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AN ASYMPTOTIC EXPANSION OF THE NON-NULL DISTRIBUTION OF HOTELLING'S GENERALIZED T_0^2-STATISTIC[1]

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1. Introduction and summary. Let $Z = \{z_1, z_2, \cdots, z_m\}$ be a $p \times m$ random matrix where z_i are independently distributed according to p-variate normal distributions with means μ_i and common covariance matrix $\Lambda = (\lambda_{ij})$ (>0, positive definite) and let $nS_n = n(s_{ij})$ be a $p \times p$ matrix which is independent of Z and is subject to a central Wishart distribution $W_p(\Lambda, n)$ with n degrees of freedom and covariance matrix Λ. Hotelling's generalized T_0^2-statistic is then defined by

$$T_0^2 = \operatorname{tr} S_n^{-1} ZZ' = \sum_{i=1}^m z_i' S_n^{-1} z_i \tag{1.1}$$

which is the statistic proposed by Hotelling [4], [5] for testing the hypothesis H: $M = \{\mu_1, \cdots, \mu_m\} = 0$ against K: $M \neq 0$.

The distribution of T_0^2 when H is true has been treated by several authors: for example, Hotelling [5], Itô [6], Siotani [13], [14], Pillai and Samson [12], and Davis [3]. Even in the null case, the exact distribution of T_0^2 is not available except for certain special values of p and m.

When $m = 1$, T_0^2 reduces to Hotelling's generalization of "Student's" t and if $p = 1$, $(1/m)T_0^2$ is simply F-statistic. Hence the non-null distributions in these cases are known exactly. For the general non-null case, Constantine [2] has obtained the exact distribution as well as moments of T_0^2/n using the generalized Laguerre polynomials of matrix argument. Unfortunately this distribution is valid only over the range $0 \leqq T_0^2/n < 1$ and hence not so useful since we are usually interested in the upper tail of the distribution. Siotani [15] has treated an asymptotic expansion for the non-null distribution of T_0^2 according to the basic idea due to Welch [17] and James [9]. The same problem has been attacked by Itô [7], using the integral representation of the characteristic function of T_0^2. However formulas of these authors are inadequate for a good approximation to the distribution since they have only the terms up to order n^{-1} (for some numerical information, see [8]) and also somewhat inconvenient terms for numerical work.

Khatri and Pillai have evaluated the moments of their statistic $U^{(p)}$ (a constant times T_0^2) and given approximate distributions of $U^{(p)}$ (and hence of T_0^2) in the light of the first four general noncentral moments, the summary of which can be obtained in their recent paper [10].

In this paper an asymptotic expansion for the non-null distribution of T_0^2 is given up to the terms of order n^{-2}, in which the effect of the noncentrality is

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contained in the form $s_j = \operatorname{tr} \Omega^j, j = 1, 2, \cdots$ where $\Omega = \Lambda^{-1}MM'$. This is carried out by expanding the characteristic function of T_0^2 along the Welch–James idea and by inverting the resultant into the corresponding expansion of the distribution function or the density function.

2. Expression of the characteristic function of T_0^2 by the differential operator. Let $\phi(t)$ be the characteristic function (ch.f.) of $T_0^2 = \operatorname{tr} S_n^{-1}ZZ'$ and $\phi(t: S_n)$ be the conditional ch.f. of T_0^2 when S_n is fixed. According to the method due to Welch [17] and James [9], $\phi(t)$ can then be expressed by using the differential operator as follows:

$$\begin{aligned} \phi(t) &= E_{S_n}[\phi(t: S_n)] = E_{S_n}[\exp\{\operatorname{tr}(S_n - \Lambda)\partial\}\phi(t: \Lambda)] \\ &= E_{S_n}[\exp\{\operatorname{tr} S_n\partial\}]\exp\{-\operatorname{tr}\Lambda\partial\}\phi(t: \Lambda) \\ &= \Theta\cdot\phi(t: \Lambda) \end{aligned} \tag{2.1}$$

where $\partial = (\partial_{ij}) = ((\frac{1}{2})(1+\delta_{ij})\partial/\partial\lambda_{ij})$, ($\delta_{ij}$ is Kronecker's symbol), a $p \times p$ symmetric matrix of differential operators and

$$\begin{aligned} \Theta &= \exp\{-\operatorname{tr}\Lambda\partial - (n/2)\log|I-(2/n)\Lambda\partial|\} \\ &= 1 + \frac{1}{n}\sum \lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu} \\ &\quad + \frac{1}{n^2}\{\tfrac{4}{3}\sum \lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw} + \tfrac{1}{2}\sum \lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\} \\ &\quad + O(n^{-3}). \end{aligned} \tag{2.2}$$

Symbol $\sum$ stands for the summation with respect to subscripts in the summand, each of which runs independently over $\{1, 2, \cdots, p\}$ and this simplification is continued throughout the paper unless otherwise specified.

It is seen from (2.1) and (2.2) that in order to obtain an asymptotic expansion of $\phi(t)$, we need to evaluate the various derivatives, $\partial_{rs}\phi(t: \Lambda)$, $\partial_{rs}\partial_{tu}\phi(t: \Lambda)$, etc. This can be done by using the idea of *perturbation* in physics in the following way: Let us consider

$$\phi(t: \Lambda + \mathscr{E}) = E[\exp\{it \operatorname{tr}(\Lambda + \mathscr{E})^{-1}ZZ'\}], \quad (\text{where } i = -1^{\frac{1}{2}}) \tag{2.3}$$

where $\mathscr{E} = (\varepsilon_{ij})$ is a $p \times p$ real symmetric matrix composed of small increments ε_{ij} to λ_{ij} such that $\Lambda + \mathscr{E}$ is still positive definite. Then we have by Taylor's expansion

$$\begin{aligned} \phi(t: \Lambda + \mathscr{E}) = [1 &+ \sum \varepsilon_{rs}\partial_{rs} + \tfrac{1}{2}\sum \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu} \\ &+ \tfrac{1}{6}\sum \varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\partial_{rs}\partial_{tu}\partial_{vw} + \tfrac{1}{24}\sum \varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy} \\ &+ \cdots]\phi(t: \Lambda). \end{aligned} \tag{2.4}$$

On the other hand, we evaluate the expectation in the right-hand side of (2.3) in the expanded form with respect to powers of ε_{ij}'s. After that, if we correctly compare both the expansions, we could obtain the desired derivatives.

3. Preliminary formulas. Let us use the following notations: $e(t) = (1-2it)^{-1}$,

$$\Omega = \Lambda^{-1}MM', \quad \omega^2 = \operatorname{tr}\Omega = s_1, \quad \Delta(t) = e(t)-1 = 2it(1-2it)^{-1},$$

$X = (\Lambda+\mathscr{E})^{-1}\Lambda - I$, $\varphi_f(t:\omega^2) = e(t)^{f/2}\exp\{(\frac{1}{2})\omega^2\Delta(t)\}$, the ch.f. of the noncentral chi-square distribution with f degrees of freedom and noncentrality parameter ω^2.

LEMMA 3.1.

$$\text{(3.1)} \qquad \phi(t:\Lambda+\mathscr{E}) = |I-\Delta(t)X|^{-m/2}e^{-\omega^2/2} \cdot \exp[(\tfrac{1}{2})e(t)\operatorname{tr}\{I-\Delta(t)X\}^{-1}\Omega]\varphi_{mp}(t:0).$$

PROOF. Since $z_\alpha(\alpha = 1,\cdots,m)$ is distributed independently according to $N_p(\mu_\alpha,\Lambda)$,

$$\text{(3.2)} \qquad \phi(t:\Lambda+\mathscr{E}) = (2\pi)^{-mp/2}|\Lambda|^{-m/2} \cdot \int_{R^{mp}}\cdots\int \exp\{it\operatorname{tr}(\Lambda+\mathscr{E})^{-1}ZZ' - \tfrac{1}{2}\operatorname{tr}\Lambda^{-1}(Z-M)(Z-M)'\}\prod_{\alpha=1}^{m}dz_\alpha$$

where R^{mp} stands for the whole mp-space. We make the nonsingular linear transformation $Z = PY$ with P such that

$$\text{(3.3)} \qquad P'(\Lambda+\mathscr{E})^{-1}P = I, \qquad P'\Lambda^{-1}P = I-\xi$$

where ξ is a diagonal matrix, i.e., $\operatorname{diag}\{\xi_1,\xi_2,\cdots,\xi_p\}$ with $|\xi_\alpha| < 1$ for all $\alpha = 1,2,\cdots,p$. Then we have after integrating out Y,

$$\text{(3.4)} \qquad \phi(t:\Lambda+\mathscr{E}) = \left\{\frac{|I-e(t)\xi|}{|I-\xi|}\right\}^{-m/2}\exp\{-\tfrac{1}{2}\operatorname{tr}(I-\xi)HH'\} \cdot \exp[\tfrac{1}{2}e(t)\operatorname{tr}(I-\xi)\{I-e(t)\xi\}^{-1}(I-\xi)HH']\varphi_{mp}(t:0),$$

where $H = P^{-1}M$. On account of the relations in (3.3),

$$(I-\xi)HH' = P'\Lambda^{-1}MM'P'^{-1} = P'\Omega P'^{-1}$$

$$(I-\xi)\{I-e(t)\xi\}^{-1} = P'\{I-\Delta(t)X\}^{-1}P'^{-1}.$$

(3.1) is obtained by substituting these results into (3.4).

In particular when $M = 0$, (3.1) reduces to the simple form

$$\text{(3.5)} \qquad \phi(t:\Lambda+\mathscr{E}) = |I-\Delta(t)X|^{-m/2}\varphi_{mp}(t:0).$$

The following are used to expand $\phi(t:\Lambda+\mathscr{E})$ in a power series of ε_{ij}'s starting with the expression of (3.1).

LEMMA 3.2. *Let A be a matrix whose characteristic roots are all less than unity in absolute value. Then*

$$(I-A)^{-1} = \textstyle\sum_{j=0}^{\infty} A^j, \tag{3.6}$$

$$\begin{aligned}|I-A|^{-m/2} &= 1+\frac{m}{2}s_1+\frac{m}{8}(2s_2+ms_1{}^2)+\frac{m}{48}(8s_3+6ms_2s_1+m^2s_1{}^3)\\ &\quad+\frac{m}{384}(48s_4+32ms_3s_1+12ms_2{}^2+12m^2s_2s_1{}^2+m^3s_1{}^4)\\ &\quad+\cdots\end{aligned} \tag{3.7}$$

where $s_j = \operatorname{tr} A^j$, $j = 1, 2, \cdots$.

In the course of the expansion, we use the following two kinds of symbols:

(I) $[rs] = \operatorname{tr}\Lambda^{-1}\Lambda_{rs}(=\lambda^{rs})$, $[rs \mid tu] = \operatorname{tr}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}(=\frac{1}{2}(\lambda^{ur}\lambda^{st}+\lambda^{us}\lambda^{rt}))$, etc.

(II) $(rs) = \operatorname{tr}\Lambda^{-1}\Lambda_{rs}\Omega$, $(rs \mid tu) = \operatorname{tr}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}\Omega$, etc.

where $\Lambda_{rs} = \partial_{rs}\Lambda$, and λ^{rs}'s are elements of Λ^{-1}. The actual displays of symbols of the first kind can be obtained by considering all pairs of letters one taken from each of two different groups.

4. Derivatives of ϕ (t: Λ). It is seen from (2.2) that in order to obtain the expansion of $\phi(t)$ up to order n^{-2}, we need to evaluate the derivatives of $\phi(t:\Lambda)$ up to the fourth degree. It turns out that $\phi(t:\Lambda+\mathscr{E})$ must be expanded explicitly up to the fourth power of ε_{rs}'s. First of all we expand (3.1) with respect to X up to the fourth degree with the aid of the formulas (3.6) and (3.7). In order to express the resultant in X in terms of ε_{rs}'s, it is convenient to expand X in the form

$$\begin{aligned}X &= (\Lambda+\mathscr{E})^{-1}\Lambda-I = (I+\Lambda^{-1}\mathscr{E})^{-1}-I = (I+\textstyle\sum\varepsilon_{rs}\Lambda^{-1}\Lambda_{rs})^{-1}-I\\ &= -\textstyle\sum\varepsilon_{rs}\Lambda^{-1}\Lambda_{rs}+\sum\varepsilon_{rs}\varepsilon_{tu}\Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}-\cdots\end{aligned} \tag{4.1}$$

since $\mathscr{E} = \sum\varepsilon_{rs}\Lambda_{rs}$.

The result of this computation is

$$\begin{aligned}\phi(t:\Lambda+\mathscr{E}) &= [1-\textstyle\sum\varepsilon_{rs}A^{(1)}_{rs}(t)+\sum\varepsilon_{rs}\varepsilon_{tu}A^{(2)}_{rs,tu}(t)\\ &\quad-\textstyle\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}A^{(3)}_{rs,tu,vw}(t)\\ &\quad+\textstyle\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}A^{(4)}_{rs,tu,vw,xy}(t)-\cdots]\varphi_{mp}(t:\omega^2),\end{aligned} \tag{4.2}$$

where

$$A^{(1)}_{rs}(t) = \frac{m}{2}\Delta(t)[rs]+\tfrac{1}{2}e(t)\Delta(t)(rs), \tag{4.3}$$

$$(4.4)\qquad A^{(2)}_{rs,tu}(t) = \frac{m}{4}\{2\Delta(t)+\Delta^2(t)\}[rs \mid tu]+\frac{m^2}{8}\Delta^2(t)[rs][tu]$$
$$+\tfrac{1}{2}e(t)\{\Delta(t)+\Delta^2(t)\}(rs \mid tu)+\tfrac{1}{8}e^2(t)\Delta^2(t)(rs)(tu)$$
$$+\frac{m}{4}e(t)\Delta^2(t)[rs](tu)$$

and the similar but much longer expressions for $A^{(3)}_{rs,tu,vw}(t)$ and $A^{(4)}_{rs,tu,vw,xy}(t)$ are omitted here to save the space but they are available in [16].

As stated in the end of Section 2, we now need the comparison of (4.2) with (2.4). In doing so, however, we have to take account of the symmetry in subscripts rs, tu, etc. Let us define

$$(4.5)\qquad B^{(j)}_{rs,\ldots}(t) = \frac{1}{j!}\sum\nolimits_{(\mathscr{P}_j)} A^{(j)}_{rs,\ldots}(t), \qquad j = 1, 2, \cdots$$

where $\sum_{(\mathscr{P}_j)}$ stands for the summation over all the permutations of subscripts rs, tu, $\cdots$ of $A^{(j)}_{rs,tu,\ldots}(t)$. Then we have

$$(4.6)\qquad \partial_{rs}\phi(t\colon \Lambda) = -B^{(1)}_{rs}(t)\varphi_{mp}(t\colon \omega^2),$$

$$(4.7)\qquad \partial_{rs}\partial_{tu}\phi(t\colon \Lambda) = 2B^{(2)}_{rs,tu}(t)\varphi_{mp}(t\colon \omega^2),$$

$$(4.8)\qquad \partial_{rs}\partial_{tu}\partial_{vw}\phi(t\colon \Lambda) = -6B^{(3)}_{rs,tu,vw}(t)\varphi_{mp}(t\colon \omega^2),$$

$$(4.9)\qquad \partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\phi(t\colon \Lambda) = 24B^{(4)}_{rs,tu,vw,xy}(t)\varphi_{mp}(t\colon \omega^2).$$

Hence, from (2.1), the ch.f. of $T_0{}^2$ can be written in the following expanded form:

$$(4.10)\qquad \phi(t) = \Bigg[1+\frac{2}{n}\sum \lambda_{ur}\lambda_{st}B^{(2)}_{rs,tu}(t)$$
$$+\frac{1}{n^2}\{-8\sum \lambda_{wr}\lambda_{st}\lambda_{uv}B^{(3)}_{rs,tu,vw}(t)+12\sum \lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}B^{(4)}_{rs,tu,vw,xy}(t)\}$$
$$+O(n^{-3})\Bigg]\varphi_{mp}(t\colon \omega^2).$$

5. The evaluation of the summations in (4.10). We explain, in this section, the outline of the computation of the summations in (4.10). First of all we simplify the terms in $B^{(j)}_{rs,tu,\ldots}(t)$'s using the properties of the trace. For example,

$$\frac{1}{3!}\sum\nolimits_{(\mathscr{P}_3)}[rs \mid tu](vw) = \tfrac{1}{3}\{[rs \mid tu](vw)+[rs \mid vw](tu)+[tu \mid vw](rs)\}.$$

Next we evaluate the values of various summations of types like

(a) $\sum \lambda_{ur}\lambda_{st}[rs \mid tu]$, (b) $\sum \lambda_{wr}\lambda_{st}\lambda_{uv}(rs \mid tu)(vw)$, and (c) $\sum \lambda_{wr}\lambda_{st}\lambda_{uv}[rs \mid tu](vw)$.

The summation of type (a) can be easily calculated using the concrete displays of symbols of the (I)-type. The summations of the mixed type (c) is obtained by firstly summing up with respect to subscripts contained in the brackets and then by using the results for type (b). As an example for the type (b), let us consider

$K = \sum \lambda_{wr}\lambda_{st}\lambda_{uv}(rs)(tu \mid vw)$. Let $\Lambda^{-1} = \{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(p)}\}$, and

$$\Omega' = \{\omega_1, \omega_2, \cdots, \omega_p\}.$$

Then

$$\begin{aligned}(rs)(tu \mid vw) &= (\text{tr}\, \Lambda^{-1}\Lambda_{rs}\Omega)(\text{tr}\, \Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\Lambda_{vw}\Omega)\\ &= \tfrac{1}{8}\{\omega_r'\lambda^{(s)}(\lambda^{(u)}\lambda^{tw}+\lambda^{(t)}\lambda^{uw})'\omega_v+\omega_r'\lambda^{(s)}(\lambda^{(u)}\lambda^{tv}+\lambda^{(t)}\lambda^{uv})'\omega_w\\ &\quad +\omega_s'\lambda^{(r)}(\lambda^{(u)}\lambda^{tw}+\lambda^{(t)}\lambda^{uw})'\omega_v+\omega_s'\lambda^{(r)}(\lambda^{(u)}\lambda^{tv}+\lambda^{(t)}\lambda^{uv})'\omega_w\}\\ &= \tfrac{1}{8}\{k_1(r, s, t, u, v, w)+k_2(r, s, t, u, v, w)+k_3(r, s, t, u, v, w)\\ &\quad +k_4(r, s, t, u, v, w)\},\end{aligned}$$

and

$$K_1 = \sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_1(r, s, t, u, v, w) = \sum_{rv} \omega_r'\delta_r\delta_v'\omega_v+\sum_{rv} \lambda_{vr}\omega_r'\Lambda^{-1}\omega_v$$

where $\delta_r' = (\delta_{1r}, \delta_{2r}, \cdots, \delta_{pr})$ and δ_{jr}'s are Kronecker's symbols. The first term is simply equal to $(\text{tr}\,\Omega)^2 = \omega^4 = s_1^2$ and the second term is equal to

$$\begin{aligned}\text{tr}\, \Lambda^{-1}(\textstyle\sum_{rv} \lambda_{vr}\omega_v\omega_r') &= \text{tr}\, \Lambda^{-1}\Omega'\Lambda\Omega = \text{tr}\, \Lambda^{-1}(\Lambda^{-1}MM')'\Lambda(\Lambda^{-1}MM')\\ &= \text{tr}\, \Omega^2 = s_2.\end{aligned}$$

Hence $K_1 = s_1^2+s_2$. Similar computation gives us $\sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_2(r, s, t, u, v, w) = (p+1)s_2 = \sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_4(r, s, t, u, v, w)$, $\sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_3(r, s, t, u, v, w) = K_1$, and we have

$$K = \tfrac{1}{4}\{s_1^2+(p+2)s_2\}.$$

If we put $\Omega = I$, K should be equal to $p(p+1)/2$, which is the value of

$$\sum \lambda_{wr}\lambda_{st}\lambda_{uv}[rs][tu \mid vw].$$

Complete list of values of individual summations of types (a), (b), and (c) is available in [16]. With the aid of these results, we can evaluate each of the summations in (4.10).

$$\begin{aligned}\sum \lambda_{ur}\lambda_{st}B^{(2)}_{rs,tu}(t) &= \sum \lambda_{ur}\lambda_{st}\left\{\frac{1}{2!} \textstyle\sum_{(\mathscr{P}_2)} A^{(2)}_{rs,tu}(t)\right\}\\ &= \frac{m}{8} p(p+1)\{2\Delta(t) +\Delta^2(t)\}+\frac{m^2}{8} p\Delta^2(t)\\ &\quad +\tfrac{1}{4}(p+1)s_1e(t)\{\Delta(t)+\Delta^2(t)\}+\tfrac{1}{8}s_2e^2(t)\Delta^2(t)+\frac{m}{4} s_1e(t)\Delta^2(t).\end{aligned}$$

Since $e(t) = \Delta(t)+1 = (1-2it)^{-1}$ and $e^k(t)$ is the characteristic function of the chi-square distribution with $2k$ degrees of freedom, it is convenient to arrange the above expression with respect to powers of $e(t)$. Then

$$\sum \lambda_{ur}\lambda_{st}B^{(2)}_{rs,tu}(t) = \tfrac{1}{8}\textstyle\sum_{k=0}^{4} a_k(m, p; \Omega)e^k(t) \tag{5.1}$$

which gives us the term of order n^{-1} in the expansion of $\phi(t)$. Coefficients of $e^k(t)$ will be given later in Theorem 6.1. Two summations in (4.10) which give the term of order n^{-2} can be calculated in the same way as the above, although a great deal of labor is necessary. The result can be written in the form

$$\begin{aligned} -8\textstyle\sum \lambda_{wr}\lambda_{st}\lambda_{uv}B^{(3)}_{rs,tu,vw}(t) + 12\textstyle\sum \lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}B^{(4)}_{rs,tu,vw,xy}(t) \\ = \tfrac{1}{96}\textstyle\sum_{k=0}^{8} b_k(m, p; \Omega)e^k(t), \end{aligned} \tag{5.2}$$

where coefficients $b_k(m, p; \Omega)$ of $e^k(t)$ will be given in Theorem 6.1.

6. The final result. The desired expanded form of $\phi(t)$ is now obtained by substituting (5.1) and (5.2) into (4.10). Since

$$e^k(t)\varphi_{mp}(t\colon \omega^2) = \varphi_{mp+2k}(t\colon \omega^2), \tag{6.1}$$

we can immediately write $\phi(t)$ in the form

$$\begin{aligned} \phi(t) = \varphi_{mp}(t\colon \omega^2) &+ \frac{1}{4n}\textstyle\sum_{k=0}^{4} a_k(m, p; \Omega)\varphi_{mp+2k}(t; \omega^2) \\ &+ \frac{1}{96n^2}\textstyle\sum_{k=0}^{8} b_k(m, p; \Omega)\varphi_{mp+2k}(t\colon \omega^2) \\ &+ O(n^{-3}). \end{aligned} \tag{6.2}$$

Let $f(x)$ be the density function of T_0^2 and let $g_r(x\colon \omega^2)$ be the density function of the noncentral chi-square distribution with r degrees of freedom and noncentrality parameter ω^2. Let $F(x) = \int_0^x f(u)\,du$, $G_r(x\colon \omega^2) = \int_0^x g_r(u\colon \omega^2)\,du$. We know that

$$g_r(x\colon \omega^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\varphi_r(t\colon \omega^2)\,dt \tag{6.3}$$

and hence from (6.2) we have immediately the following final result:

THEOREM 6.1. *An asymptotic expansion of the non-null distribution of Hotelling's generalized T_0^2 defined by* (1.1) *is given by*

$$\begin{aligned} f(x) = g_{mp}(x\colon \omega^2) &+ \frac{1}{4n}\textstyle\sum_{k=0}^{4} a_k(m, p; \Omega)g_{mp+2k}(x\colon \omega^2) \\ &+ \frac{1}{96n^2}\textstyle\sum_{k=0}^{8} b_k(m, p; \Omega)g_{mp+2k}(x\colon \omega^2) \\ &+ O(n^{-3}), \end{aligned} \tag{6.4}$$

where with the notations

$$\Omega = \Lambda^{-1}MM', \qquad s_j = \operatorname{tr}\Omega^j, \qquad j = 1, 2, \cdots, s_1 = \omega^2,$$

$$a_0(m,p;\Omega) = mp(m-p-1), \qquad a_1(m,p;\Omega) = -2m(mp-s_1),$$

$$a_2(m,p;\Omega) = mp(m+p+1)-2(2m+p+1)s_1+s_2,$$

$$a_3(m,p;\Omega) = 2\{(m+p+1)s_1-s_2\}, \qquad a_4(m,p;\Omega) = s_2,$$

$$b_0(m,p;\Omega) = mp\{m(3mp-8)(m-2p-2)+m(p+1)(3p^2+3p-4)-4(2p^2+3p-1)\},$$

$$b_1(m,p;\Omega) = -12m^2p(m-p-1)(mp-s_1),$$

$$\begin{aligned} b_2(m,p;\Omega) = {} & 6m[mp\{3m^2p+8m-(p+1)(p^2+p-4)\} \\ & -2\{(mp+2)(4m-p-1)-(p-2)(p+1)(p+3)\}s_1 \\ & +2ms_1{}^2+(mp-p^2-p+4)s_2], \end{aligned}$$

$$\begin{aligned} b_3(m,p;\Omega) = {} & -4[mp\{m(3mp+16)(m+p+1)+8m(p+1)+4(p^2+3p+4)\} \\ & -3\{3m(mp+4)(2m+p+1)-m(p+1)(p^2+p-16)+4(p^2+3p+4)\}s_1 \\ & +6\{2m^2+m(p+1)+2\}s_1{}^2+3\{(mp+8)(2m-p-1)+4(3p+4)\}s_2 \\ & -3ms_1s_2-4s_3], \end{aligned}$$

$$\begin{aligned} b_4(m,p;\Omega) = {} & 3[mp\{m(mp+8)(m+2p+2)+m(p+1)(p^2+p+4\}+4(2p^2+5p+5)\} \\ & -4\{m(mp+6)(4m+5p+5)+m(p+1)(p^2+p+14) \\ & +4(3p^2+8p+9)\}s_1+4\{6m(m+p+1)+(p^2+2p+15)\}s_1{}^2 \\ & +4\{3m^2p+36m+18p+32\}s_2-4(4m+p+1)s_1s_2-32s_3+s_2{}^2], \end{aligned}$$

$$\begin{aligned} b_5(m,p;\Omega) = {} & 12[\{m(mp+8)(m+2p+2)+m(p+1)(p^2+p+4)+4(2p^2+5p+5)\}s_1 \\ & -2\{(2m+p+1)(m+p+1)+8\}s_1{}^2-\{(mp+16)(2m+p+1) \\ & +8(p+3)\}s_2+3(2m+p+1)s_1s_2+16s_3-s_2{}^2], \end{aligned}$$

$$\begin{aligned} b_6(m,p;\Omega) = {} & 2[6\{(m+p+1)^2+6\}s_1{}^2+3\{(mp+20)(m+p+1)+12\}s_2 \\ & -6(4m+3p+3)s_1s_2-80s_3+9s_2{}^2], \end{aligned}$$

$$b_7(m,p;\Omega) = 12\{(m+p+1)s_1s_2+4s_3-s_2{}^2\},$$

$$b_8(m,p;\Omega) = 3s_2{}^2.$$

Let $\gamma_1, \gamma_2, \cdots, \gamma_q$ be the characteristic roots of Ω, where q is the rank of MM' and $q \leqq \min(m, p)$. Since

$$s_j = \operatorname{tr}\Omega^j = \textstyle\sum_{r=1}^q \gamma_r{}^j, \qquad j = 1, 2, \cdots \tag{6.5}$$

(6.4) depends on only the symmetric power-sums of γ's as it should be so.

The systematic numerical comparison between powers of $T_0{}^2$-test and the Λ-test based on the likelihood ratio criterion for testing the hypothesis $H: M = 0$ against $K: M \neq 0$ is under investigation.

The following are special cases of (6.4) in terms of cumulative distribution functions.

Case 1. When $M = 0$, i.e., when the null hypothesis H is true:

$$
\begin{aligned}
(6.6)\qquad F(x) = {} & G_{mp}(x:0) \\
& +\frac{mp}{4n}[(m-p-1)G_{mp}(x:0)-2mG_{mp+2}(x:0) \\
& \quad +(m+p+1)G_{mp+4}(x:0)] \\
& +\frac{mp}{96n^2}[\{m(3mp-8)(m-2p-2)+m(p+1)(3p^2+3p-4) \\
& \quad -4(2p^2+3p-1)\}G_{mp}(x:0) \\
& \quad -12m^2p(m-p-1)G_{mp+2}(x:0) \\
& \quad +6m\{3m^2p+8m-(p+1)(p^2+p-4)\}G_{mp+4}(x:0) \\
& \quad -4\{m(3mp+16)(m+p+1)+8m(p+1) \\
& \quad +4(p^2+3p+4)\}G_{mp+6}(x:0) \\
& \quad +3\{m(mp+8)(m+2p+2)+m(p+1)(p^2+p+4) \\
& \quad +4(2p^2+5p+5)\}G_{mp+8}(x:0)] \\
& +O(n^{-3}),
\end{aligned}
$$

which may be compared with formula (4.3) in [6] due to Itô.

Case 2. When $m = 1$: In this case, $T_0{}^2$ becomes an ordinary T^2 whose exact distribution is known (see e.g. [1]), and $\operatorname{tr}\Omega^j = \omega^{2j} = (\mu'\Lambda^{-1}\mu)^j$, $j = 1, 2, \cdots$

$$
\begin{aligned}
(6.7)\qquad F(x) = {} & G_p(x:\omega^2) \\
& -\frac{1}{4n}[p^2G_p(x:\omega^2)+2(p-\omega^2)G_{p+2}(x:\omega^2) \\
& \quad -\{(p-\omega^2)(p+2-\omega^2)-4\omega^2\}G_{p+4}(x:\omega^2) \\
& \quad -2\omega^2(p+2-\omega^2)G_{p+6}(x:\omega^2)-\omega^4G_{p+8}(x:\omega^2)] \\
& +\frac{1}{96n^2}[p(3p^3-8p^2+8)G_p(x:\omega^2)+12p^2(p-\omega^2)G_{p+2}(x:\omega^2) \\
& \quad -6\{p(p+2)(p^2-6)-2(p^3+3p^2-6p-12)\omega^2 \\
& \qquad +(p^2-6)\omega^4\}G_{p+4}(x:\omega^2) \\
& \quad -4\{7p(p+2)(p+4)+3(p^2-7p-34)\omega^2(p+2-\omega^2) \\
& \qquad -7\omega^6\}G_{p+6}(x:\omega^2) \\
& \quad +3\{p(p+2)(p+4)(p+6)-4(p+2)(p+4)(p+13)\omega^2 \\
& \qquad +4(p^2+29p+95)\omega^4-4(p+13)\omega^6+\omega^8\}G_{p+8}(x:\omega^2) \\
& \quad +12\omega^2\{(p+2)(p+4)(p+6)-(3p+25)\omega^2(p+4-\omega^2) \\
& \qquad -\omega^6\}G_{p+10}(x:\omega^2) \\
& \quad +2\omega^4\{9(p+4)(p+6)-2(9p+61)\omega^2+9\omega^4\}G_{p+12}(x:\omega^2) \\
& \quad +12\omega^6(p+6-\omega^2)G_{p+14}(x:\omega^2)+3\omega^8G_{p+16}(x:\omega^2)] \\
& +O(n^{-3}).
\end{aligned}
$$

Case 3. When $p = 1$: In this case, T_0^2/m becomes a noncentral F-statistic with m and n degrees of freedom and noncentrality parameter $\omega^2 = \sum_{r=1}^{m} \mu_j^2/\sigma^2$ and we have $\operatorname{tr}\Omega^j = \omega^{2j}, j = 1, 2, \cdots$

$$
\begin{aligned}
(6.8)\quad F(x) = {} & G_m(x:\omega^2) \\
& -\frac{1}{4n}[m(m-2)G_m(x:\omega^2) - 2m(m-\omega^2)G_{m+2}(x:\omega^2) \\
& \quad +(m-\omega^2)(m+2-\omega^2) - 2(m+1)\omega^2 G_{m+4}(x:\omega^2) \\
& \quad +2\omega^2(m+2-\omega^2)G_{m+6}(x:\omega^2) + \omega^4 G_{m+8}(x:\omega^2)] \\
& +\frac{1}{96n^2}[m(m-2)(m-4)(3m-2)G_m(x:\omega^2) \\
& \quad -12m^2(m-2)(m-\omega^2)G_{m+2}(x:\omega^2) \\
& \quad +6m\{m(m+2)(3m+2) - 4(2m^2+3m+2)\omega^2 \\
& \qquad +(3m+2)\omega^4\}G_{m+4}(x:\omega^2) \\
& \quad -4\{m(m+2)(m+4)(3m+4) - 6(3m^2+9m+8)\omega^2(m+2-\omega^2) \\
& \qquad -(3m+4)\omega^6\}G_{m+6}(x:\omega^2) \\
& \quad +3\{m(m+2)(m+4)(m+6) - 8(m+2)(m+4)(2m+5)\omega^2 \\
& \qquad +4(9m^2+48m+68)\omega^4 - 8(2m+5)\omega^6 + \omega^8\}G_{m+8}(x:\omega^2) \\
& \quad +12\omega^2\{(m+2)(m+4)(m+6) - 2(3m+11)\omega^2(m+4-\omega^2) \\
& \qquad -\omega^6\}G_{m+10}(x:\omega^2) \\
& \quad +2\omega^4\{9(m+4)(m+6) - 4(6m+29)\omega^2 + 9\omega^4\}G_{m+12}(x:\omega^2) \\
& \quad +12\omega^6(m+6-\omega^2)G_{m+14}(x:\omega^2) + 3\omega^8 G_{m+16}(x:\omega^2)] \\
& +O(n^{-3}).
\end{aligned}
$$

This can be used to check our result by comparing with the expanded form derived from the exact distribution of mF.

7. Some numerical comparison between the exact and approximate powers for $p = 2$. To see how the accuracy has improved by introducing the terms of order n^{-2}, some numerical comparison may be made from the table below between the exact and approximate powers when $p = 2$. Values of the exact powers are taken from Pillai and Jayachandran's [11] Table 10 where our T_0^2 (for $p = 2$), n, m, and $\gamma_i (i = 1, 2)$ correspond to $U^{(2)}$, $2n+3$, $2m+3$, and $\omega_i (i = 1, 2)$ in their table, respectively.

From Table 1 it is obvious that the accuracy given by the terms up to order n^{-1} is insufficient and the contribution due to the terms of order n^{-2} is considerable.

TABLE 1

The comparison between the exact and approximate powers of T_0^2-test or $p = 2$ and the significance level, $\alpha = 0.05$

n	γ_1	γ_2	Up to the order	$m = 3$	$m = 5$	$m = 7$	$m = 13$
33	0.25	0.25	$O(1)$	0.0351	0.0244	0.0180	0.0086
			$O(n^{-1})$	0.0626	0.0535	0.0472	0.0342
			$O(n^{-2})$	0.0676	0.0626	0.0602	0.0572
			Exact	0.0676	0.0624	0.0598	0.0562
	0	1	$O(1)$	0.0493	0.0326	0.0235	0.0107
			$O(n^{-1})$	0.0827	0.0676	0.0581	0.0405
			$O(n^{-2})$	0.0871	0.0765	0.0713	0.0650
			Exact	0.0871	0.0761	0.0705	0.0629
	0	3	$O(1)$	0.1266	0.0792	0.0544	0.0226
			$O(n^{-1})$	0.1793	0.1369	0.1124	0.0726
			$O(n^{-2})$	0.1801	0.1422	0.1231	0.0998
			Exact	0.1807	0.1420	0.1218	0.0940
83	0.25	0.25	$O(1)$	0.0549	0.0460	0.0404	0.0300
			$O(n^{-1})$	0.0686	0.0627	0.0593	0.0534
			$O(n^{-2})$	0.0693	0.0640	0.0613	0.0575
			Exact	0.0693	0.0640	0.0612	0.0575
	0	1	$O(1)$	0.0743	0.0592	0.0504	0.0357
			$O(n^{-1})$	0.0903	0.0784	0.0719	0.0617
			$O(n^{-2})$	0.0909	0.0795	0.0737	0.0657
			Exact	0.0909	0.0795	0.0737	0.0657
	0	3	$O(1)$	0.1730	0.1279	0.1028	0.0650
			$O(n^{-1})$	0.1953	0.1552	0.1334	0.1015
			$O(n^{-2})$	0.1952	0.1554	0.1341	0.1044
			Exact	0.1952	0.1555	0.1342	0.1044

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