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INFINITE DIVISIBILITY OF DISCRETE DISTRIBUTIONS, II

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1. Introduction. Let $\{P_i\}$, $i = 0, 1, 2 \cdots, P_0 \neq 0, P_1 \neq 0$ represent the probability of a random variable X taking on the values $0, 1, 2, \cdots$. Katti (1967) shows that a necessary and sufficient set of conditions for X to be infinitely divisible is that

$$\pi_i = iP_i/P_0 - \textstyle\sum_{j=1}^{i-1} \pi_{i-j}P_j/P_0 \geqq 0 \tag{1}$$

for $i = 1, 2, \cdots$. The aim of this paper is to derive a sufficient condition and to present some additional results.

2. A sufficient condition.

THEOREM 2.1. *A discrete distribution* $\{P_i\}$ $i = 0, 1, \cdots, P_0 \neq 0, P_1 \neq 0$ *is infinitely divisible if* $\{P_i/P_{i-1}\}$ $i = 1, 2, \cdots$ *forms a monotone increasing sequence.*

PROOF. Denote P_i/P_{i-1} by K_i. By assumption, $K_1 \leqq K_2 \leqq \cdots$. Note that since P_1 and P_0 are nonzero, $\{K_i\}$ $i = 1, 2, \cdots$ are positive and nonzero. Now,

$$\frac{P_i}{P_0} = \frac{P_i}{P_{i-1}}\frac{P_{i-1}}{P_{i-2}}\cdots\frac{P_1}{P_0} = \prod_{j=1}^{i} K_j. \tag{2}$$

From (1),

$$\pi_i = \frac{iP_i}{P_0} - \sum_{j=1}^{i-1}\frac{P_j\pi_{i-j}}{P_0} = i\left(\prod_{j=1}^{i} K_j\right) - \sum_{j=1}^{i-1}\pi_{i-j}\left(\prod_{k=1}^{j} K_k\right). \tag{3}$$

On replacing i by $(i+1)$ in (3), we get

$$\pi_{i+1} = (i+1)\left(\prod_{j=1}^{i+1} K_j\right) - \sum_{j=0}^{i-1}\pi_{i-j}\left(\prod_{k=1}^{j+1} K_k\right). \tag{4}$$

Hence,

$$\frac{\pi_{i+1}}{K_{i+1}} = (i+1)\left(\prod_{j=1}^{i} K_j\right) - \sum_{j=0}^{i-1}\frac{\pi_{i-j}}{K_{i+1}}\left(\prod_{k=1}^{j+1} K_k\right). \tag{5}$$

On using property (2), we get,

$$\frac{\pi_{i+1}}{K_{i+1}} \geqq i\left(\prod_{j=1}^{i} K_j\right) - \sum_{j=1}^{i-1}\pi_{i-j}\left(\prod_{k=1}^{j} K_k\right) - \frac{\pi_i K_1}{K_{i+1}}, \tag{6}$$

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or

$$\pi_{i+1}/K_{i+1} \geqq \pi_i[1-K_1/K_{i+1}] \geqq 0, \tag{7}$$

which leads to $\pi_{i+1} \geqq 0$ and proves that $\{P_i\}$, $i = 0, 1, \cdots$ is infinitely divisible.

Here it may be of interest to note that this inequality has been noticed in connection with renewal theory by Kendall (1967), Goldie (1967) and Horn (1970). For some additional results, reference may be made to Steutel (1967, 1968, and 1969). The importance of our paper therefore lies in dissociating this result from the particular application and proving it for the general case of integer-valued distributions.

EXAMPLE 2.1. Consider

$$P_i = c\theta^i/\{(i+a_1)(i+a_2)\cdots(i+a_n)\}, \tag{8}$$

$i = 0, 1, 2, \cdots$ where $0 < \theta < 1$, n is finite, $a_j > 0$ for $j = 1, \cdots, n$ and c is an appropriate constant so that $\sum_{i=0}^{\infty} P_i = 1$. Here,

$$K_i = \theta\left(\frac{i+a_1-1}{i+a_1}\right)\cdot\left(\frac{i+a_2-1}{i+a_2}\right)\cdot \cdots \cdot\left(\frac{i+a_n-1}{i+a_n}\right). \tag{9}$$

Clearly $\{(i+a_j-1)/(i+a_j)\}$, $i = 1, 2, \cdots$ is a monotone increasing sequence for all j and hence $\{K_i\}$ $i = 1, 2, \cdots$ is a monotone increasing sequence. This proves that $\{P_i\}$ $i = 0, 1, \cdots$ is an infinitely divisible distribution.

The logarithmic distribution with $P_i = c\theta^{i+1}/(i+1)$ for $i = 0, 1, \cdots$ can also be shown to satisfy this sufficient condition and hence, it is also infinitely divisible. Notice that the proof through the sufficient condition is far simpler than the original proof in Katti (1967).

3. Applications. Most of the infinitely divisible discrete distributions given in the literature have characteristic functions of the form $(\phi(t))^\theta$ where θ is a parameter. The importance of Theorem 2.1 is that it can be used to generate a number of infinitely divisible distributions that do not have characteristic functions of that form.

THEOREM 3.1. *Let* $\{P_i\}$ $i = 0, 1, \cdots$ *be an infinitely divisible distribution such that* $P_0 \neq 0, P_1 \neq 0, K_i = P_i/P_{i-1}$ *and* $\{f(K_i)\}$ $i = 1, 2, \cdots$ *forms a monotone increasing sequence. Let* $f(K_i)$ *be a function such that* $f(K_1) > 0$ *and* $\{f(K_i)\}$ $i = 1, 2, \cdots$ *forms a monotone increasing sequence, and* $\lim_{i\to\infty} f(K_i) < 1$. Then,

$$Q_i = \textstyle\prod_{j=1}^{i} f(K_j)/\sum_{i=1}^{\infty}\prod_{j=1}^{i} f(K_j) \tag{10}$$

forms an infinitely divisible distribution.

The proof follows as a direct consequence of Theorem 2.1. Some particular forms of $f(K)$ are given below:

COROLLARY 3.1. *If* $\{P_i\}$ $i = 0, 1, 2, \cdots$ *with* $P_0 \neq 0$, $P_1 \neq 0$ *is an infinitely divisible distribution such that* $\{K_i\} = \{P_i/P_{i-1}\}$ $i = 1, 2, \cdots$ *forms a monotone increasing sequence, then* $\{Q_i\}$ $i = 0, 1, 2, \cdots$ *defined as below form infinitely divisible distributions*:

(11) (a) $Q_0 = \dfrac{P_0 + c}{1+c}, \quad Q_i = \dfrac{P_i}{1+c}$ *for* $i = 1, 2, \cdots$

(12) (b) $Q_i = P_i^n / \sum_{i=0}^{\infty} P_i^n$ *for* $i = 0, 1, 2, \cdots$; n *finite.*

(13) (c) $Q_0 = c/(1+c), \quad Q_i = P_i/(1+c)$ *for* $i = 1, 2, \cdots$, *where* $c \geqq P_1^2/P_2$.

(14) (d) $Q_i = i^j P_i / \sum_{i=0}^{\infty} i^j P_i$ *for* $i = 0, 1, 2, \cdots$, *for any* j *for which* $\sum_{i=0}^{\infty} i^j P_i$ *converges.*

Substituting of particular functional forms in this corollary is left out for brevity.

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