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ALMOST SURE CONVERGENCE OF UNIFORM TRANSPORT PROCESSES TO BROWNIAN MOTION[1]

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Let $x_n(t)$ be the position of a particle in one dimension that switches between uniform velocities $+n$ and $-n$ at the jump times of a Poisson process with intensity n^2. In this note are constructed realizations of the processes $x_n(t)$ that converge almost surely to Brownian motion, uniformly on the unit time interval.

1. Introduction. Let $v(t)$ be a Markov chain with stationary transition probabilities, states $+1$ and -1, and with infinitesimal matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. We define a sequence of uniform transport processes by $x_n(t) = n\int_0^t v(n^2 s)\,ds$, $n = 1, 2, \cdots$. $x_n(t)$ represents the position at time $t \geqq 0$ of a particle in one dimension that switches between uniform velocities $+n$ and $-n$ at the jump times of a Poisson process $N_n(t)$ with intensity n^2. A special case of a result of Pinsky [2] shows that the processes $x_n(t)$ converge in distribution to standard one-dimensional Brownian motion. See also [5]. In this note we strengthen the convergence to convergence almost surely, uniformly on the unit time interval. The proof employs Skorokhod's result of the reproduction of independent random variables by evaluating Brownian motion at random times. More precisely, we have the following theorem.

THEOREM. *There exist realizations* $\{x_n(t),\ t \geqq 0\}$ *of the above uniform transport processes on the same probability space as a standard Brownian motion process* $\{x(t),\ t \geqq 0\}$, *with* $x(0) \equiv 0$, *so that we have*, $\lim_{n\to\infty} \max_{0 \leqq t \leqq 1} |x_n(t) - x(t)| = 0$, *almost surely*.

2. Proof of theorem. The proof depends on the construction of appropriate realizations of the uniform transport processes. Let $(\Omega, \mathscr{A}, P)$ be the probability space for a standard Brownian motion $\{x(t),\ t \geqq 0\}$, with $x(0) \equiv 0$.

On $(\Omega, \mathscr{A}, P)$ for each $n = 1, 2, \cdots$, let $\xi_1^{(n)}, \xi_2^{(n)}, \cdots$, be a sequence of independent random variables each with an exponential distribution with parameter $2n$, that is, $P(\xi_i^{(n)} > \lambda) = e^{-2n\lambda}$ for $\lambda \geqq 0$, and assume the $\xi_i^{(n)}$ random variables are independent of the process $x(t)$.

Furthermore on $(\Omega, \mathscr{A}, P)$ let $k_1, k_2, \cdots$, be a sequence of independent random variables so that $P(k_i = 1) = P(k_i = -1) = \frac{1}{2}$ for each i, and let the k_i's be independent of the $\xi_i^{(n)}$'s and the Brownian motion process. All of this is easily

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accomplished by introducing a product space and for convenience we relabel the new space $(\Omega, \mathscr{A}, P)$.

Consider the sequence of independent, identically distributed random variables $k_1\xi_1^{(n)}, k_2\xi_2^{(n)}, \cdots$, for each $n \geqq 1$. We note that $E(k_i\xi_i^{(n)}) = 0$ and $\sigma^2(k_i\xi_i^{(n)}) = E((\xi_i^{(n)})^2) = 1/2n^2$. By a theorem of Skorokhod ([4] page 163) (see also [3]) for each $n \geqq 1$ there exists a sequence $\sigma_1^{(n)}, \sigma_2^{(n)}, \cdots$, of nonnegative independent, identically distributed random variables on $(\Omega, \mathscr{A}, P)$ so that the sequence $x(\sigma_1^{(n)})$, $x(\sigma_1^{(n)}+\sigma_2^{(n)}), \cdots$, has the same distribution as $k_1\xi_1^{(n)}, k_1\xi_1^{(n)}+k_2\xi_2^{(n)}, \cdots$, and $E(\sigma_i^{(n)}) = \sigma^2(k_i\xi_i^{(n)}) = 1/2n^2$.

For a fixed n we define for $i = 1,2, \cdots$,

$$\gamma_i^{(n)} = n^{-1}\left|x\left(\textstyle\sum_{j=0}^{i}\sigma_j^{(n)}\right)-x\left(\textstyle\sum_{j=0}^{i-1}\sigma_j^{(n)}\right)\right|, \tag{1}$$

where $\sigma_0^{(n)} \equiv 0$. The random variables $\gamma_1^{(n)}, \gamma_2^{(n)}, \cdots$, are independent, each with an exponential distribution with parameter $2n^2$, so that $E(\gamma_i^{(n)}) = 1/2n^2$.

Now, let $x^{(n)}(t)$, $t \geqq 0$, be piecewise linear in such a manner so that

$$x^{(n)}\left(\textstyle\sum_{j=1}^{i}\gamma_j^{(n)}\right) = x\left(\textstyle\sum_{j=1}^{i}\sigma_j^{(n)}\right), \tag{2}$$

and $x^{(n)}(0) \equiv 0$. Thus $x^{(n)}(\cdot)$ has slope $+n$ or $-n$. Also let $\tau_i^{(n)}$ be the time of the ith discontinuity of the right-hand derivative of $x^{(n)}(\cdot)$.

We claim that $x^{(n)}(t)$ is a realization of the nth uniform transport process above. We need only check that the increments $\tau_i^{(n)}-\tau_{i-1}^{(n)}$, $i = 1, 2, \cdots$, with $\tau_0^{(n)} \equiv 0$, are independent with a common exponential distribution, parameter n^2.

Now, the probability that $x(\sum_{j=0}^{i}\sigma_j^{(n)})-x(\sum_{j=0}^{i-1}\sigma_j^{(n)})$ is positive is $\frac{1}{2}$, independent of the past up to time $\sum_{j=0}^{i-1}\sigma_j^{(n)}$. Hence $\tau_1^{(n)} = \gamma_1^{(n)}+ \cdots +\gamma_N^{(n)}$, where $P(N = i) = 2^{-i}$, $i = 1, 2, \cdots$. It is easy to see ([1] page 54, (5.6)) that $\tau_1^{(n)}$ has an exponential distribution with parameter n^2, one half of the parameter of the exponentially distributed $\gamma_i^{(n)}$'s. In the same way, each increment $\tau_i^{(n)}\tau_{i-1}^{(n)}$ has an exponential distribution, parameter n^2. The increments are independent since they are sums of disjoint blocks of the $\gamma_i^{(n)}$'s.

By Kolmogorov's inequality for each $\varepsilon > 0$ we have

$$P\left(\max_{1\leqq i\leqq 2n^2}\left|\gamma_1^{(n)}+\cdots+\gamma_i^{(n)}-i/2n^2\right| \geqq \varepsilon\right) \leqq 1/\varepsilon^2 \textstyle\sum_{i=1}^{2n^2}\sigma^2(\gamma_i^{(n)}) = 1/2\varepsilon^2 n^2. \tag{3}$$

By the Borel–Cantelli lemma it follows that

$$\lim_{n\to\infty}\max_{1\leqq i\leqq 2n^2}\left|\gamma_1^{(n)}+\cdots+\gamma_i^{(n)}-i/2n^2\right| = 0, \text{a.s.} \tag{4}$$

Similarly,

$$\lim_{n\to\infty}\max_{1\leqq i\leqq 2n^2}\left|\sigma_1^{(n)}+\cdots+\sigma_i^{(n)}-i/2n^2\right| = 0, \text{a.s.} \tag{5}$$

Finally, we have, letting $\gamma_0^{(n)} = \sigma_0^{(n)} \equiv 0$, that

$$\lim_{n\to\infty} \max_{0 \leqq t \leqq 1} |x^{(n)}(t) - x(t)|$$

$$= \lim_{n\to\infty} \max_{0 \leqq i \leqq 2n^2} |x^{(n)}(\textstyle\sum_{j=0}^{i} \gamma_j^{(n)}) - x(\sum_{j=0}^{i} \gamma_j^{(n)})| \qquad \text{(by (4))}$$

$$= \lim_{n\to\infty} \max_{0 \leqq i \leqq 2n^2} |x(\textstyle\sum_{j=0}^{i} \sigma_j^{(n)}) - x(\sum_{j=0}^{i} \gamma_j^{(n)})| \qquad \text{(by (2))}$$

$= 0$, by (4) and (5) and the uniform continuity of Brownian motion on [0, 1]. This terminates the proof.

NOTE ADDED IN PROOF. It is possible to extend our result to N-dimensions. The proof employs an N-dimensional extension of Skorokhod's result; namely, given a sequence $x_1, x_2, \ldots,$ of radially symmetric, independent, identically distributed R^N-valued random variables with finite variance (and hence mean zero) there exists a sequence $\sigma_1, \sigma_2, \ldots,$ of non-negative, independent, identically distributed random variables on the same space as an N-dimensional Brownian motion process $x^N(t)$ so that the random variables $\sum_{k=1}^{n} \sigma_k$ are stopping times and $x^N(\sum_{k=1}^{n} \sigma_k) - x^N(\sum_{k=1}^{n-1} \sigma_k)$ have the same joint distributions as the $x^1, x^2, \ldots,$ and $E(\sigma_1) = \sigma^2(x_1)/N$. Now define N-dimensional uniform transport processes as in [5]. Our extension is that the N-dimensional uniform transport processes converge to $x^N(\cdot)$ as before. As to the proof of the extension; forget $k_i\xi_i^{(n)}$; instead let $\xi_i^{(n)}$ be radially symmetric with distribution given by $P(|\xi_i^{(n)}| > \lambda) = \exp(-2^{\frac{1}{2}}n/\lambda N^{\frac{1}{2}})$. Then $\sigma^2(\xi_i^{(n)}) = N/n^2$. Choose $\sigma_i^{(n)}$ corresponding to the sequence $\xi_i^{(n)}$, $i = 1, 2, \ldots,$ as assured by Skorokhod's result and define $\gamma_i^{(n)}$ in terms of $x^N(\cdot)$ and the $\sigma_j^{(n)}$ as before. Note that $E(\sigma_i^{(n)}) = 1/n^2$. There is no problem about the process constructed from $x^N(\cdot)$ being a transport process. The rest of the proof proceeds as before.

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