permutations θ such that $\theta(i) = \varphi(j)$ and $\theta(j) = \varphi(i)$. There is a one-to-one correspondence between the permutations in Φ_1 and those in Φ_2 under which $\zeta \in \Phi_1$ and $\theta \in \Phi_2$ correspond to each other if and only if $\zeta(k) = \theta(k)$ for all values of k except $k = i$ and $k = j$. If $\zeta \in \Phi_1$ and $\theta \in \Phi_2$ correspond to each other, then

$$N(\varphi, \zeta) - N(\psi, \zeta) = 2,$$
$$N(\psi, \theta) - N(\varphi, \theta) = 2.$$

Therefore,

$$\begin{aligned}
&[N(\psi, \zeta)\exp(x'y(\zeta)) + N(\psi, \theta)\exp(x'y(\theta))] \\
&\quad - [N(\varphi, \zeta)\exp(x'y(\zeta)) + N(\varphi, \theta)\exp(x'y(\theta))] \\
&= 2[\exp(x'y(\theta)) - \exp(x'y(\zeta))] \\
&= 2C[\exp(x_i y_{\theta(i)} + x_j y_{\theta(j)}) - \exp(x_i y_{\zeta(i)} + x_j y_{\zeta(j)})] \\
&= 2C[\exp(x_i y_{\varphi(j)} + x_j y_{\varphi(i)}) - \exp(x_i y_{\varphi(i)} + x_j y_{\varphi(j)})],
\end{aligned} \tag{5.5}$$

where

$$C = \exp[\textstyle\sum_{k \neq i,j} x_k y_{\zeta(k)}] > 0. \tag{5.6}$$

Since $x_i < x_j$ and $y_{\varphi(i)} > y_{\varphi(j)}$, the expression inside the final brackets in (5.5) is positive.

Therefore,

$$\begin{aligned}
&N(\psi, \zeta)\exp(x'y(\zeta)) + N(\psi, \theta)\exp(x'y(\theta)) \\
&\qquad > N(\varphi, \zeta)\exp(x'y(\zeta)) + N(\varphi, \theta)\exp(x'y(\theta)).
\end{aligned} \tag{5.7}$$

It now follows from (5.7) that

$$S(\psi, \Phi_1 \cup \Phi_2) > S(\varphi, \Phi_1 \cup \Phi_2). \tag{5.8}$$

Next, let Φ_3 be the subset of Φ containing all permutations ζ such that $\zeta(i) = \varphi(i)$ and $\varphi(j) < \zeta(j) < \varphi(i)$. Similarly, let Φ_4 be the subset of Φ containing all permutations θ such that $\theta(j) = \varphi(i)$ and $\varphi(j) < \theta(i) < \varphi(i)$. Both Φ_3 and Φ_4 will be empty if $\varphi(i) = \varphi(j) + 1$. If Φ_3 and Φ_4 are not empty then there is a one-to-one correspondence between their members under which $\zeta \in \Phi_3$ and $\theta \in \Phi_4$ correspond to each other if $\zeta(j) = \theta(i)$ and $\zeta(k) = \theta(k)$ for all values of k except $k = i$ and $k = j$. If $\zeta \in \Phi_3$ and $\theta \in \Phi_4$ correspond to each other, then

$$N(\varphi, \zeta) - N(\psi, \zeta) = 1,$$
$$N(\psi, \theta) - N(\varphi, \theta) = 1.$$

Therefore,

$$
\begin{aligned}
&[N(\psi, \zeta)\exp(x'y(\zeta))+N(\psi, \theta)\exp(x'y(\theta))] \\
&\qquad -[N(\varphi, \zeta)\exp(x'y(\zeta))+N(\varphi, \theta)\exp(x'y(\theta))] \\
&\qquad = \exp(x'y(\theta))-\exp(x'y(\zeta)) \\
&\qquad = C[\exp(x_i y_{\theta(i)}+x_j y_{\theta(j)})-\exp(x_i y_{\zeta(i)}+x_j y_{\zeta(j)})] \\
&\qquad = C[\exp(x_i y_{\zeta(j)}+x_j y_{\varphi(i)})-\exp(x_i y_{\varphi(i)}+x_j y_{\zeta(j)})],
\end{aligned} \tag{5.9}
$$

where C is again given by (5.6). Since $x_i < x_j$ and $y_{\zeta(j)} < y_{\varphi(i)}$, it again follows that the expression in the final brackets in (5.9) is positive. Therefore,

$$S(\psi, \Phi_3 \cup \Phi_4) \geqq S(\varphi, \Phi_3 \cup \Phi_4). \tag{5.10}$$

Equality will hold in (5.10) only if $\Phi_3 \cup \Phi_4$ is empty.

Next, let Φ_5 be the subset of Φ containing all permutations ζ such that $\zeta(j) = \varphi(j)$ and $\varphi(j) < \zeta(i) < \varphi(i)$. Similarly, let Φ_6 be the subset of Φ containing all permutations θ such that $\theta(i) = \varphi(j)$ and $\varphi(j) < \theta(j) < \varphi(i)$. Again there is a one-to-one correspondence between the permutations in Φ_5 and Φ_6, and an argument similar to the one just given for Φ_3 and Φ_4 shows that

$$S(\psi, \Phi_5 \cup \Phi_6) \geqq S(\varphi, \Phi_5 \cup \Phi_6). \tag{5.11}$$

If $\Phi^* = \bigcup_{i=0}^{6} \Phi_i$, then (5.4), (5.8), (5.10), and (5.11) together imply that

$$S(\psi, \Phi^*) > S(\varphi . \Phi^*) \tag{5.12}$$

Therefore, in order to establish that $V(\psi) > V(\varphi)$, it is sufficient to establish conditions such that $S(\psi, \Phi - \Phi^*) \geqq S(\varphi, \Phi - \Phi^*)$.

The set $\Phi - \Phi^*$ contains all permutations ζ satisfying one of the following conditions:

$$
\begin{aligned}
&C_1 : \zeta(i) = \varphi(i), && \zeta(j) < \varphi(j), \\
&C_2 : \zeta(j) = \varphi(i), && \zeta(i) < \varphi(j), \\
&C_3 : \zeta(i) = \varphi(j), && \zeta(j) < \varphi(j), \\
&C_4 : \zeta(j) = \varphi(j), && \zeta(i) < \varphi(j), \\
&C_5 : \zeta(i) = \varphi(i), && \zeta(j) > \varphi(i), \\
&C_6 : \zeta(j) = \varphi(i), && \zeta(i) > \varphi(i), \\
&C_7 : \zeta(i) = \varphi(j), && \zeta(j) > \varphi(i), \\
&C_8 : \zeta(j) = \varphi(j), && \zeta(i) > \varphi(i).
\end{aligned} \tag{5.13}
$$

Suppose that ζ_1 is a permutation satisfying condition C_1. For convenience of notation, let $q = \zeta_1(j)$ and let h be the integer such that $\zeta_1(h) = \varphi(j)$.

Let ζ_2 be the permutation such that $\zeta_2(j) = \varphi(i)$, $\zeta_2(i) = q$, and $\zeta_2(k) = \zeta_1(k)$ for all other values of k. Then ζ_2 satisfies condition C_2.

Next, let ζ_3 be the permutation such that $\zeta_3(i) = \varphi(j)$, $\zeta_3(j) = q$, $\zeta_3(h) = \varphi(i)$, and $\zeta_3(k) = \zeta_1(k)$ for all other values of k. Then ζ_3 satisfies condition C_3.

Finally, let ζ_4 be the permutation such that $\zeta_4(j) = \varphi(j)$, $\zeta_4(i) = q$, $\zeta_4(h) = \varphi(i)$, and $\zeta_4(k) = \zeta_1(k)$ for all other values of k. Then ζ_4 satisfies condition C_4.

For these permutations,

$$N(\varphi, \zeta_1) - N(\psi, \zeta_1) = 1,$$

$$N(\psi, \zeta_2) - N(\varphi, \zeta_2) = 1,$$

$$N(\psi, \zeta_3) - N(\varphi, \zeta_3) = 1,$$

$$N(\varphi, \zeta_4) - N(\psi, \zeta_4) = 1.$$

We wish to determine conditions under which the value of

$$\sum_{\eta=1}^{4} [N(\psi, \zeta_\eta) - N(\varphi, \zeta_\eta)] \exp(x'y(\zeta_\eta)) \tag{5.14}$$

will be nonnegative. If a constant is subtracted from each of the values $x_1, \cdots, x_n$ in (5.14) and another constant is subtracted from each of the values $y_1, \cdots, y_n$, then the value of (5.14) will simply be multiplied by a positive constant. This modified value of (5.14) will therefore be nonnegative if and only if the original value of (5.14) was nonnegative. For convenience, we shall replace x_i by $x_i - x_h$ and y_i by $y_i - y_q$, for $i = 1, \cdots, n$. If we let

$$e_\eta = \exp[(x_i - x_h)(y_{\zeta_\eta(i)} - y_q) + (x_j - x_h)(y_{\zeta_\eta(j)} - y_q)]$$

for $\eta = 1, 2, 3, 4$, then the value A of (5.14) becomes

$$\begin{aligned} A &= C_0(e_2 + e_3 - e_1 - e_4) \\ &= C_0\{\exp[(x_j - x_h)(y_{\varphi(i)} - y_q)] + \exp[(x_i - x_h)(y_{\varphi(j)} - y_q)] \\ &\quad - \exp[(x_i - x_h)(y_{\varphi(i)} - y_q)] - \exp[(x_j - x_h)(y_{\varphi(j)} - y_q)]\}, \end{aligned} \tag{5.15}$$

where C_0 is a positive constant.

For any value of $t (-\infty < t < \infty)$, define

$$\lambda(t) = \exp[(y_{\varphi(i)} - y_q)t] - \exp[(y_{\varphi(j)} - y_q)t]. \tag{5.16}$$

Since $y_q < y_{\varphi(j)} < y_{\varphi(i)}$, it follows that $\lambda(t) \geqq 0$ for $t \geqq 0$, that $\lambda(t) \leqq 0$ for $t \leqq 0$, and that λ is strictly increasing for $t \geqq 0$. The value A in (5.15) can now be rewritten as $A = C_0[\lambda(x_j - x_h) - \lambda(x_i - x_h)]$. Thus, $A \geqq 0$ if and only if

$$\lambda(x_j - x_h) \geqq \lambda(x_i - x_h). \tag{5.17}$$

Since $x_i < x_j$, it follows from the properties of λ just mentioned that if $x_h < x_j$, then (5.17) will be satisfied. Now let Ξ_1 be the set of all permutations satisfying conditions C_1, C_2, C_3, or C_4 in (5.13). If the relation (5.17) is satisfied for every permutation ζ_1 satisfying condition C_1, then $S(\psi, \Xi_1) \geqq S(\varphi, \Xi_1)$. Thus, we have established the following lemma.

LEMMA 1. *Let λ be defined by* (5.16). *If $\lambda(x_j - x_h) \geqq \lambda(x_i - x_h)$ for every pair of integers (q, h) such that $1 \leqq q < \varphi(j)$* and *$j < h \leqq n$, then*

$$S(\psi, \Xi_1) \geqq S(\varphi, \Xi_1). \tag{5.18}$$

It should be noted that if $\varphi(j) = 1$ or if $j = n$ then the conditions of Lemma 1, and therefore the relation (5.18), will automatically be satisfied.

We shall now consider permutations satisfying condition C_5, C_6, C_7, or C_8 in (5.13). The development here is analogous to the development that has just been given so most of the details will be omitted. Suppose that ζ_5 is a permutation satisfying conditions C_5, and let h be the integer such that $\zeta_5(h) = \varphi(j)$. Let ζ_6, ζ_7, and ζ_8 be permutations whose components have the same relation to the components of ζ_5 as the components of ζ_2, ζ_3, and ζ_4 had to the components of ζ_1. Then ζ_6, ζ_7, and ζ_8 will satisfy conditions C_6, C_7, and C_8, respectively.

If $q = \zeta_5(j)$, then just as in (5.14), (5.15), (5.16), and (5.17), it is again true that

$$\textstyle\sum_{\eta=5}^{8} [N(\psi, \zeta_\eta) - N(\varphi, \zeta_\eta)] \exp(x'y(\zeta_\eta)) \geqq 0 \tag{5.19}$$

if and only if

$$\lambda(x_j - x_h) \geqq \lambda(x_i - x_h), \tag{5.20}$$

where λ is defined by (5.16). However, since it is now true that $\varphi(j) < \varphi(i) < q$, and therefore that $y_{\varphi(j)} < y_{\varphi(i)} < y_q$, the function λ will now have the following properties: $\lambda(t) \geqq 0$ for $t \geqq 0$, $\lambda(t) \leqq 0$ for $t \leqq 0$, and λ strictly increasing for $t \leqq 0$. Hence, if $x_i < x_h$, then the relation (5.20) will be satisfied.

Now let Ξ_5 be the set of all permutations satisfying C_5, C_6, C_7, or C_8. The following lemma is analogous to Lemma 1.

LEMMA 2. *Let λ be defined by* (5.16). *If $\lambda(x_j - x_h) \geqq \lambda(x_i - x_h)$ for every pair of integers (q, h) such that $\varphi(i) < q \leqq n$ and $1 \leqq h < i$, then*

$$S(\psi, \Xi_5) \geqq S(\varphi, \Xi_5). \tag{5.21}$$

It should be noted that if $\varphi(i) = n$ or if $i = 1$ then the relation (5.21) will automatically be satisfied.

Since $\Xi_1 \cup \Xi_5 = \Phi - \Phi^*$, it follows that if both (5.18) and (5.21) are satisfied, then

$$S(\psi, \Phi - \Phi^*) \geqq S(\varphi, \Phi - \Phi^*). \tag{5.22}$$

Together, (5.12) and (5.22) imply that $S(\psi, \Phi) > S(\varphi, \Phi)$. In other words, if both (5.18) and (5.21) are satisfied, then the expected number of matches $M(\psi)$ from ψ will be greater than the expected number of matches $M(\varphi)$ from φ. This result is summarized in the following theorem.

THEOREM 1. *Let $\varphi \in \Phi$ be a permutation such that $\varphi(i) > \varphi(j)$ for some integers i and j with $1 \leqq i < j \leqq n$. Let ψ be defined by the relations $\psi(i) = \varphi(j)$, $\psi(j) = \varphi(i)$, and $\psi(k) = \varphi(k)$ for all other values of k. Let*

$$(5.23)\quad \begin{aligned} A^* = &\exp\{[x_j - x_h][y_{\varphi(i)} - y_q]\} \\ &+\exp\{[x_i - x_h][y_{\varphi(j)} - y_q]\} \\ &-\exp\{[x_i - x_h][y_{\varphi(i)} - y_q]\} \\ &-\exp\{[x_j - x_h][y_{\varphi(j)} - y_q]\}. \end{aligned}$$

If $A^* \geqq 0$ for every pair of integers (q, h) such that either

$$(5.24)\quad 1 \leqq q < \varphi(j) \quad and \quad j < h \leqq n \qquad or$$

$$(5.25)\quad \varphi(i) < q \leqq n \quad and \quad 1 \leqq h < i, \quad then \quad M(\psi) > M(\varphi).$$

We can now establish the following important property.

THEOREM 2. *Let $\varphi^* \in \Phi$ be a permutation for which the expected number of correct matches is maximized. Then $\varphi^*(1) = 1$ and $\varphi^*(n) = n$.*

PROOF. Suppose that $\varphi \in \Phi$ is any permutation such that $\varphi(1) > 1$ and $\varphi(j) = 1$ for some integer $j > 1$. If we let $i = 1$ and choose the permutation ψ as in Theorem 1, then the conditions of Theorem 1 are vacuously satisfied. Hence, $M(\psi) > M(\varphi)$ and the permutation φ cannot maximize the expected number of matches.

Similarly, suppose that $\varphi \in \Phi$ is any permutation such that $\varphi(n) < n$ and $\varphi(i) = n$ for some integer $i < n$. If we let $j = n$ and choose ψ as in Theorem 1, then again $M(\psi) > M(\varphi)$. This completes the proof of the theorem.

It was shown in Section 4 that pairing v_1 with w_1 and v_n with w_n maximizes the probability that any one of these vectors will be matched correctly. It is now seen from Theorem 2 that in order to maximize the expected number of correct matches, it is again true that v_1 should be paired with w_1 and v_n should be paired with w_n.

THEOREM 3. *If $n = 3$, then the expected number of correct matches is maximized by pairing v_i with w_i for $i = 1, 2, 3$.*

PROOF. It is known from Theorem 2 that v_1 must be paired with w_1 and v_3 with w_3. Therefore, v_2 must be paired with w_2.

6. Sufficient conditions for a simple solution. We shall now develop an elementary condition under which, for an arbitrary value of $n > 3$, the expected number of correct matches will be maximized by simply pairing v_i with w_i for $i = 1, \cdots, n$.

Consider again the expression for A^* given in (5.23), and let

$$(6.1)\quad \begin{aligned} a_1 &= x_i - x_h, & a_2 &= x_j - x_h, \\ b_1 &= y_{\varphi(j)} - y_q, & b_2 &= y_{\varphi(i)} - y_q. \end{aligned}$$

Then A^* can be rewritten in the form

$$A^* = \exp(a_2 b_2) - \exp(a_1 b_2) - \exp(a_2 b_1) + \exp(a_1 b_1). \tag{6.2}$$

For every pair of integers (q, h) satisfying (5.24), we will have

$$\begin{aligned} &a_1 < a_2 < 0, \qquad x_i - x_n \leqq a_1, \qquad x_j - x_n \leqq a_2, \\ &0 < b_1 < b_2, \qquad y_{\varphi(i)} - y_1 \geqq b_2, \qquad y_{\varphi(j)} - y_1 \geqq b_1. \end{aligned} \tag{6.3}$$

Also, for every pair of integers (q, h) satisfying (5.25), we will have

$$\begin{aligned} &0 < a_1 < a_2, \qquad x_j - x_1 \geqq a_2, \qquad x_i - x_1 \geqq a_1, \\ &b_1 < b_2 < 0, \qquad y_{\varphi(j)} - y_n \leqq b_1, \qquad y_{\varphi(i)} - y_n \leqq b_2. \end{aligned} \tag{6.4}$$

The next lemma now follows from Theorem 1.

LEMMA 3. *Suppose that the permutations φ and ψ are as specified in Theorem* 1 *and that A^* is given by* (6.2). *If $A^* \geqq 0$ for all values of a_1, a_2, b_1, and b_2 satisfying either* (6.3) *or* (6.4), *then $M(\psi) > M(\varphi)$.*

Next, it should be noted that A^*, as given by (6.2), can be regarded as the second mixed difference of the function e^{ab}. Therefore, if

$$\frac{\partial^2 e^{ab}}{\partial a \, \partial b} \geqq 0 \tag{6.5}$$

for all values of a and b such that

$$x_i - x_n \leqq a < 0 \quad \text{and} \quad 0 < b \leqq y_{\varphi(i)} - y_1, \tag{6.6}$$

then it will be true that $A^* \geqq 0$ for all values of a_1, a_2, b_1, and b_2 satisfying (6.3). Similarly, if (6.5) is satisfied for all values of a and b such that

$$0 < a \leqq x_j - x_1 \quad \text{and} \quad y_{\varphi(j)} - y_n \leqq b < 0, \tag{6.7}$$

then it will be true that $A^* \geqq 0$ for all values of a_1, a_2, b_1, and b_2 satisfying (6.4).

These considerations lead to the following theorem.

THEOREM 4. *Let the permutations φ and ψ be as specified in Theorem* 1. *If*

$$j = n \quad \text{or} \quad \varphi(j) = 1 \quad \text{or} \quad (x_n - x_i)[y_{\varphi(i)} - y_1] \leqq 1 \tag{6.8}$$

and if

$$i = 1 \quad \text{or} \quad \varphi(i) = n \quad \text{or} \quad (x_j - x_1)[y_n - y_{\varphi(j)}] \leqq 1, \tag{6.9}$$

then $M(\psi) > M(\varphi)$.

PROOF. If $j = n$ or $\varphi(j) = 1$, then the relations in (5.24) are satisfied vacuously. Similarly, if $i = 1$ or $\varphi(i) = n$, then the relations in (5.25) are satisfied vacuously.

Suppose next that the final inequality in (6.8) is satisfied. Then $ab \geqq -1$ for any values of a and b satisfying (6.6). Since

$$\frac{\partial^2 e^{ab}}{\partial a\, \partial b} = (1+ab)\, e^{ab},$$

it follows that (6.5) is satisfied for all values of a and b satisfying (6.6).

Suppose next that the final inequality in (6.9) is satisfied. Then it can similarly be shown that (6.5) is satisfied for all values of a and b satisfying (6.7). The theorem now follows from Theorem 1, Lemma 3, and the remarks leading up to the present theorem.

THEOREM 5. *If*

$$(6.10) \qquad (x_n - x_2)(y_{n-1} - y_1) \leqq 1 \quad \textit{and} \quad (x_{n-1} - x_1)(y_n - y_2) \leqq 1,$$

then the expected number of correct matches is maximized by pairing v_i *with* w_i *for* $i = 1, \cdots, n$.

PROOF. It follows from Theorem 2 that in order to maximize the expected number of correct matches we need only consider permutations φ such that $\varphi(1) = 1$ and $\varphi(n) = n$.

Suppose that $\varphi \in \Phi$ is any such permutation and suppose that $2 \leqq \varphi(j) < \varphi(i) \leqq n-1$ for some integers i and j with $2 \leqq i < j \leqq n-1$. If (6.10) is satisfied, then both conditions (6.8) and (6.9) are also satisfied and it follows from Theorem 4 that φ cannot maximize the expected number of correct matches. Therefore, this expectation will be maximized only by the permutation $\varphi^* = (1, 2, \cdots, n)$.

COROLLARY 1. *If* $(x_n - x_1)(y_n - y_1) \leqq 1$, *then the expected number of correct matches is maximized by pairing* v_i *with* w_i *for* $i = 1, \cdots, n$.

PROOF. If $(x_n - x_1)(y_n - y_1) \leqq 1$, then the relations in (6.10) will be satisfied. The corollary now follows from Theorem 5.

Let us now consider again the important special example in which $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are the values of a broken random sample from a bivariate normal distribution with means m_T and m_U, variances σ_T^2 and σ_U^2, and correlation coefficient ρ. It is assumed without loss of generality that $0 \leqq \rho < 1$. By (3.2) and (3.3) we then have, for $i = 1, \cdots, n$,

$$x_i = v_i,$$

$$y_i = \frac{1}{(1-\rho^2)\sigma_T^2}\left[m_T + \frac{\rho\sigma_T}{\sigma_U}(w_i - m_U) \right].$$

In accordance with the assumption that $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, it is assumed here that $v_1 < \cdots < v_n$ and $w_1 < \cdots < w_n$. By Corollary 1 it now follows that if

$$\frac{\rho}{(1-\rho^2)\sigma_T \sigma_U}(v_n - v_1)(w_n - w_1) \leqq 1,$$

then the expected number of correct matches is maximized by pairing v_i with w_i for $i = 1, \cdots, n$.

7. Concluding examples. Let us now consider again a broken random sample from an arbitrary distribution represented by a pdf of the form (1.1). As usual, it is assumed that $x_1 \leqq \cdots \leqq x_n$ and $y_1 \leqq \cdots \leqq y_n$. It has been shown that in order to maximize the expected number of correct matches, it is always optimal to use a permutation φ such that $\varphi(1) = 1$ and $\varphi(n) = n$. Therefore, if $n = 3$, the optimal permutation must be $\varphi^* = (1, 2, 3)$. If $n = 4$, the optimal permutation must be either $\varphi = (1, 3, 2, 4)$ or $\psi = (1, 2, 3, 4)$. A sufficient condition has been given in Theorem 4 under which $M(\psi) > M(\varphi)$. We shall now present an example in which $M(\varphi) > M(\psi)$. Suppose that

$$x_1 = 0, \quad x_2 = 0.05, \quad x_3 = 0.10, \quad x_4 = B,$$

$$y_1 = 0, \quad y_2 = B-2\varepsilon, \quad y_3 = B-\varepsilon, \quad y_4 = B,$$

where $\varepsilon > 0$ is small and $B > 0$ is large. If the function V is defined as in (5.2), then it can be shown that for any sufficiently large value of B,

$$\frac{d}{d\varepsilon}[V(\varphi) - V(\psi)]_{\varepsilon=0} > 0. \tag{7.1}$$

Since $y_2 = y_3$ when $\varepsilon = 0$, it is clear that $V(\varphi) = V(\psi)$ when $\varepsilon = 0$. Therefore, it follows from (7.1) that $V(\varphi) > V(\psi)$ for sufficiently large values of B and sufficiently small but positive values for ε. Consequently, $M(\varphi) > M(\psi)$ for such values.

The sufficient conditions for $M(\psi) \geqq M(\varphi)$ given in Theorem 4 and Theorem 5 and in Corollary 1 are typically more restrictive than is necessary. As an illustration, suppose that $\varepsilon = 0.05$ in the example being considered here. Then both Theorem 4 and Theorem 5 state that $M(\psi) \geqq M(\varphi)$ for any value of B such that $B \leqq 1.05$. However, numerical computations show that actually $M(\psi) \geqq M(\varphi)$ for $B \leqq 2.7$ and $M(\psi) < M(\varphi)$ for $B > 2.7$. Similarly, in this example, Corollary 1 states that if $B \leqq 1$, then $M(\psi) \geqq M(\varphi)$ for any value of ε. Numerical computations show that if $B \leqq 2.45$, then $M(\psi) \geqq M(\varphi)$ for any value of ε. For various larger values of B, Table 1 shows the minimum value ε^* of ε such that $M(\psi) \geqq M(\varphi)$.

TABLE 1

B	ε^*
2.5	.011
2.6	.037
2.7	.049
2.8	.065
2.9	.079
3.0	.092
3.5	.142
4.0	.172
4.5	.191

As another example along similar lines, suppose again that $n = 4$ and consider the values

$$x_1 = 0, \quad x_2 = \varepsilon, \quad x_3 = 2\varepsilon, \quad x_4 = 5,$$
$$y_1 = 0, \quad y_2 = 5-2\varepsilon, \quad y_3 = 5-\varepsilon, \quad y_4 = 5.$$

If φ and ψ are again defined as before, then a direct computation shows that

$$\frac{d}{d\varepsilon}[V(\varphi)-V(\psi)]_{\varepsilon=0} = 0,$$
$$\frac{d^2}{d\varepsilon^2}[V(\varphi)-V(\psi)]_{\varepsilon=0} > 0.$$

Therefore, it again follows that $M(\varphi) > M(\psi)$ for sufficiently small but positive values of ε.

As a final example, suppose that $n = 5$ and consider the values

$$x_1 = 0, \quad x_2 = 0.05, \quad x_3 = 0.10, \quad x_4 = 0.15, \quad x_5 = 8,$$
$$y_1 = 0, \quad y_2 = 8-3\varepsilon, \quad y_3 = 8-2\varepsilon, \quad y_4 = 8-\varepsilon, \quad y_5 = 8.$$

If $\varphi = (1, 4, 3, 2, 5)$ and $\psi = (1, 2, 3, 4, 5)$, then it can be shown that

$$\frac{d}{d\varepsilon}[V(\varphi)-V(\psi)]_{\varepsilon=0} > 0.$$

Therefore, $M(\varphi) > M(\psi)$ for small but positive values of ε.

Acknowledgment. We would like to thank Professor Milton Chew for many helpful suggestions during the formative stages of this paper.

REFERENCES

[1] Du Bois, N. S. D'Andrea, Jr. (1969). A solution to the problem of linking multivariate documents. *J. Amer. Statist. Assoc.* **64** 163–174.
[2] Feller, William (1968). *An Introduction to Probability Theory and Its Applications* **1** 3rd ed. Wiley, New York.
[3] Hardy, G. H., Littlewood, J. E. and Pólya, G. (1967). *Inequalities* 2nd ed. Cambridge Univ. Press.
[4] Hodges, J. L. Jr. and Lehmann, E. L. (1964). *Basic Concepts of Probability and Statistics*. Holden-Day, San Francisco.