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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Western Regional meeting, Las Vegas, Nevada, March* 22–24, 1971.)

128-1. Some probability inequalities related to the law of large numbers. R. J. TOMKINS, University of Saskatchewan, Regina.

Let $X_1, X_2, \cdots, X_n$ $(n > 1)$ be random variables (rv) and define $S_k = X_1 + \cdots + X_k$. Upper bounds of the Hájek–Rényi type are presented for $P(\max_{1 \leqq k \leqq n} \phi_k S_k \geqq \varepsilon \mid \mathscr{G})$ where $\phi_1 \geqq \cdots \geqq \phi_n > 0$ are rv, $\varepsilon > 0$ and $\mathscr{G}$ is a σ-field. The theorems place no further assumptions on the X_k's; some, in fact, do not even require the integrability. It is shown, however, that if the X_k's are independent or form a submartingale difference sequence, then some well-known inequalities follow as consequences of these theorems. (Received December 30, 1970.)

128-2. Convergence to random binary digits when none of initial data necessarily independent. JOHN E. WALSH and GRACE J. KELLEHER, Southern Methodist University and University of Texas at Arlington.

Desired is a set of very nearly random binary digits (very closely represent independent flips of an ideal coin with sides 0 and 1). Available is $m \times n$ array of approximately random binary digits obtained experimentally. No one of these digits is necessarily independent of any of the others but the level of dependence among rows is very small. A method is given for compounding these digits to obtain a smaller set that is much more nearly random. The randomness of a set of digits is measured by its "maximum bias." A set is very nearly random if its maximum bias is very small. A maximum bias for compounded digits is determined from: The compounding method, the largest contribution to the maximum bias from within rows, and the largest contribution from the dependence among rows (very small). The maximum bias for a compounded set can be very small (even when the maximum bias for the initial set is quite large) but has a lower bound depending on the bias contribution from dependence among rows. One approach is oriented toward minimizing m for a given maximum bias for the compounded set, so that obtaining the initial set is simplified. Another approach is oriented toward having the number of compounded digits a reasonably large fraction of the number in the initial set. (Received January 5, 1971.)

128-3. Maximal resolution V fractions of 2^n designs having 2^q runs. R. C. JUOLA, Boise State College.

The maximum number of factors which can be accommodated in a resolution V fraction of a 2^n design having 2^q runs is the largest integer solution of $n^2 + n \pm 2 \leqq 2^{q+1}$. Techniques developed by John and Juola (submitted *Ann. Math. Statist.*) are utilized to construct designs which achieve this maximum number of factors for $3 \leqq q \leqq 9$. This work extends the work of many authors (see Draper and Mitchell, *Ann. Math. Statist.* **39**, (1968) 246–55) who have found this maximum for the $2^{n-p}(p+q = n)$ series of fractional factorial designs. (Received January 5, 1971.)

128-4. Nonhomogeneous filtered Poisson processes in m-dimensional Euclidean space. H. W. LORBER, EG&G, Inc.

Several topics are discussed in the theory of nonhomogeneous filtered Poisson processes (NFPP's) defined in real multidimensional Euclidean spaces R_m. The topics that are discussed are novel or else do not follow trivially from extensions of the one-dimensional theory. A set of readily applicable conditions are proved sufficient for a process to be Poisson in R_m, and relations analogous to those in Campbell's theorem are derived from the joint characteristic function of values of a NFPP at two points in R_m. The spectral density functions for a non-stationary process in R_m are defined and related to those of a corresponding stationary process, and consequent developments are applied to the problem of optimum linear stationary estimation of the intensity process of an NFPP. (Received January 13, 1971.)

128-5. Moving averages of homogeneous random fields. LAWRENCE A. BRUCKNER, Sandia Laboratories.

Let $X(g)$ be a homogeneous random field on a discrete locally compact Abelian group G. Let $H(X)$ be the linear completion of $\{X(g): g \in G\}$ in L_2 space. The following result is obtained: there exists a fundamental random field $Y(g)$ on G with values in $H(X)$ such that $X(g)$ is obtained as a moving average of $Y(g)$ if, and only if, $X(g)$ has a spectral density which is positive almost everywhere with respect to the Haar measure on the dual group of G. (Received January 13, 1971.)

128-6. Jackknifing stochastic processes. H. L. GRAY, TERRY A. WATKINS and JOSEPH E. ADAMS, Texas Tech University.

In this paper the notion of the jackknife method is extended to a general class of stochastic processes. This extension leads to a definition of a new estimator which is useful in the problem of bias reduction as well as in establishing confidence intervals for certain parameters of the associated stochastic process. The theoretical results are exemplified. (Received January 13, 1971.)

128-7. Limit theorems for the Galton–Watson process with time-dependent immigration. JAMES FOSTER and JOHN A. WILLIAMSON, University of Colorado.

Let $h_k(s)$, $k \geqq 0$, and $f(s) = \sum_{j=0}^{\infty} p_j s^j$ be generating functions of nonnegative integer valued random variables. Let Z_n be a random variable defined by $E(s^{Z_n}) = \prod_{k=0}^{n} h_{n-k}(f_k(s))$, $0 \leqq s \leqq 1$ where $f_0(s) = s$ and $f_{k+1}(s) = f(f_k(s))$ for $k \geqq 0$. Then Z_n represents the size at time n of a Galton–Watson branching process that has been modified by immigration. I_k particles enter the system at time k. They then reproduce independently of each other, and of the particles already in the system, according to probabilities defined by $f(s)$. The I_k are independent with $E(s^{I_k}) = h_k(s)$. The notation $m = f'(1)$, $\sigma^2 = f''(1)$ is used. It will be assumed that $p_0 < 1$ and $p_1 < 1$. Then: (i) If there exists a probability distribution function F such that at every continuity point of F, $\lim_{n\to\infty} P((1/n)\sum_{k=0}^{n} I_k \leqq x) = F(x)$, and if $m = 1$ and $\sigma^2 < \infty$, then Z_n/n converges in distribution to a proper limit law. $\varphi(s) = \int_0^{\infty} e^{-sx} F(dx)$ must have the form

$$\varphi(s) = \exp\{-\textstyle\int_0^{\infty}[(1-e^{-sx})/x]P(dx)\}.$$

The limit law for Z_n/n has Laplace transform $\psi(s) = \exp\{-\int_0^{\infty}[(1-e^{-sx})/x]Q(dx)\}$ where $Q = P^*(1-e^{-2x/\sigma^2})$. (ii) If $m = 1$ and $\sigma^2 < \infty$, then regardless of how the random variables I_k are chosen, the limits $\lim_{n\to\infty} P(Z_n = k)$, $k = 1, 2, \cdots$, are zero whenever they exist. That is, the limit distribution of Z_n as $n \to \infty$, when it exists, is a (possibly defective) distribution with all its mass at zero. (iii) Suppose that $m \leqq 1$ and that the random variables I_k are identically distributed with $h_k(s) = h(s)$. Assume that $h(0) < 1$ to rule out the trivial case of no immigration at all. Then Z_n has a proper limit distribution as $n \to \infty$ if and only if $\int_0^1 (1-h(s))/(f(s)-s)\,ds < \infty$. (Received January 18, 1971.)

128-8. On a stochastic inequality for the Wilks' statistic. A. K. GUPTA, The University of Arizona.

Asoh and Okamoto (*Ann. Inst. Statist. Math.* **21** 67–72) have shown that the general nonnull distribution of the Wilks' statistic, the likelihood ratio statistic in multivariate analysis of variance, can be expressed as a product of conditional beta variates and they also presented a stochastic inequality in terms of the product of certain noncentral beta variates. By making use of the convolution techniques we have obtained the distribution of the product of noncentral beta variates for $p = 2$ and 3 and therefore an upper bound for the nonnull distribution of the Wilks' statistic. Thus a conservative evaluation of the power of the likelihood ratio test is provided for the cases when the alternate hypothesis is of rank 1, 2 or even 3. The latter case is called the spacial case. The general form of the distribution of the product of noncentral beta variates for any p is also given. (Received January 19, 1971.)

128-9. Exact multiplicative functionals in duality. MICHAEL J. SHARPE, University of California, San Diego.

With terminology that of Blumenthal and Getoor (*Markov Processes and Potential Theory* (1968) Academic Press, N.Y.) we consider a standard Markov process X in duality with another standard process $\hat{X}$. THEOREM 1. *Every exact terminal time T of X such that* a.s. *T is not in $[\zeta, \infty)$ is* a.s. *equal to $J_A = \inf\{t > 0 : (X_{t-}, X_t) \in A\}$ for some $A \in \mathscr{E} \times \mathscr{E}$*. In proving Theorem 1, we make use of complete characterization of purely discontinuous additive functionals of (X, M), M an exact MF of X, in the case where the additive functional is permitted to be undefined for paths starting in a set which is polar for the subprocess (X, M). Using Theorem 1, we find THEOREM 2. *Every exact* MF *M of X is equivalent, for $t > 0$, to $\prod_{(s \leqq t)}(1 - F(X_{s-}, X_s)) . 1_{[0,R)}(t) \cdot N_t$ where $F \in b(\mathscr{E} \times \mathscr{E})$ is < 1, R is an exact terminal time and N is a continuous* MF. One may then spell out the form of the dual exact MF as defined by Getoor (*Bull. Amer. Math. Soc.* **76** (1970) 1053–1056). (Received January 19, 1971.)

128-10. On selection of the best set of predictor variates and the simultaneous interval estimation of the largest multiple correlation coefficient. M. HASEEB RIZVI and HERBERT SOLOMON, Kansas State University and Stanford University.

Consider a $(k+1)$-variate $N(\boldsymbol{\mu}, \Sigma)$ distribution of $\mathbf{X} = (X_0, X_1, \cdots, X_k)$ with unknown mean $\boldsymbol{\mu}$ and unknown covariance matrix Σ. For the problem of predicting X_0 on the basis of the best linear combination of variates in sets of fixed size t $(1 \leqq t \leqq k-1)$ of the k variates $X_1, X_2, \cdots, X_k$ we want to select those t predictor variates that have the largest multiple correlation with X_0 and simultaneously want to construct a confidence interval for this largest multiple correlation coefficient. Specifically, we want to determine the sample size n so that the simultaneous probability of a correct selection and inclusion is no smaller than a preassigned constant whenever $d(\rho_{[u]}, \rho_{[u-1]}) \geqq \delta$, where d is a suitable distance function and $\rho_{[u]}$ and $\rho_{[u-1]}$ are the largest and the second largest multiple correlation coefficients, $u = \binom{k}{t}$ and δ is a specified constant. A natural decision procedure based on sample multiple correlations is proposed and large sample results are obtained for some special Σ and for some special values of t. This extends some results of Ramberg (Cornell Ph.D. Thesis (1969)). (Received January 19, 1971.)

128-11. Estimation of the mean with finite memory. TERRY J. WAGNER, The University of Texas at Austin.

Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables with a common mean μ. Cover has observed [*Ann. Math. Statist.* **40** 828–835] that the usual sequential estimate of μ, $\mu_{n+1} = (n/(n+1))\mu_n + X_{n+1}/(n+1)$, fails to converge to a constant when μ_n may be recalled to only a finite number of decimal places. Using the ideas of Cover we show that for a given $\varepsilon > 0$ there exists an m and a finite-valued statistic $T_n \in \{1, \cdots, m\}$ where (i) $T_{n+1} = f_n(T_n, X_{n+1})$; (ii) $\mu_n = d(T_n)$; (iii) $\mu_n \to \mu^*$ (a constant) a.s. with $|\mu^* - \mu| < \varepsilon$. Thus we have a sequential estimate of μ which has a memory of m states dealing with the data and which converges to within ε of μ. (Received January 19, 1971.)

128-12. Limiting distributions of statistics similar to Student's *t*. Z. W. BIRNBAUM and I. VINCZE, University of Washington and Mathematical Institute of the Hungarian Academy of Sciences.

Let X be a random variable with distribution function $F(x)$, and $X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(2m+1)}$ an ordered sample of X; furthermore let $\mu = \inf\{x: F(x) = \frac{1}{2}\}$ and $V = X_{(m+1)}$ be the population-median and sample-median, respectively. For given r, $1 \leqq r \leqq m$, consider the statistic $S = (V-\mu)/[X_{(m+1+r)} - X_{(m+1-r)}]$. This statistic is clearly invariant with regard to location and scale, a property it shares with Student's t; it can be computed and used when only a censored sample is available, and it has some other practical advantages. Under rather mild assumptions on $F(x)$, it is shown that $\lim_{m\to\infty} P\{(2/m)^{\frac{1}{2}} S \leqq s\} = [1/(2r-1)!] \int_0^\infty \Phi(zs) z^{2r-1} e^{-z}\, dz$, where $\Phi(\cdot)$ is the standardized normal distribution function. A table of this limiting distribution has been computed. Similar results have been obtained for the more general case when μ is a given population-quantile and V the corresponding sample-quantile. (Received January 19, 1971.)

128-13. Least square estimation for continuous time autoregressive time series. HERBERT T. DAVIS, University of New Mexico.

Given a linear differential operator $L = \sum_{j=0}^{m} a_j D^j$, $a_0 = 1$, then a continuous parameter time series $\{x(t), -\infty < t < \infty\}$ is said to be an autoregressive process if $(Lx)(t) = \varepsilon(T)$ is a process with uncorrelated increments (Var $\varepsilon(t) = \sigma^2$). Further $\{x(t)\}$ is weakly stationary if the zeros of the characteristic polynomial $\sum_{j=0}^{m} a_j z^j$ have negative real parts. In this paper we present a continuous time analogue to the usual least squares theory for discrete time series. Specifically, given L the Best Linear Unbiased Estimator $\hat{x}(t)$ of $x(t)$ is the L-spline. Then the Least Squares Estimators $\hat{a}_1, \cdots, \hat{a}_m$ are the coefficients of the linear differential operator $\hat{L} = \sum_{j=0}^{m} \hat{a}_j D^j$ for which $R = \int_{-\infty}^{\infty} [\hat{L}\hat{x})(t)]^2\, dt$ is a minimum. Computing algorithms to minimize R are also discussed. If the time series is Gaussian, then the estimators $\hat{a}$ are the Maximum Likelihood Estimators and are shown to be asymptotically normally distributed as n, the number of data points collected, becomes large. Further it is shown that $R/\sigma^2 = X_n + \delta_n$ where x_n has a chi-square distribution with n degrees of freedom, and $\delta_n \to_p 0$. (Received January 19, 1971.)

128-14. An initial value method for the computation of the characteristic values and functions of symmetric integral operator. MICHAEL A. GOLDBERG, University of Nevada.

Many problems in statistics involve the calculation of the characteristic values and functions of a symmetric integral operator. This paper presents a method for their computation based on integrating a set of differential equations. If $k(t, s)$ is continuous on $[a, b] \times [a, b]$ we define an integral operator by $Kf(t) = \int_a^b k(t, s) f(s)\, ds$. A complex number λ is called a characteristic value with characteristic function $u(t)$ if $u(t) = \lambda K u(t)$. We assume that $k(t, s) = k(s, t)$ and that the characteristic values are simple. The resolvent kernel $R(t, s, \lambda)$ satisfies $R(t, s, \lambda) = k(t, s) + \lambda \int_a^b k(t, r) R(r, s, \lambda)\, ds$. It was shown recently (Kalaba, R., Zagustin, E., Reduction of Fredholm integral equations to Cauchy systems. U.S.C. Tech. Report (1970)) that if λ is not a characteristic value that: (i) $R_\lambda(t, s, \lambda) = \int_a^b R(t, r, \lambda) R(r, s, \lambda)\, ds$, (ii) $R(t, s, 0) = k(t, s)$. Using $D(\lambda) R(t, s, \lambda) = D(t, s, \lambda)$, where $D(\lambda)$ is Fredholm's determinant and $D(t, s, \lambda)$ is Fredholm's principal minor we get (iii) $D(\lambda) D_\lambda(t, s, \lambda) = D_\lambda(\lambda) D(t, s, \lambda) + \int_a^b D(t, r, \lambda) D(r, s, \lambda)\, ds$, (iv) $D(t, s, 0) = k(t, s)$. It is also known that: (v) $D_\lambda(\lambda) = \int_a^b D(s, s, \lambda)\, ds$, (vi) $D(0) = 1$. (iii)–(vi) are viewed as defininv an initial value problem for $(D(\lambda), D(t, s, \lambda))$. For numerical purposes the integrals are discretized and the resulting set of equations integrated. The zeros $\{\lambda_i\}$ of $D(\lambda)$ are the characteristic values, and the functions $D(t, s, \lambda_i)$ for fixed s are the characteristic functions. (Received January 19, 1971.)

128-15. Optimal radar signal design for target detection in clutter environment. R. C. DAVIS, General Dynamics.

We consider the problem of designing optimum signals and receivers for an active radar system the purpose of which is target detection in the presence of interference consisting of random surface reverberation (clutter) return and Gaussian noise originating in the radar receiver. The radar system has a coherent transmitter which modulates the carrier frequency subject to the practical constraints of maximum available average power, peak power, and bandwidth. Consistent with previously developed phenomenological models of clutter return, the clutter can be represented as a non-stationary Gaussian random process. The target reflected signal is assumed to be deterministic but with unknown amplitude and reflection phase and possessing in general a radial velocity with respect to the transmitting antenna. The problem consists in designing a modulation signal subject to the above three constraints which for a preassigned probability of false alarm maximizes the probability of target detection whenever a target is present. We have proved that an optimal modulation *always* uses maximum average transmitted power contrary to previous conjectures. By means of Fourier analysis, we have been able to reduce the problem to one of non-linear programming in a real Hilbert space. For a case of great practical interest; namely, a uniform distribution in range of the surface scatterers, the optimal modulation in the time domain is given explicitly. (Received January 20, 1971.)

128-16. A Rank test for two group concordance (preliminary report). W. R. SCHUCANY, Southern Methodist University.

The Kendall coefficient of concordance is used to determine the degree of association among m sets of rankings of N objects. E. B. Page introduced the L statistic to allow for an ordered alternative hypothesis in the Fiedman two-way analysis of variance using ranks. This can be employed for testing concordance within a group of judges and, simultaneously, their agreement with a single standard or a priori ranking. The statistic introduced here allows one to test the hypothesis of agreement within each of two groups of judges as well as between the groups. This rank test encompasses Page's L as a special case. The exact distribution may be tabled, though in most practical cases a normal approximation should be adequate. The two tails of the null distribution indicate that the basic problem is more properly formulated as a multiple decision problem than as a classical test of hypothesis. (Received January 20, 1971.)

128-17. The asymptotic inadmissibility of the sample distribution function. ROBERT R. READ, Naval Postgraduate School.

Given a sample of size n, a continuous estimator for a distribution F (Pyke's modified sample distribution) is shown to have the property that its expected squared error, for almost all x in the positive sample space of F, is no larger than that of the sample distribution function given F and n sufficiently large. Letting risk be given by the expected squared error integrated with respect to F, it is shown that the Pyke estimator dominates both the sample distribution and the other best invariant estimator found by Agarwal, given F and n sufficiently large. Other common estimators cannot serve in this dominating role. Explicit calculation of risk is made when F is the uniform distribution. In this case the Pyke estimator strictly dominates the sample distribution for all $n \geqq 1$. (Received January 20, 1971.)

128-18. On the difference between pairwise and mutual independence. EDWARD C. VAN DER MEULEN, University of Rochester.

A characterization is given of the difference between pairwise independence and mutual independence of a finite collection of random variables $X_1, \cdots, X_n$. Similarly, a characterization is given of the distinction between a collection $\{X_1, \cdots, X_n\}$ of exchangeable random variables and a collection $\{X_1, \cdots, X_n\}$ of random variables which are independent identically distributed. Also, a necessary and sufficient condition is obtained for three random variables X, Y, and Z, which are pairwise conditionally independent given the third one, to be mutually independent. (Received January 21, 1971.)

128-19. Non-parametric theory on abstract spaces (Preliminary report). C. B. BELL and VIKTOR KUROTSCHKA, University of Michigan.

Let $(\mathscr{X}, \mathscr{T})$ be a measurable space; $\mathscr{S}'$, a compact group of 1–1 transformations of $\mathscr{X}$ onto $\mathscr{X}$ with measurable orbits; μ, the normed Haar measure on $\mathscr{S}'$; T, a maximal invariant of $\mathscr{S}'$ such that $A = \{T(z): z \in \mathscr{X}\} \in \mathscr{T}$; $\Omega'(A)$, a family of probability measures on $(A, \mathscr{T} \cap A)$; $P^*(B) = \int_{\mathscr{S}'} P(s(B) \mid A)\, d\mu(s)$; $\{P^*: P \in \Omega'(A)\}$; and $V(z) = s(A)$ when $z \in s(A)$, $s \in \mathscr{S}'$. THEOREM. *Let* $\Omega'(A)$ *be complete. Then,* (i) $\mathscr{S}'$ *induces a complete, sufficient partition of* $\mathscr{X}$ wrt Ω'; (ii) φ *is a similar test function of size* α wrt Ω' iff $E(\varphi' \mid \mathscr{S}'(z)) = \alpha$ *for* a.a. z.; (iii) W *is* DF wrt Ω' *with* cpf Q iff $P\{W \leqq w \mid \mathscr{S}'(z)\} = Q(w)$ *for all* w, *for* a.a. z; *and for all* J *in* Ω'; (iv) $z = V(z) \cap \mathscr{S}'(z)$ *for all* z; (v) $V(Z)$ *is uniformly distributed over* $\{s(A);\ s \in \mathscr{S}'\}$; *and* (vi) $V(Z)$ *and* $\mathscr{S}'(Z)$ *are mutually independent for all* J *in* Ω'. *Finally,* (vii) *the Neyman Structure results in* (ii) *and* (iii) *above can be expressed in terms of* $\int \varphi(s(z))\, d\mu(s)$ *and* $\mu\{s: W(s(z)) \leqq w\}$, *resp.* (Received January 22, 1971.)

128-20. Confidence intervals for independent exponential series systems. GERALD J. LIEBERMAN and SHELDON M. ROSS, Stanford University and University of California, Berkeley.

Suppose $X_1, X_2, \cdots, X_n$ are independent identically distributed exponential random variables with parameter λ_1. Let $Y_1, Y_2, \cdots, Y_m$ also be independent identically distributed exponential random variables with parameter λ_2, and assume that X's and Y's are independent. The problem is to estimate $R(t) = \exp[-(\lambda_1 + \lambda_2)t]$. The motivation behind this is that if one has a series system with two independent exponential components then $R(t)$ represents the reliability of the system at time t, i.e., the probability that the system survives until time t. A procedure for determining an exact $(1-\alpha)$ level lower confidence bound for $R(t)$ is presented. Let $U = \min(\sum_{i=1}^n X_i, \sum_{i=1}^m Y_i)$ and $K = \{\text{largest } j \leqq n: \sum_{i=1}^j X_i \leqq U\} + \{\text{largest } j \leqq m: \sum_{i=1}^j Y_i \leqq U\}$. Then given $K = k$, it is shown that U has a gamma distribution with parameters k and $\lambda_1 + \lambda_2$. Hence, a lower confidence bound for the reliability, with confidence coefficient $1-\alpha$, is given by $\exp\{-\chi^2_{\alpha;2k}\, t/2U\}$. Note that an interesting characterization of the minimum of two independent gamma random variables is obtained. The suggested procedure is then compared with others presented in the literature. (Received January 22, 1971.)

128-21. On Jonckheere's k-sample test against ordered alternatives. ROBERT E. ODEH, University of Victoria.

In this paper a k-sample non-parametric test for trend is considered. The test procedure described was originally proposed by A. R. Jonckheere (*Biometrika* **41** (1954) 133–145). Given a sample of size n_i, $i = 1, \cdots, k$ respectively from each of k populations, the test rejects the hypothesis that the k populations are identical if $S = \sum_{i=2}^k S_i \geqq S_\alpha$. Here S_i is the sum of the Mann–Whitney statistics computed when each observation in the ith sample is compared with all observations from the first $(i-1)$ populations. A recurrence formula is derived for computing the exact distribution of S. Tables of exact probabilities and critical values are given for nominal values of $\alpha = 0.5, 0.2, 0.1, 0.05, 0.025, 0.01$, and 0.005 for $k = 3$ and all possible sample sizes from two to eight, and for equal sample sizes for values of $n = 2(1)6$, $k = 4(1)6$. (Received January 22, 1971.)

128-22. Extreme values in the $GI/G/1$ queue (preliminary report). DONALD L. IGLEHART, Stanford University.

Consider a $GI/G/1$ queue in which W_n is the waiting time of the nth customer, $W(t)$ is the virtual waiting time at time t, and $Q(t)$ is the number of customers in the system at time t. We let the extreme values of these processes be

$$W_n^* = \max\{W_j : 1 \leqq j \leqq n\}, \qquad W^*(t) = \sup\{W(s) : 0 \leqq s \leqq t\},$$

and $Q^*(t) = \sup\{Q(s) : 0 \leqq s \leqq t\}$. The asymptotic behavior of the queue is determined by the traffic intensity ρ, the ratio of arrival rate to service rate. When $\rho < 1$ and the service time has an exponential tail, limit theorems are obtained for W_n^* and $W^*(t)$; they grow like $\log n$ or $\log t$. When $\rho \geqq 1$, limit theorems are obtained for W_n^*, $W^*(t)$, and $Q^*(t)$; they grow like $n^{\frac{1}{2}}$ or $t^{\frac{1}{2}}$ if $\rho = 1$ and like n or t when $t > 1$. For the case $\rho < 1$, it is necessary to obtain the tail behavior of the maximum of a random walk with negative drift before it first enters the set $(-\infty, 0]$. (Received January 22, 1971.)

128-23. Best linear recursive estimation. TRUMAN LEWIS and DWANE ANDERSON, Texas Tech University.

This paper presents a matrix formulation of recursive forms for best linear unbiased estimators $\hat{X}_N$ of the parameter vector x in the linear model $Y_i = h_i x + U_i$, $i = 1, 2, \cdots, N$ when the observation vectors Y_i are correlated. If data are collected in sequence one can formulate recursive forms of the estimator $\hat{X}_N$ in such a way that it is not necessary to store all the previous data but only previous estimates and current data. This requires less storage space to obtain best linear unbiased estimators. This is especially advantageous in real time estimation problems. (Received January 22, 1971.)

(Abstracts of papers presented at the Eastern Regional meeting, University Park, Pennsylvania, April 21–23, 1971.)

129-4. On the exact distributions of the traces of $S_1(S_1+S_2)$ and $S_1S_2^{-1}$. P. R. KRISHNAIAH and T. C. CHANG, Aerospace Research Laboratories and University of Cincinnati and Aerospace Research Laboratories.

In this paper, the authors derived the exact distributions of the traces of $S_1S_2^{-1}$ and $S_1(S_1+S_2)^{-1}$ where S_1 and S_2 are independently distributed as central Wishart matrices and the expected values of these matrices are the same. The method used involves expressing the Laplace transformations of the above traces in terms of the linear combinations of products of certain one-dimensional integrals and then taking the inverse Laplace transformations. (Received December 18, 1970.)

129-5. Joint distribution of few roots of a class of random matrices. P. R. KRISHNAIAH and V. B. WAIKAR, Aerospace Research Laboratories.

Let $l_1 < l_2 < \cdots < l_p$ be the latent roots of a class of random matrices and let the joint density of these roots be of the form $f(l_1, \cdots, l_p) = \text{const} \prod_{i=1}^{p} \psi(l_i) \prod_{i>j}^{p} (l_i - l_j)$, $a < l_1 < l_2 < \cdots < l_p < b$. In this paper, the authors obtained *exact* expressions for the joint marginal density of any few consecutive ordered roots, that is, $l_r, l_{r+1}, \cdots, l_{r+s}(1 \leqq r \leqq p, 0 \leqq s \leqq p-r)$. In addition, the authors obtained an expression for the probability integral associated with the joint distribution of any two ordered roots $l_i, l_j (i < j = 1, \cdots, p)$. The class of random matrices considered in this paper included Wishart matrix, multivariate analysis of variance matrix and canonical correlation matrix. The authors also obtained an expression for the joint density of any two unordered roots of the above class of random matrices. (Received December 22, 1970.)

129-6. Generating functions in elementary probability theory. JOHN P. HOYT, Indiana University of Pennsylvania.

For years many writers of textbooks in probability and statistics have used the term "moment generating function" without first defining "generating function." Today more and more writers are using the "probability generating function" of integral-valued random variables and in some case are incorrectly calling it "the factorial moment generating function." The fact that the commonly used moment generating function and the commonly used probability generating function are two different kinds of generating functions is seldom mentioned. The reason for choosing one of the two particular types of generating functions is never mentioned. This paper defines two types of generating functions for sequences and then applies each type to the generation of three sequences: (i) the sequence of probabilities for an integral-valued random variable; (ii) the sequence of factorial moments for any random variable; (iii) the sequence of ordinary moments for any random variable. The reason for preferring one of the two types to the other is then shown for each of the three sequences involved. (Received January 12, 1971.)

129-7. On a statistic for testing correlation in a multinormal population. SURESH C. RASTOGI and V. K. ROHATGI, University of Maryland and Catholic University.

Let $X_1, X_2, \cdots, X_N$ be a random sample from a p-variate normal population $MVN(\mu, \Sigma)$, where $\mu = (\mu_1, \cdots, \mu_p)'$ and $\Sigma = ((\sigma_{ij}))$, $\sigma_{ii} = \sigma^2$, $\sigma_{ij} = \rho\sigma^2$, $i \neq j$. The mle of ρ is shown to be $\hat{\rho} = (\sum_{i \neq j=1}^{p} S_{ij})(p-1)^{-1}(\sum_1^p S_{ii})^{-1}$. It is shown that LR test of $H_0: \rho = \rho_0$ can be reduced to Snedecor's F statistic. The unbiased estimation of the contrast $\sum_{i=1}^{p} c_i\mu_i$ is considered next when for $i = k+1, \cdots, p$ and $n < N$, $(X_{in+1}, \cdots, X_{iN})$ are the missing observations in the sample. The usual estimator $T_0 = \sum_{i=1}^{k} c_i\bar{X}_i^{(N)} + \sum_{j=k+1}^{p} c_j\bar{X}_j^{(n)}$ is compared with the proposed estimator $T = \sum_{i=1}^{k} c_i[A_i\bar{X}_i^{(n)} + (1-A_i)\bar{X}_i^{(N-n)}] + \sum_{j=k+1}^{p} c_j\bar{X}_j^{(n)}$ where A_i's are so chosen that Var (T) is minimized. If ρ is known, Var $(T) <$ Var (T_0) whenever $0 < \rho < 1$. In case ρ is unknown T is modified to T_1 by substituting $\hat{\rho}$ in place of ρ. Var $(T_1) <$ Var (T_0) if $EU^2 < (2\rho EU)/(1+(k-1)\rho)$ where $U = \hat{\rho}/(1+(k-1)\hat{\rho})$. Two possible choices of $\hat{\rho}$ are considered (i) all p variables with n replications ($p \leqq n-1$), (ii) only k variables with N replications ($2 \leqq k < p$). The efficiency tables are given for some selected values of n, N, k, p, ρ and the contrast vector c. (Received January 13, 1971.)

129-8. On obtaining large-sample tests from asymptotically normal estimators. T. W. F. STROUD, Queen's University.

This is an extension of Wald's asymptotic test procedure based on unrestricted maximum-likelihood estimators. Wald showed that under certain regularity conditions the test statistic has a limiting central chi-square distribution under the hypothesis and a limiting noncentral chi-square distribution under a sequence of local alternatives. We extend this procedure, allowing it to be based on a broader class of estimators and to obey simpler and less restrictive conditions. Sufficient conditions for validity of the limiting distributions are local twice-differentiability of the left side of the hypothesis and, under a sequence of local alternatives, asymptotic normality of the estimator of the parameter defining the distribution and stochastic convergence (to the appropriate asymptotic value) of the estimator of the covariance matrix. The required asymptotic behavior is verified for the case of independent sampling from two normal distributions and formulas are presented which aid in computing the test statistic. (Received January 18, 1971.)

129-9. On the maximization of an integral of a matrix function over the group of orthogonal matrices. A. K. CHATTOPADYAY and K. C. S. PILLAI, Purdue University.

The noncentral distributions of latent roots arising in various situations in multivariate analysis involve the integration of a hypergeometric function over a group of orthogonal matrices. [James, A. T. *Ann. Math. Statist.* **35** (1964) 475–501.) Anderson [*Ann. Math. Statist.* **36** (1965) 1153–1173] has obtained the subgroup of the orthogonal group which maximizes the same. Li, Pillai and Chang [*Ann. Math. Statist.* **41** (1970) 1541–1556) generalized Anderson's results to the two sample case. In this paper the subgroup of the orthogonal group has been obtained which maximizes the hypergeometric function involved in the integral under mild restrictions and the results have been extended to cover the group of unitary matrices generalizing the results of Li and Pillai [Mimeo series No. 231, Dept. of Statistics, Purdue University and *Ann. Math. Statist.* **41** (1970) 1541–1556]. These results have been utilised in a subsequent paper to find the asymptotic expansions of the distributions of the latent roots in the MANOVA and Canonical correlation case. (Received January 22, 1971.)

129-10. A class of non-parametric tests for homogeneity against ordered alternatives. PETER V. TRYON and THOMAS P. HETTMANSPERGER, National Bureau of Standards and The Pennsylvania State University.

In this paper, the c-sample location problem with ordered or restricted alternatives is considered. Linear combinations of Chernoff–Savage type two-sample statistics computed among the $c(c-1)/2$ pairs of samples are proposed as test statistics. The Chernoff–Savage Theorem [*Ann. Math. Statist.* **29** (1958) 972–994] is extended to prove that the $c(c-1)/2$ dimensional vector of two-sample statistics has an asymptotically multivariate normal distribution. It is shown that for each linear combination of two-sample statistics there is another linear combination, using only the $c-1$ two-sample statistics based on adjacent samples as determined by the alternative, which has the same Pitman efficacy and generally greater Bahadur approximate slope. If the ordered alternative is restricted further by specifying the relative spacings in the alternative, then the weighting coefficients can be chosen to maximize the Pitman efficacy over the class of linear combinations. It is also shown that the statistics proposed by Jonckheere [*Biometrika* **41** (1954) 133–145 and Puri [*Comm. Pure Appl. Math.* **18** (1965) 51–63] have maximum Pitman efficacy when the alternative specifies equal spacings. (Received January 27, 1971.)

129-11. Some results on stochastically ordered queueing systems (preliminary report). DAVID R. JACOBS, JR. and SIEGFRIED SCHACH, The Johns Hopkins University.

In the $G/G/1$ queueing situation, which commences at an arrival epoch ending a period when the system is empty, define the following random processes: (1) $\eta(t)$, the virtual waiting time at time t; (2) η_n, the actual waiting time of the nth customer; (3) $\xi(t)$, the queue size at time t; and (4) ξ_n, the queue size just before the nth arrival epoch. When the limits exist, denote by $\eta(\infty)$, η_∞, $\xi(\infty)$, and ξ_∞ random variables which have the limiting distributions. It is shown that a stochastic increase in interarrival time and/or a stochastic decrease in service time results in a stochastic decrease in η_n and ξ_n, for each n and ∞. Furthermore, it is shown that a stochastic decrease in service results in a stochastic decrease in $\eta(t)$ and $\xi(t)$, for each t and ∞. Finally, it is shown that a stochastic increase in interarrival time results in a stochastic decrease in $\eta(\infty)$ and $\xi(\infty)$; while this is not necessarily true for $\eta(t)$ and $\xi(t)$ if t is finite. Many of these results have been extended to the $G/G/k$ queueing situation. Some of the above mentioned results have been obtained by Daley and Moran [*Theor. Probability Appl.* **13** (1968) 356–359]. (Received February 8, 1971.)

129-12. Optimal allocation of observation for selecting one among several populations. CHANDRA M. GULATI, Carnegie-Mellon University.

Consider the problem of allocating n observations to $k+1$ available populations ($k \geqq 1$). Suppose that in k of the populations a certain characteristic has density g. On the basis of the values observed, one must choose the population for which the density is g. It is assumed that when a wrong population is chosen, a certain known loss is incurred. The problem is how to allocate these observations so as either to maximize the probability of a right decision or to minimize the expected loss. Problems of this type can be considered sequentially as well as non-sequentially. For the case $k = 1$, a sufficient condition is obtained under which the optimal sequential procedure is to take all n observations from one population. General conditions on f and g have been derived for deciding which population should be samples when only one observation is available. For the problem where f and g are given by $f(x) = p^x(1-p)^{1-x}$ and $g(x) = 1-x$ for $x = 0, 1$ $(0 < p < 1)$, the optimal sequential procedure is derived and some properties of the optimal non-sequential procedure are discussed. Some of the results obtained are compared with other methods which have previously been suggested in the literature. (Received February 9, 1971.)

(Abstracts of papers presented at the Central Regional meeting, Columbia, Missouri, May 5–7, 1971.)

130-3. Bayes-fiducial inference for the Weibull distribution. DAVID A. BOGDANOFF and DONALD A. PIERCE, St. Louis University and The University of Kent.

Bayesian inferential methods for the two parameter Weibull distribution (and extreme-value distribution) are developed for the situation where the sample information is large relative to the prior information. The emphasis is on Bayesian confidence intervals, and a practical method of calculating posterior distributions for a large class of functions, which includes the two parameters, the pth quantile, the reliability, and the mean, is given for both censored and uncensored data. The Bayesian posterior distribution is shown equal to the fiducial distribution for uncensored data and Type II progressively censored data, and the Bayesian intervals are shown to have the nominal probability of coverage in the frequency sense for many parameters of interest. Simulation studies are made to find the probability of coverage with Type I censoring. Comparisons with other methods are made and numerical examples are given. (Received January 20, 1971.)

(Abstracts of papers presented by title.)

71T-25. A characterization of multinomial and negative multinomial distribution. K. G. JANARDAN, Montclair State College.

The purpose of this paper is to show that the random vectors $\mathbf{X}$ and $\mathbf{Y}$ have multinomial (or negative multinomial) distribution with the same parameter vector $\boldsymbol{\theta}$ and the other parameters being respectively m and n if and only if the conditional distribution of $\mathbf{X}$ given $\mathbf{X}+\mathbf{Y}$ is multivariate hypergeometric (or multivariate inverse hypergeometric) distribution with parameters $m+n = N$ and $\mathbf{X}+\mathbf{Y} = \mathbf{N}$. (Received December 22, 1970.)

71T-26. Least squares estimation for a class of non-linear models (preliminary report). IRWIN GUTTMAN and VICTOR PEREYRA, University of Montreal and Universidad Central de Venezuela.

Suppose $E(y) = \eta(\mathbf{t})$, where $\eta(\mathbf{t})$ is of the form (i) $\eta(\mathbf{t}) = \eta(\mathbf{t}; \boldsymbol{\alpha}, \mathbf{a}) = \sum_{i=1}^{k} a_i\phi_i(\mathbf{t}; \alpha_i)$ or (ii) $\eta(\mathbf{t}) = \eta(\mathbf{t}; \boldsymbol{\alpha}) = \sum_{i=1}^{k} a_i(\boldsymbol{\alpha})\phi_i(\mathbf{t}; \alpha_i)$, where $\{\phi_i(\mathbf{t}; \alpha_i)\}$ are non-linear functions of their arguments. In case (i), the *constant coefficients* case, $\mathbf{a}$ and $\boldsymbol{\alpha}$ are vectors of unknown parameters, while in case (ii), *the variable coefficients case* $a_i(\boldsymbol{\alpha})$ are given functions of the unknown $\boldsymbol{\alpha}$. Suppose further that we wish to estimate the parameters involved when (i) or (ii) obtains, on the basis of n independent observations $\{y_u\}$ observed at $\{\mathbf{t}_u\}$, $u = 1, \cdots, n$, where $E(y_u) = \eta_u = \eta(\mathbf{t}_u)$ and $\operatorname{Var}(y_u) = \sigma^2$. For case (i), we produce estimates of $(\boldsymbol{\alpha}, \mathbf{a})$ as follows: (a) minimize over $\boldsymbol{\alpha}$, the functional $||\mathbf{y}-P\mathbf{y}||^2 = (\mathbf{y}-P\mathbf{y})'(\mathbf{y}-P\mathbf{y})$, where the projection P is defined by $P\mathbf{y} = \sum_{i=1}^{k} < \mathbf{y}, \mathbf{e}_i > \mathbf{e}_i$, with $\mathbf{e}_i$ an orthonomal basis for the span $S(\boldsymbol{\alpha})$ of the vectors $\boldsymbol{\phi}_i = \boldsymbol{\phi}_i(\alpha_i) = (\cdots, \phi_i(\mathbf{t}_u; \alpha_i), \cdots)'$, $i = 1, \cdots, k$. Denote the estimate of $\boldsymbol{\alpha}$ so produced by $\hat{\boldsymbol{\alpha}}$, that is min $r_1(\boldsymbol{\alpha}) =$

$r_1(\hat{\alpha}) = m_1$, say. Notice that this first part of the procedure eliminates, for the moment, consideration of the parameters **a**. To find estimates of **a**, we proceed as follows: (b) Minimize over **a**, the functional $r_3(\mathbf{a}) = ||\mathbf{y} - \sum_{i=1}^{k} a_i \phi_i(\alpha_i)||^2$. Let $\hat{\mathbf{a}}$ denote the estimate so produced, i.e., $r_3(\hat{\mathbf{a}}) = \min r_3(\mathbf{a}) = m_3$, say. We prove the following. THEOREM. *If* $(\alpha^*, \mathbf{a}^*)$ *denotes the least squares estimator of* $(\alpha, \mathbf{a})$, *that is* $\min_{(\alpha, \mathbf{a})} r_2(\alpha, \mathbf{a}) = r_2(\alpha^*, \mathbf{a}^*) = m_2$, *where* $r_2(\alpha, \mathbf{a}) = ||\mathbf{y} - \eta(\alpha, \mathbf{a})||^2$, *with* $\eta(\alpha, \mathbf{a}) = (\cdots, \eta(\mathbf{t}_u; \alpha, \mathbf{a}), \cdots)'$, *then* $(\alpha^*, \mathbf{a}^*) = (\hat{\alpha}, \hat{\mathbf{a}})$. An additional assumption of normality of the y_u's allows a discussion of confidence regions for $(\alpha, \mathbf{a})$ and $\eta(\mathbf{t}; \alpha, \mathbf{a})$. For the variable coefficients case (ii), the parameters $a_i = a_i(\alpha)$ are given functions of α, and are not free. Nevertheless, we may obtain an estimate of $\hat{\alpha}$ by using part (a) of the above procedure. A criterion for judging the suitability of $\hat{\alpha}$ so produced is given. If $\hat{\alpha}$ is adjudged suitable, a reasonable point estimate of $\eta(\mathbf{t}; \alpha)$ is $\eta(\mathbf{t}; \hat{\alpha}) = \sum_{i=1}^{k} a_i(\hat{\alpha}) \phi_i(\mathbf{t}', \hat{\alpha}_i)$. If $\hat{\alpha}$ is not adjudged suitable, it could of course be utilized as a sensible starting guess in any iterative procedure to minimize $r_4(\alpha) = ||\mathbf{y} - \eta(\alpha)||^2$, especially if the quantity $b = (r_4(\hat{\alpha}) - m_1)/m_1$ is small. (Received December 22, 1970.)

71T-27. Asymptotic normality of *H-L* estimator for dependent data. HIRA LAL KOUL, Michigan State University.

Suppose we are observing a strictly stationary strongly mixing stochastic process $\{X_i\}$, with F as cdf of X_0, say. Suppose θ is the location parameter of F. We obtain asymptotic normality of a class of Hodges–Lehmann estimators of θ under the assumptions that the density of F be bounded and F have finite Fisher's information and the score function that generates the estimators of θ have bounded second derivative. Our results generalize some of the results of J. Gastwirth and Herman Rubin "Behaviour of robust estimators on dependent data," *Mimeo Ser. No.* 197 [1969], Department of Statistics, Purdue Univ. Finally, we also obtain asymptotic normality of a class of rank statistics. (Received January 12, 1971.)

71T-28. A note on monotonicity in the gamma distributions. GARY C. MCDONALD, General Motors Research Laboratories.

Let $X_1, \cdots, X_k$ be k ($\geqq 2$) independent identically distributed random variables having the gamma distribution with scale and shape parameters 1 and s ($\geqq 1$) respectively. Denote the density and cumulative distribution function of X_1 by $g_s(\cdot)$ and $G_s(\cdot)$ respectively. A quantity which naturally arises in certain ranking and selection procedures is

$$P = P(k, b, s) = \int_0^\infty [G_s(x/b)]^{k-1} g_s(x)\, dx,$$

where $0 < b \leqq 1$. The quantity P is the probability that a specified one of the X_j, say X_1, is at least as large as b times the maximum of the X_j. It is known that for a fixed k and P, $k^{-1} \leqq P < 1$, an increase in s implies an increase in b. This note gives an elementary self-contained proof of this fact for $k = 2$ and integral values of s without recourse to a convex ordering property of the gamma family employed in earlier proofs. (Received January 12, 1971.)

71T-29. On sequential estimation of a linear function of the mean of multivariate normal population. V. K. ROHATGI and SURESH C. RASTOGI, Catholic University and University of Maryland.

Let $X_1, \cdots, X_n$ be a random sample from a k-variate normal population MVN(μ, Σ) where $\mu = (\mu_1, \cdots, \mu_k)'$ is the mean vector, $\Sigma = \text{diag}\,(\sigma_1^2, \cdots, \sigma_k^2)\sigma_i^2 > 0$ is the diagonal dispersion matrix. Sequential estimation of μ has been considered by Khan (*Sankhyā A* **30** 331–334). We have considered the problem of sequential estimation of a linear function $\nu = \Sigma\alpha_i\mu_i$ where α_i, $i = 1, \cdots, k$ are known real numbers. The loss function is taken to be $L(n) = |Y - \nu|^s + n^t$ where $s > 0$, $t > 0$ are given real numbers; and $Y = \Sigma\alpha_i\bar{X}_{in}$ is the unbiased estimator of ν. By adopting the methods of Starr and Woodroofe (*Proc. Nat. Acad. Sci. USA* **63** 285–288) the stopping rule used is shown to have some interesting limiting properties when σ_i's become infinite. (Received January 13, 1971.)

71T-30. Large sample estimation of restricted parameters. J. K. GHOSH and K. SUBRAMANYAM, Indian Statistical Institute, Calcutta.

The results of J. M. Hammersley (On estimating restricted parameters (*J. Roy. Statist. Soc. Ser. B* [1950])) have been generalized using Chernoff's (A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations (*Ann. Math. Statist.* [1952])) theorem on large deviations. The above results are further generalized to the case of two parameters one of which is continuous and the other is discrete. The case of several parameters is similar. Our results are then applied to various examples of which the most important is the classical problem of estimating the parameter n of a binomial distribution. (Received January 14, 1971.)