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A PROBLEM CONCERNING GENERALIZED AGE-DEPENDENT BRANCHING PROCESSES WITH IMMIGRATION

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In a generalized age-dependent branching process, immigrants are allowed to arrive according to a non-homogeneous Poisson process. A limit theorem is given for the critical case.

1. Introduction. We consider a generalized age-dependent branching process in which new objects arrive from outside the process according to a non-homogeneous Poisson process. It is assumed that, once they arrive, new objects reproduce and die according to the laws of the original process and, for simplicity, that all new objects are at age 0 when they arrive. Thus each new object begins a new branching process which behaves independently of all other objects present. Letting $I(t)$ be the number of immigrants arrived by time t and $Y(t)$ the total number of objects alive at time t, we may write:

$$Y(t) = \sum_{i=1}^{I(t)} Z_i(t - T_i), \qquad \text{where } Z_i(t),\ i = 1, 2, 3, \cdots, \tag{1}$$

are independent identically distributed branching processes, each generated by a single (immigrant) object and $T_1, T_2, T_3, \cdots$, are the successive arrival-times for the immigration process, $I(t)$. We adopt the usual conventions that $Y(t) = 0$ if $I(t) = 0$ and $Z_i(t) = 0$ for $t < 0$.

The main result will be the derivation of a (gamma) limit law for $Y(t)/t$, $t \to \infty$, in the case that the underlying branching process is critical (mean number of offspring per object $= 1$) and $\theta(t)/t$ converges to a (finite) constant, where $\theta(t) = EI(t)$. Similar results have been given by B. A. Sevast'yanov [9], for Markov branching processes and by J. Foster [5], for Galton–Watson processes, using different methods.

We shall not go into the details of the generalized age-dependent branching process here considered except to note that, while keeping to the basic independence assumptions common to all branching process theories, it is sufficiently general to allow an object to give birth throughout its life time and includes as special cases such processes as Kendall's age-dependent birth-and-death process [7] and Bellman and Harris' age-dependent branching process [6]. The reader is referred to Crump and Mode [1], Ryan [8] or the author [3] for a more complete discussion.

2. A basic equation. Letting the immigration process $I(t)$ be of the non-homogeneous Poisson type with $\theta(t) = EI(t)$, we may express its moment generating function $E[\exp(sI(t))]$ as $\exp\{(e^s - 1)\theta(t)\}$. We will assume that $\theta(t)$ is continuous

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and that the common mgf $K(s, t) \equiv E[\exp(sZ_i(t))]$ converges on a positive s-radius, for all $t \geqq 0$. Then it is not hard to establish that,

$$H(s,t) = \exp\{\textstyle\int_0^t [K(s,t-u)-1]\, d\theta(u)\}, \tag{2}$$

where $H(s, t) \equiv E[\exp sY(t)]$ is the mgf for the resultant process given in equation (1). The proof is essentially contained in [2], Section 4, and will not be repeated here.

3. A critical limit theorem. The behavior of the processes, $Z_i(t)$ are determined by the life-data of a typical object (say the initial object, for definiteness). We define the relevant quantities in terms of the random variable L = life-length, the random process $N(x)$ = the number of direct descendents born to the initial object by the time it reaches the age x, and $N \equiv N(\infty)$, the total number of direct descendents of the initial object. We now define constants: $b = \int_0^\infty x dEN(x)$, $\mu = EL/b$ and $a = \mu EN(N-1)/2b$. Then, from [3], Corollary 3.1, we have:

PROPOSITION. *Suppose* $EN = 1$, $EN^k < \infty$ *for* $k = 1, 2, 3, \cdots$, $EN(0+) = 0$, *and the constants* a, b, *and* μ *are finite and positive. Then*

$$\lim_{t\to\infty} t^{-n+1}\mu_n(t) = n!\mu a^{n-1}; \qquad n = 1, 2, 3, \cdots, \tag{4}$$

where $\mu_n(t) = EZ_i^{\,n}(t)$ *are the (common) moments of the underlying branching processes.*

We may now state a limit theorem for the process with immigration, $Y(t)$.

THEOREM. *Suppose the conditions of the Proposition hold and, in addition, that* $\theta(t)/t$ *converges to a finite, positive constant,* λ, *as* $t \to \infty$. *Then* $Y(t)/t$ *converges in distribution to a (gamma) law with mgf*: $(1-as)^{-\alpha}$, *where* $\alpha = \lambda\mu/a$.

PROOF. We consider

$$L(s,t) \equiv \log H(s/t,t) = \textstyle\int_0^t (K(s/t,t-u)-1)\, d\theta.$$

Expanding in moments and interchanging the order of summation and integration, we obtain:

$$L(s,t) = \textstyle\sum_{n=1}^\infty t^{-n} \int_0^t \mu_n(t-u)\, d\theta(u) s^n/n!.$$

By the Proposition and a standard Tauberian relation [4], the nth cumulant, $c_n(t) = t^{-n} \int_0^t \mu_n(t-u)\, d\theta(u)$, of $Y(t)/t$ converges to $c_n = \alpha a^n(n-1)!$, as $t \to \infty$, $n = 1, 2, 3, \cdots$. The sequence of cumulants, c_n, are (uniquely) those of a gamma law with *mgf*: $(1-as)^{-\alpha}$. Thus the corresponding moments $E(Y(t)/t)^n$ converge to the moments of this gamma law and, by [4], page 263, so do the distributions.

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