The Annals of Mathematical Statistics
1971, Vol. 42, No. 3, 1124-1128

ON THE EXISTENCE OF ABSOLUTE MOMENTS FOR THE EXTINCTION TIME OF A GALTON-WATSON PROCESS

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If $\{Z_n\}$ is a Galton-Watson process with mean one, and τ is the extinction time, it is shown that $EZ^{1+\alpha} < \infty$ implies $E\tau^\beta = \infty$ for $\beta > 1/\alpha$, $0 < \alpha < 1$. Conditions which imply $EZ^{1+\alpha} = \infty$ and $E\tau^\beta < \infty$ for $\beta < 1/\alpha$, $0 < \alpha < 1$ are given. Necessary and sufficient conditions for $EX^{m+\alpha} < \infty$ or $EX^m \log X < \infty$ are given in terms of the Laplace transform of a general nonnegative random variable X, $0 < \alpha < 1$, $m = 0, 1, \cdots$.

1. Introduction. Let Z_n be the number of individuals in the nth generation of a reproducing system so that $\{Z_n\}$ is a Galton-Watson process, and let $\tau = \inf\{n \mid Z_n = 0\}$ be the extinction time for the process. It is well known that $\mu_1 = EZ_1 > 1$ iff $P(\tau < \infty) < 1$, and that if $\mu_1 < 1$ then $P(\tau > n) \leqq \mu_1{}^n$. Thus, in the non-critical case, $\mu_1 \neq 1$, either $E\tau^\alpha < \infty$ for no $\alpha > 0$ or all $\alpha > 0$. There is a drastic difference in the critical case.

If $\mu_1 = 1$ (see Kesten, Ney, Spitzer [4]) then $P(\tau > n) \sim \sigma^2/2n$, when $\sigma^2 = \operatorname{Var} Z_1 < \infty$, and in this case $E\tau^\alpha < \infty$ iff $0 < \alpha < 1$. It is pointed out in the above reference that $nP(\tau > n) \to 0$ if $\sigma^2 = \infty$. This fact is what led us to ask about higher moments of τ when Z_1 has worse behavior, and we have the following negative answer.

THEOREM 1. *Let* $\mu_1 = 1$. *If* $EZ_1{}^{1+\alpha} < \infty$ *then* $E\tau^\beta = \infty$ *for all* $\beta > 1/\alpha$, $0 < \alpha < 1$.

We have been unable to determine the validity of the next

ASSERTION. *Let* $\mu_1 = 1$. *If* $EZ_1{}^{1+\alpha} = \infty$ *then* $E\tau^\beta < \infty$ *for all* $\beta < 1/\alpha$, $0 < \alpha < 1$.

However, if stronger restrictions than $EZ_1{}^{1+\alpha} = \infty$ are placed on the distribution of Z_1 results of the above type do hold. To make these precise we need additional notation.

Let $f(s) = \sum_0^\infty s^n P(Z_1 = n)$ be the probability generating function (pgf) of Z_1, so that the pgf of Z_n is f_n, the nth functional iterate of f, i.e. $f_n = f \circ f_{n-1}$, $f_0 =$ identity. Then $P(\tau > n) = 1 - f_n(0)$ and $\mu_1 = f'(1-)$.

From now on assume $\mu_1 = 1$.

We introduce the following conditions on f, where K is some finite positive constant.

$$A_\beta: f(s) - s \leqq K(1-s)^{1+\beta} \quad \text{for} \quad s_0 \leqq s \leqq 1, \quad \text{some} \quad s_0 < 1.$$

$$B_\beta: f(s) - s \geqq K(1-s)^{1+\beta} \quad \text{or} \quad s_0 \leqq s \leqq 1, \quad \text{some} \quad s_0 < 1.$$

It is easy to show that $EZ_1{}^{1+\beta} < \infty$ implies A_β, (see e.g. Loève [5] page 199).

Received September 1970.

Further, A_β implies $EZ_1^{1+\alpha} < \infty$ for each $\alpha < \beta$, $0 < \beta < 1$, while B_β implies $EZ_1^{1+\beta} = \infty$, $0 < \beta < 1$ (see Remark 2 following Theorem 5 below). We know of no moment condition which implies B_β, but we do have the following sufficient condition (Theorem 7): If $P(Z_1 > x)$ is asymptotic to $x^{-1-\alpha}L(x)$, where L is slowly varying, then B_β holds for each $\beta > \alpha$.

THEOREM 2. *Let* $0 < \beta < 1$. *Condition* B_β *implies* $EZ_1^{1+\beta} = \infty$ *and* $E\tau^\gamma < \infty$ *for all* $\gamma < 1/\beta$.

Since $E\tau^\gamma < \infty$ iff $\Sigma n^{\gamma-1}(1-f_n(0)) < \infty$, $\gamma > 0$, Theorems 1 and 2 are corollaries of

THEOREM 3. *Let* $\beta > 0$ *and fix* v_0 *in* $[0, 1)$. *Then* A_β *implies* $n^\gamma(1-f_n(v_0)) \to \infty$ *for all* $\gamma > 1/\beta$, *and* B_β *implies* $n^\gamma(1-f_n(v_0)) \to 0$ *for all* $\gamma < 1/\beta$.

PROOF. Define $h(u) = 1-f(1-u)$; then $h_n(u) = 1-f_n(1-u)$ $(h_n = h_{n-1} \circ h)$, and h is continuous, strictly increasing, concave downward and $0 < h(u) < u \leqq 1$. Let $u_0 = 1-v_0$ and define $u_{n+1} = h(u_n) = 1-f_n(v_0)$. Further, define $a_n = (u_n - u_{n+1})u_n^{-1-\beta}$, $r_n = u_{n+1}/u_n \leqq 1$ and $u_n^{-\beta} = \rho_1 + \cdots + \rho_n$. Then

$$\rho_{n+1} = a_n(r_n)^{-\beta}(1-r_n^{\beta})/(1-r_n).$$

Since $0 < a \leqq (1-x^\beta)/(1-x) \leqq b < \infty$ for $0 \leqq x \leqq 1$, B_β implies $\rho_{n+1} \geqq K_1$ for some $K_1 > 0$ and all n. Under hypothesis A_β, $1 \geqq r_n = 1-a_n u_n^\beta \geqq 1-K u_n^\beta \uparrow 1$ so that $\rho_{n+1} \leqq K_2$ for some $K_2 < \infty$ and all n.

The pattern for the above proof is found in the paper by Szekeres ([7] see Theorem 1c) which contains further information on the behavior of the iterates h_n.

Another interesting proof of at least part of Theorem 3 can be given by applying the A_β and B_β conditions to

$$\frac{1}{1-f_n(s_0)} - \frac{1}{1-s_0} = \sum_{k=0}^{n-1} \Delta(f_k(s_0))$$

where

$$\Delta(s) = \frac{1}{1-f(s)} - \frac{1}{1-s} = \frac{f(s)-s}{(1-s)^2} \cdot \frac{1-s}{1-f(s)}.$$

Assume that $0 < \beta < 1$. If $P_\gamma(Q_\gamma)$ is the proposition "$n^\gamma(1-f_n(s_0)) \to \infty$ (0)," then under hypothesis A_β, P_γ implies P_α for all $\alpha > 1+(1-\beta)\gamma$, while under hypothesis B_β, Q_γ implies Q_α for all $\alpha < 1+(1-\beta)\gamma$. Iteration produces: A_β and P_γ imply P_α for all $\alpha > 1/\beta$; B_β and Q_γ imply Q_α for all $\alpha < 1/\beta$. Now Q_0 is well known, but we know of no way to get a P_γ except by using the arguments of the given proof.

While trying to get an insight into condition B_β and to prove the "assertion" above, we found necessary and sufficient conditions, in terms of Laplace transforms, for finiteness of non-integral absolute moments of general random variables (Theorem 5).

2. Moments and Laplace transforms. Let X be a nonnegative rv with df F and Laplace transform $\varphi(\lambda) = \int_{0-}^{\infty} e^{-\lambda x} F(dx)$. Set $\mu_\alpha = EX^\alpha$ and introduce inductively the notation

$$F_0 := F, F_{n+1}(x) := \int_0^x \left[\frac{\mu_n}{n!} - F_n(y)\right] dy$$

for each n such that $\mu_n < \infty$.

The following theorem (perhaps well known) gives a representation for μ_α and φ in terms of F_n.

THEOREM 4. *Assume* $\mu_m < \infty$, *m a nonnegative integer. Then for* $n = 0, 1, \cdots, m$

(a) $F_n(x) \uparrow \mu_n/n!$ as $x \uparrow \infty$;

(b) $\mu_{n+\alpha} = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-k)} \displaystyle\int_0^\infty x^{n+\alpha-k-1} \left[\frac{\mu_k}{k!} - F_k(x)\right] dx,$

$k = 0, 1, \cdots, n$, $\alpha \geqq 0$, $n+\alpha-k > 0$, *in the sense that if one side is finite so is the other*;

(c) $(-1/\lambda)^{n+1}\left[\varphi(\lambda) - \sum_0^n \mu_k(-\lambda)^k/k!\right] = \displaystyle\int_0^\infty e^{-\lambda x}\left[\frac{\mu_n}{n!} - F_n(x)\right] dx.$

The proof uses induction and integration by parts and is omitted. See Feller [2] for the first step.

For some reason Harkness and Shantaram [3] have been led to study the df's $n!F_n/\mu_n = G_n$ and they give a version of (*c*) for characteristic functions. Professor James Hannan has pointed out (oral communication) that an induction shows that

$$1 - G_n(x) = \int_0^\infty \left[\left(1 - \frac{x}{y}\right)^+\right]^n H_n(dy), \qquad x \geqq 0$$

where $H_n(y) = \int_0^y u^n F(du)/\mu_n$, so that G_n is the mixture of the minimums of n independent rv's with uniform distribution on $(0, y)$, where H_n is the mixing distribution.

In what follows we assume $\mu_m < \infty$ and define

$$\eta_{m,\alpha}(\lambda) = \lambda^{-m-1-\alpha}\left|\varphi(\lambda) - \textstyle\sum_0^m \mu_k(-\lambda)^k/k!\right|.$$

THEOREM 5. (a) *For* $0 < \alpha < 1$, $\mu_{m+\alpha} < \infty$ iff *for some* (*and thus all*) $c > 0$ $\int_0^c \eta_{m,\alpha} < \infty$, *and then* (*whether finite or not*)

$$\mu_{m+\alpha} = \frac{\Gamma(m+\alpha+1)}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^\infty \eta_{m,\alpha}.$$

(b) $\nu_m := EX^m \log X < \infty$ iff *for some* (*and all*) $c > 0$ $\int_0^c \eta_{m0} < \infty$ *and then* (*whether finite or not*)

$$\nu_m = m! \textstyle\int_0^\infty \eta_{m,0}.$$

PROOF. Since $\eta_{m,\alpha}(\lambda) \sim \lambda^{-1-\alpha}$ as $\lambda \to \infty$, $\alpha > -1$, part (a) is a simple consequence of Theorem 4 and Fubini's theorem. To prove part (b) note that

$$\int_0^c \eta_{m,0} = \int_0^\infty \frac{1-e^{-xc}}{x}\left[\frac{\mu_m}{m!} - F_m(x)\right] dx,$$

so if it is finite for some c it is finite for $c = \infty$. To complete the proof, use the following formula which follows by induction and integration by parts:

$$\nu_m := \int_0^\infty x^m \log x F(dx) = \int_0^\infty L^{(k+1)}(x)\left[\frac{\mu_k}{k!} - F_k(x)\right] dx,$$

where $L(x) = x^m \log x$, so that $L^{(k)}(x) = (m!/(m-k)!)\, x^{m-k} (\log x + a_k)$ $k = 0, \cdots, m$, and $L^{(m+1)}(x) = m!/x$.

REMARKS. (1) The above extends the results of Athreya [1] who shows that $EX \log X < \infty$ iff $\int_0^c \eta_{1,0} < \infty$ some $c > 0$.

(2) When X is integer valued (identify X and Z_1 and all corresponding notation) $f(e^{-\lambda}) = \varphi(\lambda)$ and the change of variable $s = e^{-\lambda}$ shows that

$$\eta_{1,\alpha-1}(\lambda) = \frac{\varphi(\lambda) - (1-\lambda)}{\lambda^{1+\alpha}} \sim \frac{f(s) - s}{(1-s)^{1+\alpha}} \quad \text{as} \quad \lambda \to 0, s \to 1.$$

Hence the B_α condition is equivalent to $\eta_{1,\alpha-1}(\lambda) \geqq K > 0$, $0 \leqq \lambda \leqq \lambda_0$. It is now clear that A_β implies $EZ_1^{\,1+\alpha} < \infty$, $0 \leqq \alpha < \beta < 1$ and that B_β implies $EZ_1^{\,1+\beta} = \infty$.

We now state the Abelian–Tauberian theorems of Laplace transforms in the form we need. Recall that a function $L: (0, \infty) \to (0, \infty)$ is *slowly varying* iff $\lim_{t \to \infty} L(xt)/L(t) = 1$ for all $x > 0$.

THEOREM 6. *Let X be a nonnegative* rv *with* df *F and Laplace transformation φ. Assume that* $0 < \alpha < 1$ *and that m is the nonnegative integer such that* $\mu_m < \infty = \mu_{m+1}$. *The following are equivalent* (*where* $k = 0, 1, \cdots, m$, *and* $x = 1/\lambda \to \infty$):

$$(a_k) \frac{\mu_k}{k!} - F_k(x) \sim \frac{\Gamma(m+\alpha-k)}{\Gamma(m+\alpha)} x^{-m-\alpha+k} L(x)$$

$$(a_{m+1}) F_{m+1}(x) \sim \frac{\Gamma(\alpha)}{(1-\alpha)\Gamma(m+\alpha)} x^{1-\alpha} L(x)$$

$$(b_k) \eta_{k,\alpha-1}(\lambda) \sim \frac{\Gamma(1-\alpha)\Gamma(\alpha)}{\Gamma(m+\alpha)} \lambda^{m-k} L(1/\lambda).$$

PROOF. The equivalence of the (a_k)'s is given in Feller [2], Theorem VIII.9.1 and the lemma to Theorem XIII.5.4 (which holds for $0 < |\rho| < \infty$). The equivalence of (a_m) and (b_m) is the familiar Abelian–Tauberian theorem, see Feller [2], Theorem XIII.5.4. The equivalence of the (b_k)'s is trivial.

This gives our final

THEOREM 7. *If* $P(Z_1 > x) \sim x^{-1-\alpha}L(x)$, *where* L *is slowly varying, then* f *satisfies condition* B_β *for all* $\beta > \alpha$, *whence* $E\tau^\gamma < \infty$ *for all* $\gamma < 1/\alpha$ *and* $EZ_1{}^\beta = \infty$ *for all* $\beta > \alpha$.

PROOF. For $s = e^{-\lambda}$, as $\lambda \to 0$

$$\frac{f(s)-s}{(1-s)^{1+\alpha}} \sim \eta_{1,\alpha-1}(\lambda) \sim L\left(\frac{1}{\lambda}\right).$$

By the representation of $L(x) = a(x) \exp \int_1^x \varepsilon(y)/y \, dy$ where $a(x) \to c \in (0, \infty)$ and $\varepsilon(y) \to 0$ (see Feller [2], corollary to Theorem VIII.9.1) we see that $x^\varepsilon L(x) \to \infty$ if $\varepsilon > 0$, whence $B_{\alpha+\varepsilon}$ holds.

REMARKS. It was hoped that $\int_0^\infty \eta_{1,\alpha-1} = \infty$(iff $EZ_1{}^{1+\alpha} = \infty$) would shed light on the "assertion" at the beginning of this note, but it has failed to do so as yet.

Seneta [6] did show that

$$E\tau < \infty \quad \text{iff} \quad \int_0^1 \frac{1-u}{f(u)-u} du < \infty,$$

but his approach does not seem to work for non-integral moments.

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