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RECURRENCE RELATIONS FOR THE MIXED MOMENTS OF ORDER STATISTICS[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a continuous distribution with cdf $P(x)$ and pdf $p(x)$. Let $X_{1:n} \leqq X_{2:n} \leqq \cdots \leqq X_{n:n}$ be the corresponding order statistics. Denote the first moment $E(X_{r:n})$ by $\mu_{r:n}$ $(1 \leqq r \leqq n)$ and the mixed moment $E(X_{r:n}, X_{s:n})$ by $\mu_{r,s:n}$ $(1 \leqq r \leqq s \leqq n)$. We assume that all these moments exist. Several recurrence relations between these moments are summarized by Govindarajulu [1]. In this note, we give a simple argument which generalizes some of the results given in [1]. These generalizations then lead to some modifications in the theorems given by Govindarajulu.

2. Recurrence relations. Let

$$W_1 = \{(u, v) : 0 \leqq u \leqq v \leqq 1\}$$

$$W_2 = \{(u, v) : 0 \leqq v \leqq u \leqq 1\}$$

$$R = W_1 \cup W_2$$

and

$$B(p, q, r) = \frac{\Gamma(p) \cdot \Gamma(q) \cdot \Gamma(r)}{\Gamma(p+q+r)} \quad \text{for} \quad p, q, r > 0.$$

Then by using the probability integral transformation $u = P(x)$ and $v = P(y)$ we can write for $1 \leqq r \leqq n$

$$\mu_{r:n} = \frac{1}{B(r, n-r+1)} \int_0^1 x(u) u^{r-1} (1-u)^{n-r} \, du, \tag{1}$$

and for $1 \leqq r < s \leqq n$

$$\begin{aligned} \mu_{r,s:n} &= \frac{1}{B(r, s-r, n-s+1)} \int_{W_1} x(u) y(v) u^{r-1} (v-u)^{s-r-1} (1-v)^{n-s} \, du \, dv \\ &= \frac{1}{B(r, s-r, n-s+1)} \int_{W_2} x(u) y(v) v^{r-1} (u-v)^{s-r-1} (1-u)^{n-s} \, du \, dv, \end{aligned}$$

where $x(u)$ and $y(v)$ denote that x and y are expressed as a function of u and v respectively.

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THEOREM 1. *For* $1 \leqq k \leqq n-1$

$$B(1, n-k, k)\mu_{k,n:n}+\sum_{i=0}^{k-1}(-1)^{n-i}\binom{k-1}{i}B(1, n-k, k-i)\mu_{1,n-k+1:n-i} \tag{2}$$
$$=\sum_{i=1}^{n-k}(-1)^{n-k-i}\binom{n-k-1}{i-1}\frac{\mu_{i:i}}{i}\cdot\frac{\mu_{n-i:n-i}}{n-i}.$$

PROOF. Let

$$\mathscr{I}=\int_R x(u)y(v)u^{k-1}(v-u)^{n-k-1}\,du\,dv.$$

Then on expanding $(v-u)^{n-k-1}$ we get

$$\mathscr{I}=\sum_{j=0}^{n-k-1}(-1)^{n-k-1-j}\binom{n-k-1}{j}\int_R x(u)y(v)u^{n-j-2}v^j du\,dv$$
$$=\sum_{i=1}^{n-k}(-1)^{n-k-i}\binom{n-k-1}{i-1}\int_0^1 x(u)u^{n-i-1}\,du\int_0^1 y(v)v^{i-1}\,dv,$$

which on using equation (1) reduces to the rhs of (2). Further we can also write

$$\mathscr{I}=\int_{W_1}x(u)y(v)u^{k-1}(v-u)^{n-k-1}\,du\,dv$$
$$+\int_{W_2}x(u)y(v)(-1)^{n-k-1}(u-v)^{n-k-1}[1-(1-u)]^{k-1}\,du\,dv$$
$$=B(k, n-k, 1)\mu_{k,n:n}+\sum_{j=0}^{k-1}(-1)^{n-j}\binom{k-1}{j}\int_{W_2}x(u)y(v)$$
$$\cdot(u-v)^{n-k-1}(1-u)^{k-1-j}\,du\,dv$$
$$=B(1, n-k, k)\mu_{k,n:n}+\sum_{j=0}^{k-1}(-1)^{n-j}\binom{k-1}{j}B(1, n-k, k-j)\mu_{1,n-k+1:n-j},$$

which is the lhs of (2). This completes the proof of the theorem.

It should be noted that equation (2) contains both $\mu_{1,n-k+1:n}$ and $\mu_{k,n:n}$. Hence for arbitrary parent distributions, Theorem 1 is useful only for $k = 1$ and in this case it is equivalent to Theorem 4.9 of [1].

COROLLARY 1. *If the parent distribution is symmetric about zero, then*

$$(1+(-1)^n)B(1, n-k, k)\mu_{1,n-k+1:n}$$
$$=\sum_{i=1}^{k-1}(-1)^{n-i+1}\binom{k-1}{i}B(1, n-k, k-i)\mu_{1,n-k+1:n-i} \tag{3}$$
$$+\sum_{i=2}^{n-k}(-1)^{n-k-i}\binom{n-k-1}{i-1}\frac{\mu_{i:i}}{i}\cdot\frac{\mu_{n-i:n-i}}{n-i}.$$

PROOF. We need only note that $\mu_{1:1}=0$ and $\mu_{k,n:n}=\mu_{1,n-k+1:n}$.

From Corollary 1 it follows that if the parent distribution is symmetric about zero, then for even values of n all the mixed moments $\mu_{1,s:n}$ $(s = 2, 3, \cdots, n)$ can be obtained provided that all the first and mixed moments in samples of sizes less than n are available. In particular by setting $k = n-1$ we have

$$2\mu_{1,2:n}=\sum_{i=1}^{n-2}(-1)^{i-1}\binom{n}{i}\mu_{1,2:n-i}.$$

This has been proved in [1] for the class of distributions satisfying $p'(x) = -xp(x)$.

Note that for odd values of n, the lhs of (3) vanishes and we get a relation involving the first and mixed moments in samples of sizes $n-1$ and less. For example, by setting $k = 2$ and $n = 2m+1$ in (3) we get

$$B(1, 2m-1, 1)\mu_{1,\,2m:\,2m} = \sum_{i=2}^{2m-1} (-1)^{i-1}\binom{2m-2}{i-1}\frac{\mu_{i:\,i}}{i}\cdot\frac{\mu_{2m+1-i:\,2m+1-i}}{2m+1-i}\,. \tag{4}$$

Equation (4) gives a different expression for $\mu_{1,2m:2m}$ than what is obtained by setting $k = 1$ and $n = 2m$ in (3), viz.,

$$2B(1, 2m-1, 1)\mu_{1,\,2m:\,2m} = \sum_{i=2}^{2m-2} (-1)^{i-1}\binom{2m-2}{i-1}\frac{\mu_{i:\,i}}{i}\cdot\frac{\mu_{2m-i:\,2m-i}}{2m-i},$$

because $\mu_{1:1} = 0$.

The implications of Corollary 1 in Theorems 4.12–4.14 of [1] are now obvious. These theorems essentially deal with the number of independent constraints among the first and mixed moments. We here give a modified version of Theorem 4.14 of [1].

THEOREM 2. *In order to find the first, second and mixed moments of order statistics in a sample of size n drawn from an arbitrary distribution symmetric about zero, given these moments for all sample sizes less than n, one has to evaluate at most one single integral if n is even; and one single integral and* $(n-1)/2$ *double integrals if n is odd.*

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REFERENCE

[1] GOVINDARAJULU, Z. (1963). On moments of order statistics and quasi-ranges from normal populations. *Ann. Math. Statist.* **34** 633–651.