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A HÁJEK-RÉNYI TYPE INEQUALITY FOR STOCHASTIC VECTORS WITH APPLICATIONS TO SIMULTANEOUS CONFIDENCE REGIONS[1]

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For a sequence of stochastic vectors forming either a forward or a reverse martingale, a Hájek-Rényi type inequality is derived, and its applications in some problems of simultaneous confidence regions are stressed.

1. Statement of the result. A variety of multivariate Chebyshev inequalities is available in the literature; we refer to Karlin and Studden (1966) and to Mudholkar and Rao (1967) which include earlier references. In the present note, for stochastic vectors, a simultaneous inequality comparable to Chow's (1960) semi-martingale extension of the Hájek-Rényi (1955) inequality is considered.

Let $\{\mathbf{Z}_i, i \geqq 1\}$ be a sequence of stochastic p-dimensional column vectors, where $p \geqq 1$. Let $\mathscr{B}_i = \mathscr{B}(\mathbf{Z}_1, \cdots, \mathbf{Z}_i)$ and $\mathscr{C}_i = \mathscr{C}(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \cdots)$ be the σ-fields generated by $(\mathbf{Z}_1, \cdots, \mathbf{Z}_i)$ and $(\mathbf{Z}_i, \mathbf{Z}_{i+1}, \cdots)$ respectively, $i \geqq 1$; clearly, $\mathscr{B}_i$ is $\uparrow$ while $\mathscr{C}_i$ is $\downarrow$ in i. Suppose that $E\mathbf{Z}_i = \mathbf{0}$ and $E\mathbf{Z}_i\mathbf{Z}_i'$ exists for all $i \geqq 1$. Also, assume that either of the following two conditions holds:

$$E(\mathbf{Z}_n \mid \mathscr{B}_k) = \mathbf{Z}_k \quad \text{almost surely (a.s.)} \qquad \text{for all} \quad n \geqq k \geqq 1, \tag{1}$$

$$E(\mathbf{Z}_k \mid \mathscr{C}_n) = \mathbf{Z}_n \text{ a.s.,} \qquad \text{for all} \quad n \geqq k \geqq 1. \tag{2}$$

Let $\mathbf{A}$ be an arbitrary $(p \times p)$ positive definite (p.d.) matrix, and let

$$\zeta_n = E(\mathbf{Z}_n'\mathbf{A}^{-1}\mathbf{Z}_n), \qquad \zeta_{n+1}^* = E[(\mathbf{Z}_{n+1}-\mathbf{Z}_n)'\mathbf{A}^{-1}(\mathbf{Z}_{n+1}-\mathbf{Z}_n)], \qquad n \geqq 1, \tag{3}$$

where $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_p$. Then, we have the following theorem.

THEOREM 1. *For a non-increasing sequence $\{c_i\}$ of positive constants, under* (1), *for every $\varepsilon > 0$, $n \geqq 1$, and $N \geqq 1$,*

$$P[\max_{n \leqq k \leqq n+N} c_k\{\sup_{\lambda \neq 0} (\boldsymbol{\lambda}'\mathbf{A}\boldsymbol{\lambda})^{-\frac{1}{2}}|\boldsymbol{\lambda}'\mathbf{Z}_k|\} > \varepsilon] \leqq \varepsilon^{-2}\{c_n^{\,2}\zeta_n + \textstyle\sum_{k=n+1}^{n+N} c_k^{\,2}\zeta_k^{\,*}\}; \tag{4}$$

for a non-decreasing sequence $\{c_i\}$ of positive constants, under (2),

$$P[\max_{n \leqq k \leqq n+N} c_k\{\sup_{\lambda \neq 0} (\boldsymbol{\lambda}'\mathbf{A}\boldsymbol{\lambda})^{-\frac{1}{2}}|\boldsymbol{\lambda}'\mathbf{Z}_k|\} > \varepsilon] \leqq \varepsilon^{-2}\{c_{n+N}^2\zeta_{n+N} + \textstyle\sum_{k=n+1}^{n+N} c_k^{\,2}\zeta_k^{\,*}\}. \tag{5}$$

It may be noted that in (4) or (5), when $N = 0$, the second term on the right-hand side should be taken as equal to zero. If we let $c_k = c_n$, $n \leqq k \leqq n+N$, we obtain the Kolmogorov-type inequality, while for $N = 0$, this reduces to the Chebyshev-type inequality. Some applications are considered in Section 3.

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2. Proof of the theorem. By the Schwarz-inequality

(6) $$\sup_{\lambda \neq 0} (\lambda' \mathbf{A} \lambda)^{-\frac{1}{2}} |\lambda' \mathbf{Z}_k| = (\mathbf{Z}_k' \mathbf{A}^{-1} \mathbf{Z}_k)^{\frac{1}{2}} (\geqq 0), \qquad \text{for all } k \geqq 1.$$

Hence, from (6), we have

(7) $$P\{\max_{n \leqq k \leqq n+N} c_k [\sup_{\lambda \neq 0} (\lambda' \mathbf{A} \lambda)^{-\frac{1}{2}} |\lambda' \mathbf{Z}_k|] > \varepsilon\} = P\{\max_{n \leqq k \leqq n+N} c_k^2 Y_k > \varepsilon^2\},$$

where $Y_k = (\mathbf{Z}_k' \mathbf{A}^{-1} \mathbf{Z}_k)$, $k \geqq 1$. Now, under (1),

(8) $$\begin{aligned} E[Y_n \mid \mathscr{B}_k] &= Y_k + 2E\{(\mathbf{Z}_n - \mathbf{Z}_k)' \mid \mathscr{B}_k\} \mathbf{A}^{-1} \mathbf{Z}_k + E[(\mathbf{Z}_n - \mathbf{Z}_k)' \mathbf{A}^{-1} (\mathbf{Z}_n - \mathbf{Z}_k) \mid \mathscr{B}_k] \\ &= Y_k + E[(\mathbf{Z}_n - \mathbf{Z}_k)' \mathbf{A}^{-1} (\mathbf{Z}_n - \mathbf{Z}_k)' \mid \mathscr{B}_k] \geqq Y_k \text{ a.s.}, \quad \text{for all } n \geqq k \geqq 1. \end{aligned}$$

Hence, $\{Y_k, \mathscr{B}_k, k \geqq 1\}$ forms a nonnegative semi-martingale sequence. Consequently, on using the second inequality in Theorem 1 of Chow (1960) [which provides the semi-martingale extension of the Hâjek–Rényi (1955) inequality], the right-hand side of (4) directly follows from (3) and (7).

By reversing the ordering of the index set $\{i\}$ in (2), the reverse martingale property of $\{\mathbf{Z}_i, \mathscr{C}_i, i \geqq 1\}$ can be converted into forward martingale property, and hence, the same proof as in (4) applies. This completes the proof of (5). □

3. Some applications to simultaneous confidence regions. We consider here the following three problems.

(I) Let $\omega = \{\mathbf{X}_1, \mathbf{X}_2, \cdots, \text{ad inf}\}$ be a sequence of independent stochastic p dimensional column vectors, where $E\mathbf{X}_i = \boldsymbol{\mu}_i$ and $\mathbf{V}(\mathbf{X}_i) = \boldsymbol{\Sigma}_i$, $i \geqq 1$. Let $\mathbf{X}_n = n^{-1} \sum_{i=1}^n \mathbf{X}_i$, $\bar{\boldsymbol{\mu}}_n = n^{-1} \sum_{i=1}^n \boldsymbol{\mu}_i$, and let

(9) $$\mathbf{T}_n = n(\bar{\mathbf{X}}_n - \bar{\boldsymbol{\mu}}_n) = \sum_{i=1}^n (\mathbf{X}_i - \boldsymbol{\mu}_i), \qquad n \geqq 1.$$

It follows that $\{\mathbf{T}_n, \mathscr{B}_n, n \geqq 1\}$ forms a forward martingale sequence, i.e., (1) holds. If we let $v_i^* = \text{Trace}(\boldsymbol{\Sigma}_i \mathbf{A}^{-1})$, $i \geqq 1$, we have from (3), $\zeta_{n+1}^* = v_{n+1}^*$ and $\zeta_n = v_n = v_1^* + \cdots + v_n^*$, $n \geqq 1$. Hence, from (4), we obtain that

(10) $$P[\max_{n \leqq k \leqq n+N} (kc_k)[\sup_{\lambda \neq 0} (\lambda' \mathbf{A} \lambda)^{-\frac{1}{2}} |\lambda' (\bar{\mathbf{X}}_k - \bar{\boldsymbol{\mu}}_k)|] > t] \leqq t^{-2} \cdot [c_n^2 v_n + \textstyle\sum_{k=n+1}^{n+N} c_k^2 v_k^*].$$

In particular, if $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$, $\forall\, i \geqq 1$, and we let $c_k = k^{-1}$, $\mathbf{A} = \boldsymbol{\Sigma}$, we obtain from (10) that

(11) $$\begin{aligned} &P[|\lambda'(\bar{\mathbf{X}}_k - \bar{\boldsymbol{\mu}}_k)| \leqq t(\lambda' \Sigma \lambda)^{\frac{1}{2}}, \forall\, \lambda \neq 0, n < k < n+N] \\ &\qquad \geqq 1 - pt^{-2}[n^{-1} + \textstyle\sum_{k=n+1}^{n+N} k^{-2}] \geqq 1 - p(2N+n)/[n(n+N)t^2]. \end{aligned}$$

For $N = 0$, (11) is analogous to the Scheffé-type (cf. [7] page 68) simultaneous confidence region for $\lambda' \bar{\boldsymbol{\mu}}_n$ (or $\lambda' \boldsymbol{\mu}$ when all the $\boldsymbol{\mu}_i$ are equal) under the Chebyshev set up (i.e., under no assumption of normality, inherent in [7]). For $N \geqq 1$, it is an extension along the lines of the Kolmogorov inequality.

(II) If the $\mathbf{X}_i$ are identically distributed with mean $\boldsymbol{\mu}$ and dispersion matrix $\boldsymbol{\Sigma}$, $\{(\bar{\mathbf{X}}_k-\bar{\mathbf{X}}_{n+N}), \mathscr{C}_k, n < k < n+N\}$ has the reverse martingale property, for all $n \geqq 1$; that is, (2) holds for $\mathbf{Z}_k = \bar{\mathbf{X}}_k-\bar{\mathbf{X}}_{n+N}, n \leqq k \leqq n+N$. Hence, by (5),

$$(12) \quad P[\max_{n\leqq k\leqq n+N} n^{\frac{1}{2}}|\boldsymbol{\lambda}'(\bar{\mathbf{X}}_k-\bar{\mathbf{X}}_{n+N})| < \varepsilon(\boldsymbol{\lambda}'\Sigma\boldsymbol{\lambda})^{\frac{1}{2}}, \forall \boldsymbol{\lambda} \neq \mathbf{0}] \geqq 1-p\varepsilon^{-2}N/(n+N) \geqq 1-\eta,$$

whenever $N \leqq \delta n$ and $\delta/\varepsilon^2 \leqq \eta(> 0)$. (12) establishes the 'uniform continuity in probability' with respect to $n^{-\frac{1}{2}}$ [in the sense of Anscombe (1952)] for all possible linear compounds $\{\boldsymbol{\lambda}'\mathbf{X}_k, \boldsymbol{\lambda} \neq \mathbf{0}\}$. This result is useful for the study of sequential (simultaneous) confidence regions for all possible linear compounds of $\boldsymbol{\mu}$.

(III) Consider now a p-variate separable semi-martingale $\{\mathbf{Z}_t, t \geqq 0\}$, such that (i) $E\|\mathbf{Z}_t\| < \infty$, $\forall\, t > 0$, where $\|\mathbf{x}\|$ stands for the Euclidean norm of a vector $\mathbf{x}$. Let $f(t)$ be a non-decreasing positive function on $[0, \tau]$ where $\tau > 0$, and let $E\mathbf{Z}_t = 0, \forall\, t \geqq 0$,

$$(13) \qquad \zeta(t) = E[\mathbf{Z}_t'A^{-1}\mathbf{Z}_t], \qquad t \geqq 0,$$

where $\mathbf{A}$ is p.d., and assume that (i) $\zeta(t)/f^2(t) \to a_0(< \infty)$ as $t \to 0$, and (ii) $\int_0^\tau [f(t)]^{-2}\, d\zeta(t)$ exists. Then, by virtue of (6), we obtain on proceeding as in Theorem 5.1 of Birnbaum and Marshall (1961) that

$$(14) \quad P\{\sup_{t\varepsilon[0,\tau]}[\sup_{\boldsymbol{\lambda}\neq 0}(\boldsymbol{\lambda}'\mathbf{A}\boldsymbol{\lambda})^{-\frac{1}{2}}|\boldsymbol{\lambda}'\mathbf{Z}_t|/f(t)] \geqq 1\} \leqq a_0+\textstyle\int_0^\tau [f(t)]^{-2}\, d\zeta(t).$$

The last inequality provides a multivariate extension of Theorem 5.1 of Birnbaum and Marshall and also an extension of [4] to separable semi-martingale processes.

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