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ON NONPARAMETRIC ASYMPTOTICALLY MINIMAX TESTS

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0. Summary. In this paper are studied asymptotically minimax properties of nonparametric tests. It is shown that for two different definitions of asymptotically minimax test, sequences of Wilcoxon tests are asymptotically minimax. The first definition concerns asymptotic power of tests with fixed level, and the other is a more decision-theoretic one.

1. Introduction, definitions. Minimax and asymptotically minimax properties of distribution-free tests have been studied by Hoeffding [9], Ruist [11], Chapman [3], Bell, Moser and Thompson [1], Doksum [5], [6], [7], and others.

Suppose X is a vector random variable, and let Ω be a set of probability measures P, $P(X \in A)$ being defined for all A belonging to a common σ-algebra $\mathscr{B}$. A test of the hypothesis H that P belongs to a subset ω of Ω is a function Φ, $0 \leqq \Phi(x) \leqq 1$, measurable with respect to $\mathscr{B}$ and interpreted as the probability of rejecting H when $X = x$. The power function of the test is defined as a function of P by

$$\beta(P) = E_P(\Phi) = \int \Phi(x)\, dP(x).$$

If Ω is quite general and not restricted to simple parametric sets of measures (such as e.g. the normal measures) the set Ω is often called nonparametric. When considering power of tests in a non-parametric set Ω Hoeffding [9] introduced partitions $\{\Omega(d): d \in D\}$ of the alternative set $\Omega - \omega$ into disjoint sets $\Omega(d)$, $d \in D$, and defined a test Φ to be of maximin power in a set I of level α tests with respect to a partition $\{\Omega(d): d \in D\}$ if

$$\inf_{P \in \Omega(d)} E_P(\Phi) \geqq \inf_{P \in \Omega(d)} E_P(\Phi')$$

for all $\Phi' \in I$ and all $d \in D$.

For a nonnegative weight (loss) function $W(P)$ defined for all $P \in \Omega - \omega$, the maximum risk associated with a test Φ is

$$R_{\max}(\Phi) = \sup_{P \in \Omega - \omega} (W(P) \cdot (1 - E_P(\Phi))).$$

A test which minimizes the maximum risk within a set I of level α tests is defined to be of minimax risk in I with respect to $W(p)$.

It was shown by Hoeffding [9] that if a test is of maximin power in a set I with respect to a partition $\{\Omega(d): d \in D\}$ it is also of minimax risk in I if $W(P)$ depends on P only through the partition parameter d. He also showed that the size α sign

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test is a maximin power (and minimax risk) test in the set I of all level α tests for testing the hypothesis $P \in \omega$ against $P \in \Omega - \omega$ with respect to a partition

$$\{\Omega(d): d \in (0, 1-q)\}$$

where Ω = the set of all n-product measures whose component measures have a common density $f(y)$ satisfying

$$F(0) = \int_{-\infty}^{0} f(u)\, du \geqq q$$

$$\omega = \{P \in \Omega: F(0) = q\}$$

$$\Omega(d) = \{P \in \Omega: F(0) = q + d\}.$$

Now let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent samples with component densities $f(x)$ and $g(y)$ and cumulative distribution functions

$$F(x) = \int_{-\infty}^{x} f(u)\, du \quad \text{and} \quad G(y) = \int_{-\infty}^{y} g(u)\, du.$$

For studying asymptotically minimax properties in the two-sample case we suppose throughout this paper that we have a sequence of independent samples $X_1, \cdots, X_m\ Y_1, \cdots, Y_n$, where $N = m+n$ tends to infinity and

$$0 < \lim_{N\to\infty} m/N = \lambda < 1.$$

We let Ω_N denote the set of all $m+n$ product measures where the first m components have density f and the second n components have density g, and $\omega_N \subseteq \Omega_N$ the hypothesis set in this case.

If we have a partition $\{\Omega_N(d)\}$ of the set Ω_N, where $\{\Omega_N(d): d > 0\}$ is a partition of the alternative set $\Omega_N - \omega_N$ and $\omega_N = \Omega_N(0)$ or $\omega_N = \bigcup_{d \leqq 0} \Omega_N(d)$ it is quite natural to define the set $\Omega_N'(\Delta_N) = \bigcup_{d \geqq \Delta_N} \Omega_N(d)$ of alternatives separated from the hypothesis by a "distance" Δ_N, and study the minimum power against alternatives in such sets for tests belonging to some set I_N of tests.

Following Doksum [5] we define a sequence $\{\varphi_0^{(N)}\}$ of level α tests $\varphi_0^{(N)} \in I_N$ to be *(power) asymptotically minimax* in a sequence $\{I_N\}$ of sets I_N of level α tests against families $\{\Omega_N'(\Delta): \Delta \in D\}$ of alternative sets for powers $\beta \in B$ if

$$\lim_{N\to\infty} \inf_{P_N \in \Omega_N'(\Delta_N)} E_{P_N}(\varphi_0^{(N)})$$
$$= \sup_{\{\varphi^{(N)}:\, \varphi^{(N)} \in I_N\}} \limsup_{N\to\infty} \inf_{P_N \in \Omega_N'(\Delta_N)} E_{P_N}(\varphi^{(N)})$$

for each sequence $\{\Delta_N\}$ satisying

$$\lim_{N\to\infty} \inf_{P_N \in \Omega_N'(\Delta_N')} E_{P_N}(\varphi_0^{(N)}) = \beta \qquad \text{for some } \beta \in B.$$

Doksum showed that the sequence of size α Wilcoxon tests is (power) asymptotically minimax in a sequence of large sets of nonparametric level α tests of $F(x) = G(x)$ against families $\{\Omega_N'(\Delta): \Delta > 0\}$ for powers $\beta \in (\alpha, 1]$ with alternative sets $\Omega_N'(\Delta)$ defined by

$$\Omega_N'(\Delta) = \{P_N \in \Omega_N: G(x) \leqq F(x) \quad \text{for} \quad x \in R \quad \text{and} \quad \sup_{x \in R}[F(x) - G(x)] \geqq \Delta\}$$

which are often used in studies of power of nonparametric tests (e.g. in [1] and [3]).

In the present paper we use the partition $\{\Omega_N(p):0 \leqq p \leqq 1\}$ of Ω_N where

$$\Omega_N(P) = \{P_N \in \Omega_N : \int_{-\infty}^{\infty} F(u)\,dG(u) = p\}$$

and show that a certain sequence of Wilcoxon tests is (power) asymptotically minimax in the sequence $\{I_N\}$, where I_N is the set of all level α tests of the hypothesis $P_N \in \omega_N = \bigcup_{q \leqq p_0} \Omega_N(q) = \{P_N \in \Omega_N : p \leqq p_0\}$, against families $\{\Omega_N'(p_1):p_1 > p_0\}$, where $\Omega_N'(p_1) = \bigcup_{q \geqq p_1} \Omega_N(q) = \{P_N \in \Omega_N : p \geqq p_1\}$ for powers $\beta \in (\frac{1}{2}, 1]$ if $\alpha \leqq \frac{1}{2}$.

We also introduce a more decision-theoretic definition of the concept asymptotically minimax test.

Let a_0, a_1 be the two possible actions in the test, and $L(P; a_0)$, $L(P; a_1)$ the losses when taking actions a_0 and a_1 respectively when the probability measure is P. The test is here interpreted as the probability of choosing action a_1 as a function on the outcome space.

A sequence $\{\varphi_0^{(N)} : \varphi_0^{(N)} \in I_N\}$ of tests $\varphi_0^{(N)} \in I_N$ is defined to be *risk asymptotically minimax* in a sequence $\{I_N\}$ of sets I_N of tests when the probability measure belongs to Ω_N if

$$\lim_{N \to \infty} \frac{\sup_{P_N \in \Omega_N} R(P_N; \varphi_0^{(N)})}{\inf_{\{\varphi^{(N)} : \varphi^{(N)} \in I_N\}} \sup_{P_N \in \Omega_N} R(P_N; \varphi^{(N)})} = 1$$

where $R(P; \varphi) = L(P; a_0)(1 - E_p(\varphi)) + L(P; a_1)E_p(\varphi)$.

When the losses $L(P; a_0)$ and $L(P; a_1)$ depend on P only through p, i.e. $L(P; a_0) = L(p; a_0)$ and $L(P; a_1) = L(p; a_1)$, and there exists a p_0, $0 < p_0 < 1$, such that

$$L(p; a_0) = 0 \quad \text{for} \quad p \leqq p_0$$
$$L(p; a_1) = 0 \quad \text{for} \quad p \geqq p_0$$

and some regularity conditions on L are satisfied, a sequence of Wilcoxon tests is shown to be risk asymptotically minimax in the sequence $\{I_N\}$ where I_N is the set of all tests when the probability measure belongs to Ω_N.

2. Preliminaries. In this section we are going to state some lemmas, which are used to prove the main theorems in Section 3.

For the two independent samples $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ in Section 1 define random variables

$$U_{ij} = 1 \quad \text{for} \quad Y_j \geqq X_i$$
$$\quad = 0 \quad \text{for} \quad X_i > Y_j$$
$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij}$$
$$W = U/mn.$$

Other notations used are defined in Section 1.

LEMMA 1. *If W is the random variable defined above then*

$$\sup_{P_N \in \Omega_N(p)} \operatorname{Var} W = \frac{p(1-p)}{\min(m, n)} \leqq \frac{1}{4\min(m, n)}$$

$$\inf_{P_N \in \Omega_N(p)} \operatorname{Var} W = \kappa(r, \mu, \nu)$$

$$= \frac{r(1-r)}{\nu} - \frac{(\mu-1)^2}{12\mu\nu(\nu-1)} \qquad \textit{for} \quad (\mu-1)/(\nu-1) \leqq 2r$$

$$= \tfrac{4}{3} r^{\frac{3}{2}} \frac{(2(\mu-1)(\nu-1))^{\frac{1}{2}}}{\mu\nu} - \frac{\mu+\nu-2}{\mu\nu} r^2 + \frac{r(1-r)}{\mu\nu}$$

$$\textit{for} \quad (\mu-1)/(\nu-1) \geqq 2r$$

where $r = \min(p, (1-p))$; $\mu = \min(m, n)$; $\nu = \max(m, n)$.

PROOF. See Birnbaum and Klose [2].

LEMMA 2. *For each* p_0, $0 < p_0 < 1$, $\varepsilon > 0$ *and* $a \geqq 0$ *there exists a* $\delta > 0$ *such that*

$$P\left(W \leqq p_0 + \frac{a}{[\min(m, n)]^{\frac{1}{2}}}\right) \leqq \varepsilon$$

for all F and G such that

$$\int_{-\infty}^{\infty} F(u)\,dG(u) = p \geqq p_0 + \frac{\delta}{[\min(m, n)]^{\frac{1}{2}}} \qquad \textit{and}$$

$$P\left(W \geqq p_0 - \frac{a}{[\min(m, n)]^{\frac{1}{2}}}\right) \leqq \varepsilon$$

for all F and G such that

$$\int_{-\infty}^{\infty} F(u)\,dG(u) = p \leqq p_0 - \frac{\delta}{[\min(m, n)]^{\frac{1}{2}}}.$$

PROOF. Lemma 1 gives

$$\sup_{P \in \Omega_N(p)} \operatorname{Var} W = \frac{p(1-p)}{\min(m, n)} \leqq \frac{1}{4\min(m, n)}$$

and thus according to the Chebychev inequality for $p > p_0 + a/[\min(m, n)]^{-\frac{1}{2}}$

$$P\left(W \leqq p_0 + \frac{a}{[\min(m, n)]^{\frac{1}{2}}}\right) \leqq \frac{1}{4(-a + (p-p_0)[\min(m, n)]^{\frac{1}{2}})}.$$

The first part follows by making $\delta = a + 1/2\varepsilon^{\frac{1}{2}}$. The second part is shown in the same way. □

LEMMA 3. *For fixed F and G*

$$\lim_{N\to\infty} P\left(\frac{W-p}{[\operatorname{Var} W]^{\frac{1}{2}}} \leqq x\right) = \Phi(x)$$

where Φ *is the Gaussian* (0, 1) *distribution function.*

The convergence is uniform in x and in F and G belonging to a set where $0 < a < p < b < 1$.

PROOF. The first part is an immediate consequence of the Chernoff–Savage theorem [4]. The uniformity follows from Govindrajulu, Le Cam and Raghavachari [8] Theorem 1 and the observation that for $0 < a < p < b < 1$ and $\lambda-\delta \leqq \lambda_N = m/N \leqq \lambda+\delta$ with a fixed δ, $0 < \delta < \min(1-\lambda, \lambda)$ the variance $\sigma_N^2 = \operatorname{Var} W$ stays bounded away from zero, since $\kappa(r, \mu, \nu)$ of Lemma 1 is bounded away from zero in this case. □

3. Asymptotically minimax properties of the Wilcoxon test. In this section we are going to prove that the Wilcoxon test is asymptotically minimax in two different senses. The notations are the same as in Section 1 and Section 2. Let $\varphi_0^{(N)}$, $N = m+n$, be the sequence

$$\varphi_0^{(N)} = 1 \quad \text{if} \quad W > p_0 + \frac{a_N}{[\min(m, n)]^{\frac{1}{2}}}$$

$$= 0 \quad \text{otherwise,}$$

of Wilcoxon tests of the hypothesis $P \in \omega_N = \bigcup_{q \leqq p_0} \Omega_N(q)$ where a_N is a sequence of numbers given in the proof of Theorem 1, having the limit

$$\lim_{N\to\infty} a_N = [p_0(1-p_0)]^{\frac{1}{2}} \Phi^{-1}(1-\alpha).$$

THEOREM 1. *The test sequence* $\varphi_0^{(N)}$ *is (power) asymptotically minimax in the class of all level* α *tests of the hypothesis* $P \in \omega_N = \bigcup_{q \leqq p_0} \Omega_N(q)$ *against the family* $\{\Omega_N'(p_1): p_1 > p_0\}$ *where* $\Omega_N'(p_1) = \bigcup_{q \geqq p_1} \Omega_N(q)$ *for powers* $\beta \in (\frac{1}{2}, 1)$ if $\alpha \leqq \frac{1}{2}$.

PROOF. By Lemma 3 we have for all x and for $p \in (a, b)$, where $a > 0$, $b < 1$,

$$\left| P\left(\frac{W-p}{[\operatorname{Var} W]^{\frac{1}{2}}} \leqq \quad \right) - \Phi(x) \right| \leqq \varepsilon_1(N)$$

where $\lim_{N\to\infty} \varepsilon_1(N) = 0$. We here chose $a < p_0$ and $b > p_0$ and thus have for $a < p \leqq p_0$

$$P\left(\frac{W-p}{[\operatorname{Var} W]^{\frac{1}{2}}} > x\right) \leqq 1 - \Phi(x) + \varepsilon_1(N).$$

Now by Lemma 1

$$\sup_{P \in \Omega_N(p)} \operatorname{Var} W = \frac{p(1-p)}{\min(m, n)}$$

and thus for $p \in (a, p_0]$ and $a_N \geqq 0$

$$P\left(W > p_0 + \frac{a_N}{[\min(m, n)]^{\frac{1}{2}}}\right) \tag{1}$$
$$\leqq 1 - \Phi\left(\frac{a_N}{[p(1-p)]^{\frac{1}{2}}} + \frac{(p_0 - p)[\min(m, n)]^{\frac{1}{2}}}{[p(1-p)]^{\frac{1}{2}}}\right) + \varepsilon_1(N).$$

If a_N is determined by the equation

$$1 - \Phi\left(\frac{a_N}{[p_0(1-p_0)]^{\frac{1}{2}}}\right) + \varepsilon_1(N) = \alpha \tag{2}$$

then $\lim_{N\to\infty} a_N = [p_0(1-p_0)]^{\frac{1}{2}}\Phi^{-1}(1-\alpha)$ because Φ^{-1} is continuous, and for $\alpha \leqq \frac{1}{2}$ is $a_N \geqq 0$.

Further there exists an $L > 0$ such that

$$P\left(W > p_0 + \frac{a_N}{[\min(m, n)]^{\frac{1}{2}}}\right) \leqq \alpha$$

for $p \in [0; p_0]$ and $\min(m, n) \geqq L$.
To see this, we first observe that by the second part of Lemma 2 there exists a δ such that

$$P\left(W > p_0 + \frac{a_N}{[\min(m, n)]^{\frac{1}{2}}}\right) \leqq \alpha$$

for $p \in [0; p_0 - \delta[\min(m, n)]^{-\frac{1}{2}}]$ and $\min(m, n) \geqq L_1$ for some L_1. But for $\min(m, n) \geqq L_2$, for some L_2, the function

$$\frac{a_N}{[p(1-p)]^{\frac{1}{2}}} + \frac{(p_0 - p)[\min(m, n)]^{\frac{1}{2}}}{[p(1-p)]^{\frac{1}{2}}}$$

is decreasing in the interval $(p_0 - \delta[\min(m, n)]^{-\frac{1}{2}}; p_0]$ and thus by (1) and (2)

$$P\left(W > p_0 + \frac{a_N}{[\min(m, n)]^{\frac{1}{2}}}\right) \leqq \alpha$$

for $\min(m, n) \geqq L = \max(L_1; L_2)$, and the tests ${\varphi_0}^{(N)}$ are of level α for $\min(m, n) \geqq L$. Now consider the power $E_P({\varphi_0}^{(N)})$ of the ${\varphi_0}^{(N)}$ tests.

For a fixed β, $\alpha < \beta < 1$, there exists by Lemma 2 a $\delta > 0$ such that

$$\inf_{p \geqq p_0 + \delta[\min(m, n)]^{-1/2}} E_p({\varphi_0}^{(N)}) \geqq (1+\beta)/2.$$

By Lemma 1 and Lemma 3 we get

$$\left| \inf_{P_N \in \Omega_N(p)} E_{P_N}({\varphi_0}^{(N)}) - \left(1 - \Phi\left(\frac{p_0 - p}{[p(1-p)]^{\frac{1}{2}}}[\min(m, n)]^{\frac{1}{2}} + \frac{a_N}{[p(1-p)]^{\frac{1}{2}}}\right)\right)\right| \leqq \varepsilon_1(N).$$

for $(p_0 - p)[\min(m, n)]^{\frac{1}{2}} + a_N \leqq 0$.

Again there exists an $L' > 0$ such that

$$\frac{p_0-p}{[p(1-p)]^{\frac{1}{2}}}[\min(m,n)]^{\frac{1}{2}}+\frac{a_N}{[p(1-p)]^{\frac{1}{2}}}$$

is a monotone decreasing function of p in the interval $[p_0; p_0+\delta[\min(m,n)]^{-\frac{1}{2}}]$ for $\min(m,n) \geqq L'$, and it follows that

$$\left|\inf_{P_N \in \Omega_N'(p_N)} E_{P_N}(\varphi_0^{(N)}) - \left(1-\Phi\left(\frac{p_0-p_N}{[p_N(1-p_N)]^{\frac{1}{2}}}[\min(m,n)]^{\frac{1}{2}}+\frac{a_N}{[p_N(1-p_N)]^{\frac{1}{2}}}\right)\right)\right| \leqq \varepsilon_1(N)$$

for $\min(m,n) \geqq L'$ and $(p_0-p_N)[\min(m,n)]^{\frac{1}{2}}+a_N \leqq 0$.

In order to get

$$\lim_{N\to\infty} \inf_{P_N \in \Omega_N'(p_N)} E_{P_N}(\varphi_0^{(N)}) = \beta \qquad \text{for some } \beta > \tfrac{1}{2}$$

the sequence $\{p_N\}$ must satisfy

$$\lim_{N\to\infty}[(p_0-p_N)[\min(m,n)]^{\frac{1}{2}}] = [p_0(1-p_0)]^{\frac{1}{2}}(\Phi^{-1}(1-\beta)-\Phi^{-1}(1-\alpha)).$$

Let c_1 and c_2 be any two fixed numbers with $c_1 < c_2$, and f^* and g^* any probability density functions satisfying

$$f^*(x) = 0 \quad \text{for} \quad c_1 \leqq x \leqq c_2$$

$$g^*(x) = 0 \quad \text{for} \quad x \leqq c_1 \quad \text{and} \quad x \geqq c_2$$

$$\int_{-\infty}^{c_1} f^*(x)\,dx = F^*(c_1) = p_0.$$

Further let $P_0^{(N)}$ be the probability measure corresponding to $N = m+n$ independent random variables, the first m having the density f^* and the last n having the density g^*, and $P_N^{(N)}$ the probability measure corresponding in a similar way to random variables having densities

$$f_N(x) = \frac{p_N}{p_0}f^*(x) \qquad \text{for} \quad x \leqq c_1$$

$$= \frac{1-p_N}{1-p_0}f^*(x) \qquad \text{for} \quad x > c_1, \quad \text{and}$$

$$g_N(x) = g^*(x).$$

Then $P_0^{(N)} \in \Omega_N(p_0)$ and $P_N^{(N)} \in \Omega_N(p_N)$. The most powerful level α test of $P = P_0^{(N)}$ against $P = P_N^{(N)}$ is given by the Neyman–Pearson lemma. It has a test function of the form

$$\Psi_N = 1 \quad \text{for} \quad Z > k$$

$$= \gamma \quad \text{for} \quad Z = k$$

$$= 0 \quad \text{for} \quad Z < k$$

where Z is the number of the variables $X_1, \cdots, X_m$ which are $<c_1$ and k and γ are chosen to give the test the size α.

But in this case $Z = l \Leftrightarrow U = l\,n \Leftrightarrow W = l/m$ and the test function may be written in the form

$$\begin{aligned} \Psi_N &= 1 && \text{for} \quad W > k/m \\ &= \gamma && \text{for} \quad W = k/m \\ &= 0 && \text{for} \quad W < k/m. \end{aligned}$$

Now according to Lemma 3

$$\alpha = E_{P_0^{(N)}}(\Psi_N) \geqq P_0^{(N)}(W > k/m) \geqq 1 - \Phi\left(\frac{k/m - p_0}{[p_0(1-p_0)(1/m)]^{\frac{1}{2}}}\right) - \varepsilon_1(N)$$

and if $\beta_N{}^*$ denotes the power of the test for $P = P_N{}^{(N)}$ then

$$\beta_N{}^* = E_{P_N{}^{(N)}}(\Psi_N) \leqq P_N{}^{(N)}\left(W > \frac{k-1}{m}\right) \leqq 1 - \Phi\left(\frac{(k-1/m) - P_N}{[p_N(1-p_N)(1/m)]^{\frac{1}{2}}}\right) + \varepsilon_1(N)$$

from which follows that

$$\limsup_{N\to\infty} \beta_N{}^* \leqq 1 - \Phi(\Phi^{-1}(1-\alpha) + \frac{\lambda}{\min(\lambda, 1-\lambda)}(\Phi^{-1}(1-\beta) - \Phi^{-1}(1-\alpha))).$$

If $\min(\lambda, 1-\lambda) = \lambda$ this equals β and the theorem is proved since for each sequence $\{\varphi^{(N)}\}$ we have

$$\inf_{P_N \in \Omega_N'(\Delta_N)} E_{P_N}(\varphi^{(N)}) \leqq \beta_N{}^*$$

and

$$\begin{aligned} &\limsup_{N\to\infty} \inf_{P_N \in \Omega_N'(\Delta_N)} E_{P_N}(\varphi^{(N)}) \\ &\qquad \leqq \limsup_{N\to\infty} \beta_N{}^*. \end{aligned}$$

If $\min(\lambda, 1-\lambda) = 1-\lambda$ the theorem is proved in the same way by introducing instead of $P_0{}^{(N)}$ and $P_N{}^{(N)}$ the measures $Q_0{}^{(N)}$ and $Q_N{}^{(N)}$ corresponding to densities f^*, g^*, f_N and g_N satisfying

$$\begin{aligned} f^*(x) &= 0 \quad \text{for} \quad x \leqq c_1 \quad \text{and} \quad x \geqq c_2 \\ g^*(x) &= 0 \quad \text{for} \quad c_1 \leqq x \leqq c_2 \\ \textstyle\int_{-\infty}^{c_1} g^*(x)\,dx &= G^*(c_1) = 1 - p_0 \\ f_N(x) &= f^*(x) \\ g_N(x) &= \frac{1-p_N}{1-p_0} g^*(x) \quad \text{for} \quad x \leqq c_1 \\ &= \frac{p_N}{p_0} g^*(x) \quad \text{for} \quad x \geqq c_2 \quad \square \end{aligned}$$

THEOREM 2. *The sequence* $\{\varphi_0^{(N)}\}$ *of Wilcoxon tests*

$$\varphi_0^{(N)} = 1 \quad \text{if} \quad W \geqq p_0$$
$$= 0 \quad \text{if} \quad W < p_0$$

is risk asymptotically minimax in the sequence $\{I_N\}$ *of the sets* I_N *of all tests when the probability measure belongs to* Ω_N *if* $L(P; a_0) = L(p; a_0)$ *and* $L(P; a_1) = L(p; a_1)$ *depends on P only through p, and* $L(p; a_0)$ *and* $L(p; a_1)$ *satisfies*

(A) $L(p; a_0)$ *and* $L(p; a_1)$ *are nonnegative continous functions of p.*

(B) $L(p; a_0) = 0$ *for* $p \leqq p_0$ *and* $L(p; a_1) = 0$ *for* $p \geqq p_0$ *for some* p_0.

(C) $\lim_{p \downarrow p_0} \dfrac{L(p; a_0)}{p-p_0} = \lim_{p \downarrow p_0} \dfrac{L(p; a_1)}{p_0-p} = k$ *where* $0 < k < \infty$.

PROOF. Let x_0 be the unique nonnegative solution of the equation

$$\Phi(-x_0) - x_0 \cdot \varphi(x_0) = 0$$

and $R_0 = x_0 \cdot \Phi(-x_0)$ where Φ and φ are the Gaussian (0, 1) distribution and density functions.

From conditions A and C of the theorem it follows that there exists a constant K such that $L(p; a_0)/(p-p_0) \leqq K$ for $p_0 < p \leqq 1$ and $L(p; a_1)/(p_0-p) \leqq K$ for $0 \leqq p < p_0$ and then by the Chebychev inequality and Lemma 1

$$R(p; \varphi_0^{(N)}) = L(p; a_0) \cdot P(W < p_0)$$
$$\leqq K \cdot (p-p_0) \frac{\operatorname{Var} W}{(p-p_0)^2} \leqq \frac{K}{4 \cdot \min(m, n) \cdot (p-p_0)}$$

for $p > p_0$. A similar inequality holds for $p < p_0$, and thus there exists a $\delta > 0$ such that

$$R(P; \varphi_0^{(N)}) \leqq \tfrac{1}{2} \frac{k[p_0(1-p_0)]^{\frac{1}{2}} R_0}{[\min(m, n)]^{\frac{1}{2}}}$$

if $|p-p_0| \geqq \delta[\min(m, n)]^{\frac{1}{2}}$.

Now by the uniform convergence stated in Lemma 3, the risk function $R(p; \varphi_0^{(N)}) = L(p; a_0) \cdot P(W < p_0)$ in the interval $(p_0; p_0 + \delta[\min(m, n)]^{-\frac{1}{2}})$ satisfies

$$\left| \sup_{P_N \in \Omega_N R(p)} (P_N; \varphi_0^{(N)}) - k(p-p_0)\Phi\left(\frac{p_0-p}{[p_0(1-p_0)]^{\frac{1}{2}}}[\min(m, n)]^{\frac{1}{2}}\right) \right| \leqq \varepsilon_2(\min(m, n))$$

where $\lim_{N \to \infty} [\min(m, n)]^{\frac{1}{2}} \cdot \varepsilon_2(\min(m, n)) = 0$.

The unique maximum for $p > p_0$ of

$$k(p-p_0)\Phi\left(\frac{p_0-p}{[p_0(1-p_0)]^{\frac{1}{2}}}[\min(m, n)]^{\frac{1}{2}}\right)$$

is $k[p_0(1-p_0)]^{\frac{1}{2}}R_0/[\min(m, n)]^{\frac{1}{2}}$ which is attained for

$$p = p_0 + x_0 \frac{[p_0(1-p_0)]^{\frac{1}{2}}}{[\min(m, n)]^{\frac{1}{2}}}$$

and thus both the supremum risk for $p > p_0$ and the supremum risk for

$$p = p_N = p_0 + x_0 \frac{[p_0(1-p_0)]^{\frac{1}{2}}}{[\min(m, n)]^{\frac{1}{2}}}$$

are in the interval

$$I_N = \left[\frac{k[p_0(1-p_0)]^{\frac{1}{2}}R_0}{[\min(m, n)]^{\frac{1}{2}}} - \varepsilon_2; \frac{k[p_0(1-p_0)]^{\frac{1}{2}}R_0}{[\min(m, n)]^{\frac{1}{2}}} + \varepsilon_2\right].$$

In the same way it is shown that the supremum risk for

$$p = q_N = p_0 - x_0[p_0(1-p_0)]^{\frac{1}{2}}/[\min(m, n)]^{\frac{1}{2}}$$

and the supremum risk for all $p < p_0$ are in the interval I_N too.

If $\min(\lambda; 1-\lambda) = \lambda$ we introduce as in the proof of Theorem 1 densities f^* and g^* satisfying

$$f^*(x) = 0 \quad \text{for} \quad c_1 \leqq x \leqq c_2$$

$$g^*(x) = 0 \quad \text{for} \quad x \leqq c_1 \quad \text{and} \quad x \geqq c_2$$

$$\int_{-\infty}^{c_1} f^*(x)\,dx = F^*(c_1) = q_n$$

for some c_1 and c_2. Let Q_N denote the probability measure corresponding to those densities and P_N the measures corresponding to the densities

$$f'(x) = \frac{p_N}{q_N} f^*(x) \qquad \text{for} \qquad x \leqq c_1$$

$$= \frac{1-p_N}{1-q_N} f^*(x) \qquad \text{for} \qquad x \geqq c_2$$

and $g'(x) = g^*(x)$. The minimax test Ψ_N when we take into consideration only the measures Q_N and P_N is given by the Neyman–Pearson lemma if the constants are chosen to give $R(P_N; \Psi_N) = R(Q_N; \Psi_N)$ (see e.g. Hogg and Craig [10]). As in the proof of Theorem 1 we get

$$\Psi_N(x, y) = 1 \qquad \text{for} \qquad W > b_N$$

$$= \gamma_N \qquad \text{for} \qquad W = b_N$$

$$= 0 \qquad \text{for} \qquad W < b_N$$

for some b_N and γ_N.

But for $P \in \Omega_N(p_N)$, Var W attains its maximum for $P = P_N$, and for $P \in \Omega_N(q_N)$, Var W attains its maximum for $P = Q_N$ when $m = \min(m, n)$.

Now $R(P_N; \Psi_N) = R(Q_N; \Psi_N)$ and depending on if $b > p_0$ or $b \leqq p_0$ we have either

$$R(P_N; \Psi_N) \leqq R(P_N; \varphi_0^{(N)}) \quad \text{and} \quad R(Q_N; \Psi_N) \geqq R(Q_N; \varphi_0^{(N)})$$

or

$$R(P_N; \Psi_N) \geqq R(P_N; \varphi_0^{(N)}) \quad \text{and} \quad R(Q_N; \Psi_N) \leqq R(Q_N; \varphi_0^{(N)}).$$

Since $R(P_N; \varphi_0^{(N)}) \in I_N$ and $R(Q_N; \varphi_0^{(N)}) \in I_N$ it follows that $R(P_N; \Psi_N) = R(Q_N; \Psi_N) \in I_N$ which proves the theorem when $\min(\lambda; 1-\lambda) = \lambda$.

If $\min(\lambda; 1-\lambda) = 1-\lambda$, the theorem is proved in the same way using densities f^*, g^*, f' and g' satisfying

$$f^*(x) = 0 \quad \text{for} \quad x \leqq c_1 \quad \text{and} \quad x \geqq c_2$$

$$g^*(x) = 0 \quad \text{for} \quad c_1 \leqq x \leqq c_2$$

$$\int_{-\infty}^{c_1} g^*(x)\,dx = G^*(c_1) = 1-q_N$$

$$f'(x) = f^*(x)$$

$$g'(x) = \frac{1-p_N}{1-q_N} g^*(x) \quad \text{for} \quad x \leqq c_1$$

$$g'(x) = \frac{p_N}{q_N} g^*(x) \qquad \text{for} \quad x \geqq c_2. \qquad \square$$

If the conditions C in the theorem is replaced by

$$\lim_{p \downarrow p_0} \frac{L(p; a_0)}{p-p_0} = k_0 \quad \text{and} \quad \lim_{p \downarrow p_0} \frac{L(p; a_1)}{p_0-p} = k_1$$

where $k_0 > 0$, $k_1 > 0$ and $k_0 \neq k_1$ the sequence $\{\varphi_0^{(N)}\}$ of Wilcoxon tests

$$\varphi_0^{(N)} = 1 \qquad \text{if} \qquad W \geqq p_0 + \frac{a}{[\min(m, n)]^{\frac{1}{2}}}$$

$$= 0 \qquad \text{if} \qquad W < p_0 + \frac{a}{[\min(m, n)]^{\frac{1}{2}}}$$

where a is determined by k_1 and k_2 is risk asymptotically minimax. The proof of this is essentially the same as the proof of Theorem 2, only a little more complicated at some points, and will not be given.

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